Homework #9 Assigned on April 4, 2024
due April 11, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 113

Problem 1 *(Poisson Integral Formula from Mean Value Property and Biholomorphic Map of Open Unit Disk – from Stein & Shakarchi, p.109, #2).* A harmonic function h on an open subset Ω of $\mathbb{C}$ means that the second-order partial derivatives of h are continuous and h satisfies the Laplace equation

$$\Delta u := \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \equiv 0$$

on Ω. Let u be a harmonic function in the open unit disk $\mathbb{D} = \left\{ z \in \mathbb{C} \,\Big|\, |z| < 1 \right\}$ which is continuous on the topological closure $\overline{\mathbb{D}}$ of $\mathbb{D}$. Prove the Poisson integral formula

$$u(z_0) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \frac{1 - |z_0|^2}{\left|e^{i\theta} - z_0\right|^2} u(e^{i\theta}) d\theta$$

for $z_0 \in \mathbb{D}$ by using the following two steps.

(i) Prove the the mean value property

$$u(0) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} u(e^{i\theta}) d\theta$$

of a harmonic function u on $\mathbb{D}$ which is continuous up to the boundary by integrating

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

over $\theta \in [0, 2\pi]$ with the use of polar coordinates (r, θ) to show that the average of u over the circle $\{|z| = r\}$ is independent of $0 < r \leq 1$.

(ii) Apply **(i)** to $u_0(z) = u(T(z))$ with a change of variables in the integral, where $T : \mathbb{D} \to \mathbb{D}$ is a biholomorphic map of $\mathbb{D}$ defined by

$$T(z) = \frac{z_0 - z}{1 - \overline{z_0} z}.$$

Problem 2 *(Poisson Integral Formula for Upper Half Plane).* Let u be a harmonic function on the open upper half plane $\mathbb{H}$ which is continuous up to the boundary of $\mathbb{H}$ in the Riemann sphere. Prove the Poisson integral formula

$$u(x+iy) = \frac{1}{\pi}\int_{t=-\infty}^{\infty} u(t)\,\frac{y}{(x-t)^2+y^2}\,dt$$

with $y \geq 0$ and $x \in \mathbb{R}$ on the upper half-plane $\mathbb{H}$ by using the biholomorphic map between $\mathbb{H}$ and the open unit disk $\mathbb{D}$ and using the Poisson integral formula for the open unit disk $\mathbb{D}$.

Problem 3 *(Derivation of Poisson Kernel for Upper Half Plane by Using Normal Derivative of Green's Function).* Derive the Poisson kernel for the upper half-plane $\mathbb{H}$ (given in Problem 2 above) as the normal derivative of Green's function for $\mathbb{H}$ and $z \in \mathbb{H}|$, where Green's function $G(\zeta, z)$ for $\mathbb{H}$ and $z \in \mathbb{H}$ is obtained as the electrostatic potential due to a point charge at $z \in \mathbb{H}$ and another point charge at $\bar{z}$ of opposite sign so that $\Delta_\zeta G(\zeta, z)$ equals the Dirac delta at z and the value of $G(\zeta, z)$ vanishes at every boundary point of $\mathbb{H}$.

Problem 4 *(Derivation of Poisson Kernel from Harmonic Measure in Terms of Argument Function).* The *harmonic measure* $\omega_{\alpha,\beta}(z)$, for a point $z \in \mathbb{D}$ and the arc

$$\widehat{\mathrm{arc}}\,(e^{i\alpha}, e^{i\beta})$$

in the unit circle between $e^{i\alpha}$ and $e^{i\beta}$ (with $\alpha, \beta \in \mathbb{R}$), is defined as the value at z of the harmonic function on $\mathbb{D}$ whose boundary value is the characteristic function of the arc

$$\widehat{\mathrm{arc}}\,(e^{i\alpha}, e^{i\beta}).$$

Denote by $\arg(\cdot)$ the argument function which is the imaginary part of the logarithmic function $\log(\cdot)$ and which is defined only after an appropriate cut is taken and an appropriate range is given for its value.

(a) For $0 \leq \alpha < \beta < \frac{\pi}{2}$, use the harmonic property of the argument function $\arg(\cdot)$ to verify that

$$\omega_{\alpha,\beta}(z) = \frac{1}{\pi}\left(\arg\left(\frac{z-e^{i\beta}}{z-e^{i\alpha}}\right) - \frac{1}{2}(\beta-\alpha)\right)$$

after appropriate cuts and ranges of values are chosen to define the argument function $\arg(\cdot)$.

(b) Derive the Poisson integral formula by verifying that

$$\lim_{\beta\to\alpha}\frac{\omega_{\alpha,\beta}(z)}{\beta-\alpha}=\frac{1}{2\pi}\frac{1-|z|^2}{|e^{i\alpha}-z|^2},$$

with the use of arguments from plane Euclidean geometry and polynomial equations in Cartesian coordinates for objects in plane Euclidean geometry.

Problem 5 *(Elliptic Integrals as Lengths of Sides of Rectangle in Schwarz-Christoffel Transformation – from Stein & Shakarchi, p.254, #24).* For $0<k<1$, in the Schwarz-Christoffel transformation

$$w=\int_{\zeta=0}^{z}\frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}},$$

which maps the upper half-plane (with coordinate z) to the rectangle whose vertices in the w-plane correspond to the points

$$-\frac{1}{k},-1,1,\frac{1}{k}$$

on the real axis of the z-plane, the length of the horizontal side of the rectangle is given by $2K$ and the length of the vertical side of the rectangle is given by K', where

$$K(k)=\int_{x=0}^{1}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}\quad\text{and}\quad K'(k)=\int_{x=0}^{\frac{1}{k}}\frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}$$

(defined for $0<k<1$) are known as *elliptic integrals*, which will be discussed later in the course in the context of periods of elliptic functions when we talk about elliptic functions as special functions. These elliptic integrals $K(k)$ and $K'(k)$ satisfy a number of interesting identities given below.

(a) Show that if $\tilde{k}^2=1-k^2$ and $0<\tilde{k}<1$, then

$$K'(k)=K(\tilde{k}).$$

Hint: Change variables

$$x=\frac{1}{\sqrt{1-\tilde{k}^2y^2}}$$

in the integral defining $K'(k)$.

(b) Show that if $\tilde{k}^2 = 1 - k^2$ and $0 < \tilde{k} < 1$, then

$$K(k) = \frac{2}{1+\tilde{k}} K\left(\frac{1-\tilde{k}}{1+\tilde{k}}\right).$$

Hint: Change variables

$$x = \frac{2t}{1 + \tilde{k} + (1 - \tilde{k})t^2}.$$

(c) Show that for $0 < k < 1$ one has

$$K(k) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right),$$

where $w = F(\alpha, \beta, \gamma; z)$ is the hypergeometric function

$$\begin{aligned} F(\alpha, \beta, \gamma; z) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_{t=0}^{1} t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt \\ &= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n \end{aligned}$$

for $\alpha > 0$, $\gamma > \beta > 0$, $|z| < 1$, which satisfies the *Gauss hypergeometric differential equation*

$$z(z-1)\frac{d^2w}{dz^2} + ((\alpha + \beta + 1)z - \gamma)\frac{dw}{dz} + \alpha\beta w = 0.$$