Homework #8 Assigned on March 28, 2024
due April 4, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 113

For Problems 1, 2, and 3 below concerning Schwarz-Christoffel transformations, recall the material on Gamma function and Beta function on pp.11-15 in the posted lecture notes of Lecture 8 on "Evaluation of Denite Integrals by Residues and the Use of Branches of Holomorphic Functions".

Problem 1 *(Two Examples of Schwarz-Christoffel Transformations for Rectangles – From #20 on p.252 of Stein's Complex Analysis).* Verify the following two statements.

(a) The function

$$\int_{\zeta=0}^{z} \frac{d\zeta}{\sqrt{\zeta(\zeta-1)(\zeta-\lambda)}} \quad \text{with} \quad \lambda \in \mathbb{R} \quad \text{and} \quad \lambda \neq 0, 1$$

maps the upper half-plane $\mathbb{H}$ conformally to a rectangle, one of whose vertices is the image of the point at infinity.

(b) In the case $\lambda = -1$, the image of

$$\int_{\zeta=0}^{z} \frac{d\zeta}{\sqrt{\zeta(\zeta^2-1)}}$$

is a square whose side lengths are

$$\frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{2\pi}}.$$

Problem 2 *(Example of Schwarz-Christoffel Transformations for Triangles – From #21 on p.253 of Stein's Complex Analysis).* **(a)** Show that

$$\int_{\zeta=0}^{z} \zeta^{-\beta_1}(1-\zeta)^{-\beta_2} d\zeta$$

with $0 < \beta_1 < 1$, $0 < \beta_2 < 1$, and $1 < \beta_1 + \beta_2 < 2$, maps the upper half-plane $\mathbb{H}$ conformally to a triangle whose vertices are the images of 0, 1, and ∞, and with angles $\alpha_1\pi$, $\alpha_2\pi$, and $\alpha_3\pi$, where $\alpha_j + \beta_j = 1$ and $\beta_1 + \beta_2 + \beta_3 = 2$.

(b) What happens when $\beta_1 + \beta_2 = 1$?

(c) What happens when $0 < \beta_1 + \beta_2 < 1$?

(d) In Part **(a)** the length of the side of the triangle opposite angle $\alpha_j \pi$ is

$$\frac{\sin(\alpha_j \pi)}{\pi} \Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3).$$

Problem 3 *(Example of Schwarz-Christoffel Transformation for Regular Polygon – from Stein & Shakarchi, p.253, #23).* Prove that if

$$F(z) = \int_{\zeta=1}^{z} \frac{d\zeta}{(1-\zeta^n)^{\frac{2}{n}}},$$

then F maps the open unit disk $\mathbb{D}$ conformally onto the interior of a regular polygon with n sides and perimeter

$$2^{\frac{n-2}{n}} \int_{\theta=0}^{\pi} (\sin\theta)^{-\frac{2}{n}}\, d\theta.$$

Hint: The biholomorphic property of F follows from tracing its behavior along the boundary of $\mathbb{D}$ and the argument principle.

Problem 4. **(a)** *(Biholomorphic Map of Bounded Domain Fixing 1-Jet at One Point – from Stein & Shakarchi, p.66, #9).* Let Ω be a bounded open subset of $\mathbb{C}$ and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\varphi(z_0) = z_0 \quad \text{and} \quad \varphi'(z_0) = 1,$$

then φ is the identity.

Hint: Reduce the general case to $z_0 = 0$. Write $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0, and prove that if $\varphi_k = \varphi \circ \cdots \circ \varphi$ (where φ appears k times), then $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$ as $z \to 0$. Apply the Cauchy inequalities and let $k \to \infty$ to conclude the proof. Here the standard order O notation $f(z) = O(g(z))$ as $z \to 0$ means "at least of the order of" in the sense that $|f(z)| \le C|g(z)|$ for some constant C as $z \to 0$.

(b) *(Technique of Square Root Map to Obtain Uniform Bound on Compact Subsets).* Let $R > 0$ and let $\mathbb{D}_R$ be the open unit disk of radius R center at 0. For $n \in \mathbb{N}$ let $g_n : \mathbb{D}_R \to \mathbb{C}$ be holomorphic and injective such that $g_n(0) = 0$. Let c_n be a point in $\mathbb{C} - g_n(\mathbb{D}_R)$ such that $|c_n|$ is the distance between 0 and $\mathbb{C} - g_n(\mathbb{D}_R)$. Assume that $\sup_{n\in\mathbb{N}} |c_n| < \infty$. Prove that g_n is uniformly bounded on any compact subset of $\mathbb{D}_R$.

Hint: Let $h_n(z) = \frac{g_n(z)}{c_n}$. It suffices to show that h_n is uniformly bounded on any compact subset of $\mathbb{D}_R$. The image of h_n contains $\mathbb{D}$ and does not contain 1. Let $\varphi_n(z)$ be the branch of $\sqrt{1 - h_n(z)}$ on the simply connected domain $\mathbb{D}_R$ whose value is 1 at 0. Let $\psi(w)$ be the branch of $\sqrt{1-w}$ on the simply connected domain $\mathbb{D}$ in $\mathbb{C}$ (with coordinate w) whose value at $w = 1$ is -1. The image of φ_n is disjoint from the image of ψ. The image of ψ contains a closed disk of radius $r > 0$ centered at b. Consider the bound of $\frac{1}{\varphi_n(z) - b}$.

Remark: The technique of using the square root map here is analogous to its use in the proof of the Riemann mapping theorem.

(c) *(Biholomorphism of Proper Subdomain of $\mathbb{C}$ Not Necessarily Simply Connected).* Let Ω be a connected open subset of $\mathbb{C}$ not equal to $\mathbb{C}$. Let $a \in \Omega$. Prove that if f is a biholomorphic self-map of Ω such that $f(z_0) = z_0$ and $f'(z_0) > 0$, then $f(z) \equiv z$.

Hint: First reduce to the special case $z_0 = 0$ and $f'(z_0) \geq 1$. Let R be the radius of the largest disk contained in Ω centered at 0. Let g_n be the map obtained by composing n copies of the map f. Apply Part**(b)** to the restriction g_n to $\mathbb{D}_R$ to conclude that $f'(0) = 1$ by observing that $\sup_{n \in \mathbb{N}} |c_n| \leq R$ with c_n defined in Part**(b)**. Apply the argument of Cauchy's inequality in Part**(a)** to g_n on $\mathbb{D}_R$.

Problem 5 *(Uplifting Force on Joukowski Airfoil).* Recall the following from pp.16-17 in the posted lecture notes of Lecture 14 on "Application to Fluid Flow, Temperature Distribution, Electrostatic Potential and Airfoil Lift".

On $\mathbb{C}$ with coordinate z, the computation of the force in the direction $\alpha + \frac{\pi}{2}$ on the airfoil $[-2, 2]$ from the irrotational incompressible fluid flow of constant density 1 with velocity $e^{i\alpha}$ at infinity is transformed by the Joukowski transformation $z \mapsto z + \frac{1}{z}$ to the case where the airfoil is the closed unit disk $\overline{\mathbb{D}}$ and the complex velocity potential of the fluid flow is

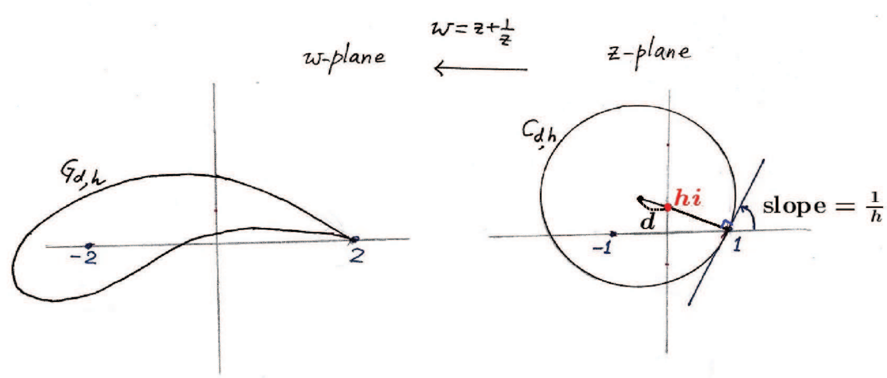
$$f(z) = e^{-i\theta} z + \frac{1}{e^{-i\theta} z} + a \log |z|$$

for some positive constant a, which is determined to be $2 \sin \alpha$ by the Kutta condition of the vanishing of the fluid velocity at the sharp rear end $z = 1$ of the closed unit disk $\overline{\mathbb{D}}$. The force is computed from the circulation of the fluid flow around the boundary of the closed unit disk $\overline{\mathbb{D}}$ and the theorem of Kutta-Joukowski which expresses the force in terms of the fluid density, the circulation, and the velocity at infinity.

The *Joukowski airfoil* $G_{d,h}$ (with positive parameters d and h) is defined as the image, under the Joukowski transformation

$$z \mapsto z + \frac{1}{z},$$

of the closed disk $C_{d,h}$ which is tangential to the line through $z = 1$ with slope $\frac{1}{h}$ and whose center is of distance d from $z = hi$. The center of $C_{d,h}$ is on the line joining the two points $z = hi$ and $z = 1$. (see the figure below)



(a) Consider the irrotational incompressible fluid flow of constant density 1 outside the disk airfoil $C_{d,h}$ with velocity $e^{i\alpha}$ at infinity. Assume the Kutta condition of the vanishing of the fluid velocity at the sharp rear end $z = 1$ of $C_{d,h}$. Compute the force in the direction $\alpha + \frac{\pi}{2}$ on the disk airfoil $C_{d,h}$ from the fluid flow.

Hint: Compute by modifying the case where the irrotational incompressible fluid flow of constant density 1 outside the airfoil $\overline{\mathbb{D}}$ with sharp rear end at $z = 1$ and with $e^{i\alpha}$ as the velocity at infinity, which is discussed on p.17 of the posted lecture notes of Lecture 14 on "Application to Fluid Flow, Temperature Distribution, Electrostatic Potential and Airfoil Lift". The modification is that now the disk $C_{d,h}$ replaces the closed unit disk $\overline{\mathbb{D}}$.

(b) Consider the irrotational incompressible fluid flow of constant density 1 outside the Joukowski airfoil $G_{d,h}$ with velocity $e^{i\alpha}$ at infinity. Assume the Kutta condition of the vanishing of the fluid velocity at its sharp rear end $z = 2$ of $G_{d,h}$. Compute the force in the direction $\alpha + \frac{\pi}{2}$ on the Joukowski airfoil $G_{d,h}$ from the fluid flow.

Hint: Use Part **(a)** and the Joukowski transformation $z \mapsto w = z + \frac{1}{z}$, which can be rewritten as

$$\left(\frac{z+1}{z-1}\right)^2 = \frac{w+2}{w-2}.$$