Evaluation of Definite Integrals by Residues and the Use of Branches of Holomorphic Functions

Some definite integrals which can be evaluated by the theory of residues involve branches of holomorphic functions. Examples are

$$\int_{x=0}^{\infty} \frac{x^a\,dx}{1+x} \quad \text{with} \quad -1<a<0,$$
$$\int_{x=0}^{1} \frac{dx}{x^\alpha(1-x)^{1-\alpha}} \quad \text{with} \quad 0<\alpha<1,$$
$$\int_{x=0}^{\infty} \frac{\log x\,dx}{x^2+1}, \qquad \int_{x=0}^{\infty} \frac{(\log x)^2\,dx}{x^4+1}.$$

Branches of Holomorphic Functions. Before we look at such definite integrals, we would like to discuss how to define a branch of a holomorphic function. Sometimes a holomorphic function is defined as the inverse of another holomorphic function, for example, $z=\log w$ as the inverse of $w=e^z$ and $z=w^{\frac{1}{n}}$ as the inverse of $w=z^n$ for an integer $n\geq 2$, and such an inverse is multi-valued and cannot be considered as a function unless choices of the many possible values of the multi-valued inverse function are made to give single-valuedness.

The choices of the many possible values of the multi-valued inverse function need to be done in such a way that the resulting single-valued function is continuous so that it is holomorphic as the inverse of a holomorphic function, because to conclude that the limit of the difference quotient

$$\lim_{w\to w_0} \frac{w-w_0}{z-z_0}$$

exists and is equal to the reciprocal of the nonzero number

$$\lim_{z\to z_0} \frac{z-z_0}{w-w_0}$$

we need to be able to use the fact that $w\to w_0$ implies $z\to z_0$, which means that the inverse function $z=z(w)$ of the holomorphic function $w=w(z)$ is continuous (and the complex derivative of $w=z(z)$ is nonzero).

Branches of the Logarithmic Function. Let us consider the logarithmic function $z = \log w$ as the inverse of the exponential function $w = e^z = e^x e^{iy}$. Since the polar coordinates of $w = e^z = e^x e^{iy}$ are $(e^x, y) = (|w|, \arg w)$, it means that in polar coordinates for w, $z = \log w$ is equal to $x+iy = \log|w|+i\arg w$. What makes $z = \log w$ multi-valued is its imaginary part $y = \arg w$. The choice of the numerical value of $\arg w$ is additively up to an integer times 2π. No matter what choice for the numerical value of $\arg w$ we make at a nonzero point $w = w_0$, when we go around the origin once in a counter-clockwise sense along the circle of radius $|w_0|$ centered at the origin, by the required condition of continuity, the value chosen for $\arg w_0$ after going around the origin once must be equal to 2π plus the the value chosen for $\arg w_0$ before going around the origin once. This is clearly impossible unless we restrict the domain of definition for $z = z(w)$ so that going around the origin once is not allowed. One easy way to achieve this is to restrict the domain to be in the open sector $\alpha < \arg w < \alpha + 2\pi$ for some $\alpha \in \mathbb{R}$. This restriction means that we have chosen a *branch* of the logarithmic function.

For any real number γ, we can define a branch of $z = w^\gamma$ as $z = e^{\gamma \log w}$ after we define a branch of $\log w$.

To illustrate how to use branches of holomorphic functions ad residue theory to evaluate definite integrals, we work out now two examples.

First Example. The first example involves only a straightforward way of defining a branch of the holomorphic function used. We will work it out twice, each time with a different contour and a different way of choosing the holomorphic function and defining its branch used. The second example involves a more complicated way of defining a branch of the holomorphic function used.

First Way of Doing First Example. The first example is

$$\int_{x=0}^{\infty} \frac{\log x \, dx}{x^2+1}.$$

We use the meromorphic function

$$f(z) = \frac{\log z}{z^2+1},$$

where the branch of the holomorphic function $\log z$ is defined by $\log z = \log r + i\theta$ when $z = re^{i\theta}$ with $0 \leq \theta \leq \pi$. The contour used is the boundary of the indented upper half disk

$$\left\{ z \in \mathbb{C} \,\middle|\, \varepsilon < |z| < R,\ \mathrm{Im}\, z > 0 \right\}$$

with radius R and an indentation of radius $\varepsilon > 0$. Then

$$\begin{aligned}
&\int_{x=-R}^{-\varepsilon} \frac{\log(-x) + \pi i}{x^2+1} dx - \int_{\theta=0}^{\pi} \frac{\log \varepsilon + i\theta}{\varepsilon^2 e^{2i\theta}+1} d\left(\varepsilon e^{i\theta}\right) + \int_{x=\varepsilon}^{R} \frac{\log x}{x^2+1} dx \\
&= 2\pi i \,\mathrm{Res}_{z=i} \frac{\log z}{z^2+1} = 2\pi i \lim_{z \to i} \frac{(z-i)\log z}{z^2+1} \\
&= 2\pi i \lim_{z \to i} \frac{\log z}{z+i} = 2\pi i \, \frac{\frac{\pi}{2} i}{2i} = \frac{\pi^2 i}{2}
\end{aligned}$$

for $0 < \varepsilon < 1 < R$. The middle term

$$\int_{\theta=0}^{\pi} \frac{\log \varepsilon + i\theta}{\varepsilon^2 e^{2i\theta}+1} d\left(\varepsilon e^{i\theta}\right)$$

on the left-hand side approaches 0 as $\varepsilon \to 0$. The first term

$$\int_{x=-R}^{-\varepsilon} \frac{\log(-x) + \pi i}{x^2+1} dx$$

on the left-hand side can be rewritten as

$$\int_{x=-R}^{-\varepsilon} \frac{\log(-x) + \pi i}{x^2+1} dx = \int_{x=\varepsilon}^{R} \frac{\log x + \pi i}{x^2+1} dx$$

by the change $x \mapsto -x$ of the dummy variable and an interchange of the upper and lower limits of integration. Note that in the change $x \mapsto -x$ of the dummy variable the expression dx contributes to a change of sign, but the interchange of the upper and lower limits of integration contributes another change of sign, resulting in no net change of sign. By passing to limit, as $\varepsilon \to 0$ and $R \to \infty$, of the real part of the equation

$$\int_{x=-R}^{-\varepsilon} \frac{\log(-x) + \pi i}{x^2+1} dx - \int_{\theta=0}^{\pi} \frac{\log \varepsilon + i\theta}{\varepsilon^2 e^{2i\theta}+1} d\left(\varepsilon e^{i\theta}\right) + \int_{x=\varepsilon}^{R} \frac{\log x}{x^2+1} dx = \pi^2 i$$

we get

$$\int_{x=0}^{\infty} \frac{\log x \, dx}{x^2+1} = 0.$$

As a bonus of the residue computation, by taking the imaginary part of the same equation and passing to limit as $\varepsilon \to 0$ and $R \to \infty$, we get

$$\int_{x=0}^{\infty} \frac{\pi}{x^2+1}\,dx = \frac{\pi^2}{2}$$

or

$$\int_{x=0}^{\infty} \frac{1}{x^2+1}\,dx = \frac{\pi}{2}$$

which is the same result as we would get by applying the fundamental theorem of calculus to

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{x^2+1}.$$

Second Way of Doing First Example. We now redo the same example with a different choice of the meromorphic function and a different choice of the contour of integration. We use the meromorphic function

$$f(z) = \frac{(\log z)^2}{z^2+1},$$

where the branch of the holomorphic function $\log z$ is defined by $\log z = \log r + i\theta$ when $z = re^{i\theta}$ with $0 < \theta < 2\pi$. The contour used is the boundary of the open annulus of radii $0 < \varepsilon < 1 < R$ minus the horizontal closed right half-strip of width 2ε centered at the x-axis

$$\left\{ z \in \mathbb{C} \,\middle|\, \varepsilon < |z| < R \right\} - \left\{ z \in \mathbb{C} \,\middle|\, -\varepsilon \leq \operatorname{Im} z \leq \varepsilon,\ \operatorname{Re} z \geq 0 \right\}.$$

Then

$$\int_{0 \leq x \leq R,\, y=\varepsilon} \frac{(\log z)^2}{z^2+1}\,dz + \int_{|z|=R,\, |y|\geq\varepsilon \text{ if } x>0} \frac{(\log z)^2}{z^2+1}\,dz - \int_{0 \leq x \leq R,\, y=-\varepsilon} \frac{(\log z)^2}{z^2+1}\,dz$$

$$= 2\pi i \operatorname{Res}_{z=i} \frac{(\log z)^2}{z^2+1} + 2\pi i \operatorname{Res}_{z=-i} \frac{(\log z)^2}{z^2+1}$$

$$= 2\pi i \lim_{z\to i} \frac{(z-i)(\log z)^2}{z^2+1} + 2\pi i \lim_{z\to -i} \frac{(z+i)(\log z)^2}{z^2+1}$$

$$= 2\pi i \lim_{z\to i} \frac{(\log z)^2}{z+i} + 2\pi i \lim_{z\to -i} \frac{(\log z)^2}{z-i}$$

$$= 2\pi i \frac{\left(\frac{\pi}{2} i\right)^2}{2i} + 2\pi i \frac{\left(\frac{3\pi}{2} i\right)^2}{-2i} = -\frac{\pi^3}{4} + \frac{9\pi^3}{4} = 2\pi^3.$$

The middle term
$$\int_{|z|=R,\,|y|\geq\varepsilon\text{ if }x>0}\frac{(\log z)^2}{z^2+1}\,dz$$
on the left-hand side approaches 0 as $\varepsilon\to 0$ and $R\to\infty$. As $\varepsilon\to 0$ and $R\to\infty$, the limit of the first term
$$\int_{0\leq x\leq R,\,y=\varepsilon}\frac{(\log z)^2}{z^2+1}\,dz$$
on the left-hand side is
$$\int_{x=0}^{\infty}\frac{(\log x)^2}{x^2+1}dx.$$
As $\varepsilon\to 0$ and $R\to\infty$, the limit of the third term
$$-\int_{\varepsilon\leq x\leq R,\,y=-\varepsilon}\frac{(\log z)^2}{z^2+1}\,dz$$
on the left-hand side is
$$-\int_{x=0}^{\infty}\frac{(\log x+2\pi i)^2}{x^2+1}dx=-\int_{x=0}^{\infty}\frac{(\log x)^2+2\pi i\log x-4\pi^2}{x^2+1}dx$$
whose imaginary part is
$$-2\pi\int_{x=0}^{\infty}\frac{\log x}{x^2+1}\,dx.$$
By taking the imaginary part of
$$\int_{\varepsilon\leq x\leq R,\,y=\varepsilon}\frac{(\log z)^2}{z^2+1}\,dz+\int_{|z|+R,\,|y|\geq\varepsilon}\frac{(\log z)^2}{z^2+1}\,dz-\int_{\varepsilon\leq x\leq R,\,y=-\varepsilon}\frac{(\log z)^2}{z^2+1}\,dz=2\pi^3$$
and passing to limit as $\varepsilon\to 0$ and $R\to\infty$, we obtain again the conclusion that
$$\int_{x=0}^{\infty}\frac{\log x}{x^2+1}\,dx=0.$$
As a bonus, when we take the real part of
$$\int_{\varepsilon\leq x\leq R,\,y=\varepsilon}\frac{(\log z)^2}{z^2+1}\,dz+\int_{|z|+R,\,|y|\geq\varepsilon}\frac{(\log z)^2}{z^2+1}\,dz-\int_{\varepsilon\leq x\leq R,\,y=-\varepsilon}\frac{(\log z)^2}{z^2+1}\,dz=2\pi^3$$

and passing to limit as $\varepsilon \to 0$ and $R \to \infty$, we obtain

$$-\int_{x=0}^{\infty} \frac{-4\pi^2}{x^2+1}\, dx = 2\pi^3,$$

which yields

$$\int_{x=0}^{\infty} \frac{1}{x^2+1}\, dx = \frac{\pi}{2}.$$

Again this is the same result as we would get by applying the fundamental theorem of calculus to

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{x^2+1}.$$

Second Example. Now we look at the second example of a definite integral whose evaluation involves a more complicated way of defining a branch of the holomorphic function used.

$$\int_{x=0}^{1} \frac{dx}{x^\alpha (1-x)^{1-\alpha}} \quad \text{with} \ \ 0 < \alpha < 1.$$

For the evaluation of this definite integral by residue theory, we introduce the following function $f(z) = z^{-\alpha}(1-z)^{\alpha-1}$. We have to define an appropriate branch for this function. First we choose a branch for the function $z^{-\alpha}$ and then choose a branch for the function $(1-z)^{1-\alpha}$ and then put the two branches together.

To define a branch for $z^{-\alpha}$, we take away the slit $[0,\infty) \subset \mathbb{R}$ so that we restrict the numerical value for the angle θ in the polar representation of $z = re^{i\theta}$ to $0 < \theta < 2\pi$ and for such a restriction of the value of θ the value of $z^{-\alpha}$ is defined to be $r^{-\alpha}e^{-i\alpha\theta}$.

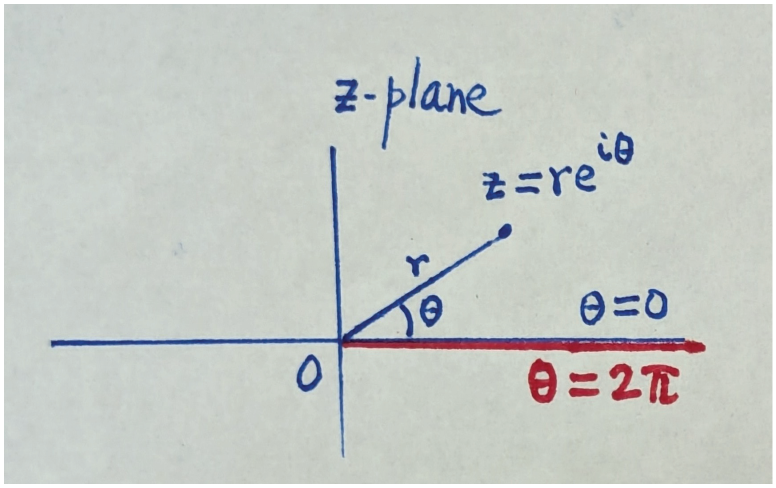

Figure 1: Using the cut $[0, \infty)$ in $\mathbb{R}$ to define a branch of $z^{-\alpha}$

To define a branch for $(1-z)^{\alpha-1}$, we take away the slit $(-\infty, 0] \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $1-z$ so that we restrict the numerical value for the angle φ in the polar representation of $1-z = \rho e^{i\varphi}$ to $-\pi < \varphi < \pi$ and for such a restriction of the value of φ the value of $(1-z)^{\alpha-1}$ is defined to be $\rho^{\alpha-1}e^{i(\alpha-1)\varphi}$. Note that taking away the slit $(-\infty, 0] \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $1-z$ is the same as taking away $[1, \infty) \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable z, because the reflection $(1-z) \mapsto (1-z)$ with respect to the origin moves the slit $(-\infty, 0] \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $1-z$ to the slit $[0, \infty) \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $z-1$ and the translation of adding -1 to the variable $z = -1$ moves the slit $[0, \infty) \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $z-1$ to the slit $[1, \infty) \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable z.

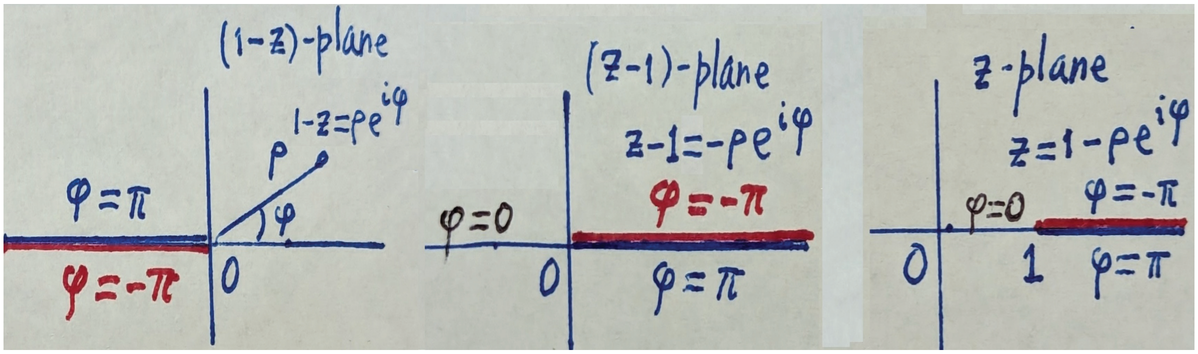

Figure 2: Using the cut $[1, \infty)$ in $\mathbb{R}$ to define a branch of $(1-z)^{\alpha-1}$

When we take the product of the branch $z^{-\alpha}$ and the branch $(1-z)^{\alpha-1}$, the slit $[0, \infty) \subset \mathbb{R}$ in $\mathbb{C}$ for the variable z has to be excluded. However, we can put back the slit $(1, \infty) \subset \mathbb{R}$ in $\mathbb{C}$ for the variable z into the domain of definition of the product of the branch $z^{-\alpha}$ and the branch $(1-z)^{\alpha-1}$ for the following reason. When z is just above the slit $(1, \infty) \subset \mathbb{R}$, the value of θ is 0 and the value of φ is $-\pi$ (corresponding to the numerical value of the angle of $1-z$ being $-\pi$) and as a consequence the value of $f(z) = z^{-\alpha}(1-z)^{\alpha-1}$ is $x^{-\alpha}(x-1)^{\alpha-1}e^{-i(\alpha-1)\pi}$. Now we consider the situation when z is just below the slit $(1, \infty) \subset \mathbb{R}$. When z is just below the slit $(1, \infty) \subset \mathbb{R}$, the value of θ is 2π and the value of φ is π (corresponding to the numerical value of the angle of $1-z$ being π) and as a consequence the value of $f(z) = z^{-\alpha}(1-z)^{\alpha-1}$ is

$$\begin{aligned} &x^{-\alpha}e^{-i2\alpha\pi}(x-1)^{\alpha-1}e^{i(\alpha-1)\pi} \\ &= x^{-\alpha}(x-1)^{\alpha-1}e^{-i\alpha\pi - i\pi} \\ &= x^{-\alpha}(x-1)^{\alpha-1}e^{-i\alpha\pi + i\pi} \end{aligned}$$

which again is equal to $x^{-\alpha}(x-1)^{\alpha-1}e^{-i(\alpha-1)\pi}$. It means that the function $f(z) = z^{-\alpha}(1-z)^{\alpha-1}$ which is holomorphic on $\mathbb{C} - [0, \infty)$ can be extended to be a continuous function on $\mathbb{C} - [0, 1]$.

Here the function $f(z) = z^{-\alpha}(1-z)^{\alpha-1}$, whose branch we would like to define by by removing a cut (*i.e.,* a slit) from $\mathbb{C}$, is the product of two functions $z^{-\alpha}$ and $(1-z)^{\alpha-1}$. We consider each of the two factors separately to construct a branch for each and then form the product of the two constructed branches. The domain for the product of the two branches at least contains the intersection of the domains, $\mathbb{C} - [0, \infty)$ and $\mathbb{C} - [1, \infty)$, one for each branch. The key point here is that the product of the two branches turns out to assume the same value at the two sides of the segment $(1, \infty)$ common to both cuts so that we can add it back to the domain of the product of the two branches to end up with the new domain $\mathbb{C} - [0, 1]$ for the product of the two branches.

For later use, we would like to remark that, according to the computation above for the value of $f(x)$ with x just above $(1, \infty)$, we have

$$\begin{aligned}\lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} x\, f(x) &= \lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} x \left(x^{-\alpha}(x-1)^{\alpha-1}e^{-i(\alpha-1)\pi}\right) \\ &= e^{-i(\alpha-1)\pi} = -e^{-i\alpha\pi}.\end{aligned}$$

We now go back to the computation of our definite integral

$$\int_{x=0}^{1} \frac{dx}{x^{\alpha}(1-x)^{1-\alpha}} \quad (0 < \alpha < 1).$$

Let C_R be the circle of radius R centered at the origin in the counterclockwise sense and let Γ_r be composed of the following four pieces: the right-half of the circle $|z-1| = r$ in the counterclockwise sense, the line-segment joining $1 + ri$ to ri, the left-half of the circle $|z| = r$ in the counterclockwise sense, and the line-segment joining $-ri$ to $1 - ri$. We apply Cauchy's theorem to the holomorphic function $f(z) = z^{-\alpha}(1-z)^{\alpha-1}$ on the domain enclosed by C_R and Γ_r for $R > 0$ sufficiently large and for $r > 0$ sufficiently small. Then

$$\int_{C_R} f(z)dz = \int_{\Gamma_r} f(z)dz.$$

We now compute the left-hand side as $R \to \infty$ by using the substitution $z = \frac{1}{w}$. We get

$$\int_{C_R} f(z)dz = -\int_{C_{\frac{1}{R}}} f\left(\frac{1}{w}\right)\left(-\frac{dw}{w^2}\right),$$

where on the right-hand we have a minus sign in front of the integral because of the counterclockwise orientation of $z \in C_R$ corresponds to the clockwise orientation of $w \in C_{\frac{1}{R}}$. Since

$$\lim_{w\to 0} \frac{1}{w} f\left(\frac{1}{w}\right) = \lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} x\, f(x) = -e^{-i\alpha\pi},$$

it follows that

$$\frac{1}{w^2} f\left(\frac{1}{w}\right)$$

has a simple pole at $w = 0$ whose residue is $-e^{-i\alpha\pi}$.

In order to compute the limit of

$$\int_{\Gamma_r} f(z)dz$$

as $r \to 0$, we determine the value of $f(x)$ for x just above $(0, 1)$ and the value of $f(x)$ for x just below $(0, 1)$. To compute the value of $f(x)$ for x just above $(0, 1)$, we observe that for x just above $(0, 1)$ the value of θ is 0 and the value of φ is 0 (corresponding to the numerical value of the angle of $1 - z$ being 0) and as a consequence the value of $f(z) = z^{-\alpha}(1 - z)^{\alpha-1}$ is $x^{-\alpha}(x-1)^{\alpha-1}$. Likewise, to compute the value of $f(x)$ for x just below $(0, 1)$, we observe that for x just below $(0, 1)$ the value of θ is 2π and the value of φ is 0 (corresponding to the numerical value of the angle of $1 - z$ being 0) and as a consequence the value of $f(z) = z^{-\alpha}(1 - z)^{\alpha-1}$ is $x^{-\alpha}e^{-i\alpha 2\pi}(x - 1)^{\alpha-1}$. Thus

$$\lim_{r\to 0} \int_{\Gamma_r} f(z)dz = e^{-i\alpha 2\pi} \int_{x=0}^{1} \frac{dx}{x^\alpha(1 - x)^{1-\alpha}} - \int_{x=0}^{1} \frac{dx}{x^\alpha(1 - x)^{1-\alpha}}.$$

Thus we end up with

$$\begin{aligned} \int_{x=0}^{1} \frac{dx}{x^\alpha(1 - x)^{1-\alpha}} &= \frac{2\pi i\left(-e^{-i\alpha\pi}\right)}{e^{-i\alpha 2\pi} - 1} \\ &= \frac{2\pi i}{e^{i\alpha\pi} - e^{-i\alpha\pi}} = \frac{\pi}{\sin \alpha\pi}. \end{aligned}$$

Background of Second Example of Evaluation of a Definite Integral by Contour Integration of a Branch of a Holomorphic Function

The second example of the evaluation of a definite integral by contour integration of a branch of a holomorphic function originates from the proof of *Euler's reflection law for the Gamma function* which is derived from the relation between the Gamma function and the Beta function. We would like to explain how this particular example arises. First we introduce the Gamma function, which is a function defined by integration so that the value of the function at n is equal to $(n-1)!$.

Definition of Gamma Function. The Gamma function in a real variable is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

for $x > 0$ to make sure that the integral converges at $t = 0$. When $x > 1$, by integration by parts we get

$$\Gamma(x) = \left[-t^{x-1}e^{-t}\right]_{t=0}^{t=\infty} + (x-1)\int_0^\infty t^{x-2}e^{-t}dt = (x-1)\,\Gamma(x-1)\,.$$

From $\Gamma(1) = \int_0^\infty e^{-t} = 1$ it follows that

$$\Gamma(n) = (n-1)!\ .$$

So the Gamma function is the generalization of the factorial function from integer values to real values. The defining formula

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

actually defined $\Gamma(z)$ for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. We would like to comment on how the integral defining the Gamma function comes about. The number $(n-1)!$ can be obtained as the final coefficient after differentiating $(n-1)$-times the monomial x^{n-1} with respect to x. This is obtained by induction on n with one differentiation giving the coefficient $(n-1)$ with x^{n-1} replaced by x^{n-2}. Actually we are looking for an integral formula with a parameter x so that at $x = n$ we can get $(n-1)!$ and not looking for a differentiation process. The differential process can be embedded in the process of an integral when integration by parts is performed. Since we need the induction situation in the process of integration by parts, besides using t^{n-1} as a factor in the integrand (in the variable t) we should use another factor which integration is equal to its own negative, because a negative sign is added to the new integral after the step of integration by parts. A choice for the second factor should be e^{-t}. We also have to worry about the difference, evaluated at the two limits of the integral, the product of the derivative of one factor and the primitive of the other factor. We want this difference evaluated at the two limits of the integral to vanish. This means that one limit of the integral could be chosen to be 0 where a positive power of t vanishes and the other limit of the integral could be chosen to be ∞ where the primitive of the other factor e^{-1} vanishes. Because of these considerations, the definition of the Gamma function comes about as

$$\Gamma\left(x\right)=\int_{t=0}^{\infty}t^{x-1}e^{-t}dt.$$

The reason of choosing t^{x-1} instead of t^x is to get convergence for the more natural choice of $x>0$ instead of $x>-1$.

Beta Function. A similar analogue of the generalization of the binomial coefficient

$$\binom{m+n}{m}=\frac{(m+n)!}{m!\,n!}$$

is the Beta function defined by

$$B\left(x,y\right)=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma\left(x+y\right)}.$$

We are going to derive the formula for the Beta function as a definite integral whose integrand depends on the variables x and y. This is done by reversing the order of integration of a double integral. For $x>0$ and $y>0$ we have

$$\Gamma(x)\Gamma(y)=\left(\int_{t=0}^{\infty}t^{x-1}e^{-t}dt\right)\left(\int_{u=0}^{\infty}u^{y-1}e^{-u}du\right).$$

To relate this to $\Gamma(x+y)$, we would like to end up with integral with t^{x+y-1} as a factor in its integrand. One way to get from t^{x-1} to t^{x+y-1} to get the factor from u^{y-1}. Using the transformation $u = tv$, we obtain

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \left(\int_{t=0}^{\infty} t^{x-1}e^{-t}dt\right)\left(\int_{t=0}^{\infty} t^{y}v^{y-1}e^{-tv}dv\right)\\ &= \int_{t=0}^{\infty}\int_{v=0}^{\infty} t^{x+y-1}e^{-t(1+v)}v^{y-1}dvdt.\end{aligned}$$

We would like the part $t^{x+y-1}e^{-t(1+v)}$ of the integrand in the last integral to look like $w^{x+y-1}e^{-w}$ and for that purpose we introduce the transformation $w = t\,(1+v)$, with

$$dv \wedge dw = dv \wedge (dt\,(1+v) + tdv) = (1+v)\,dv \wedge dt,$$

to go from the pair of variables (v,t) to the pair of variables (v,w), in order to get

$$\begin{aligned}&\int_{t=0}^{\infty}\int_{v=0}^{\infty} t^{x+y-1}e^{-t(1+v)}v^{y-1}dvdt\\ &= \int_{w=0}^{\infty}\int_{v=0}^{\infty} \frac{w^{x+y-1}}{(1+v)^{x+y-1}}\,e^{-w}\,v^{y-1}\frac{1}{1+v}\,dv\,dw\\ &= \left(\int_{w=0}^{\infty} w^{x+y-1}e^{-w}dw\right)\left(\int_{v=0}^{\infty}\frac{v^{y-1}dv}{(1+v)^{x+y}}\right)\\ &= \Gamma\,(x+y)\int_{v=0}^{\infty}\frac{v^{y-1}dv}{(1+v)^{x+y}},\end{aligned}$$

from which it follows that

$$(\dagger) \qquad B\,(x,y) = \int_{v=0}^{\infty}\frac{v^{y-1}dv}{(1+v)^{x+y}}.$$

The above use of the exterior product $dv \wedge dw$ can be alternatively formulated in terms of the Jacobian determinant in the change of variables $(v,t) \mapsto (v,w)$ in the double integral. The Jacobian determinant for the change of variables $(v,t) \mapsto (v,w)$ is $1+v$. The reason why the formulation in terms of exterior products is used here is that the partial differentiation of the first variable v in (v,t) with respect to the first variable v in (v,w), which is normally written as $\frac{\partial v}{\partial v}$, can be confusing.

The integral on the right-hand side of (†) is not symmetric in x and y though the Beta function $B(x,y)$ is. In order to transform the integral to make it symmetric in x and y, we apply the linear fractional transformation $v = \frac{\lambda}{1-\lambda}$ to change the interval of integration from $[0,\infty)$ to $[0,1]$ and get

$$B\left(x,y\right) = \int_{t=0}^{1} t^{x-1}\left(1-t\right)^{y-1} dt,$$

which is symmetric in x and y.

Relation Between Gamma Function and Sine Function. A very useful case for the Beta function is when $x+y=1$ in the above formula, in which case

$$\Gamma(x)\Gamma(1-x) = \int_{t=0}^{1} t^{x-1}\left(1-t\right)^{y-1} dt,$$

which is the example we worked out above. With the result of the evaluation of that example, we have the following important formula relating the gamma function to the sine function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

for $0 < x < 1$, which is also known as *Euler's reflection law for the Gamma function*, because the transformation $x \mapsto 1-x$ which can be rewritten as $\xi \mapsto -\xi$ with the coordinate change $\xi = x - \frac{1}{2}$ describes the reflection with respect to $x = \frac{1}{2}$.

Duplication Formula for Gamma Function. Another consequence of the Beta function is the *duplication formula.* By using $\lambda = \sin^2\theta$, we get

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \int_{\lambda=0}^{1} \frac{d\lambda}{\sqrt{\lambda(1-\lambda)}}$$
$$= \int_{\theta=0}^{\frac{\pi}{2}} \frac{d\left(\sin^2\theta\right)}{\sqrt{\sin^2\theta\,\cos^2\theta}} = 2\int_{\theta=0}^{\frac{\pi}{2}} d\theta = \pi.$$

Thus $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Using the substitution $\lambda = \frac{1}{2} - \frac{1}{2}\sqrt{\mu}$ so that $\lambda(1-\lambda) = \frac{1}{4} - \frac{1}{4}\mu$ and $d\lambda = -\frac{d\mu}{4\sqrt{\mu}}$, we get

$$B\left(x,x\right) = \int_{\lambda=0}^{1} \lambda^{x-1}(1-\lambda)^{x-1} d\lambda$$
$$= 2\int_{\lambda=0}^{\frac{1}{2}} \lambda^{x-1}(1-\lambda)^{x-1} d\lambda = \frac{1}{2}\int_{0}^{1} \left(\frac{1}{4} - \frac{1}{4}\mu\right)^{x-1} \mu^{-\frac{1}{2}}$$

$$= 2^{1-2x} \int_0^1 (1-\mu)^{x-1} \, \mu^{-\frac{1}{2}} \, d\mu = 2^{1-2x} \, B\left(x, \frac{1}{2}\right).$$

Thus we have the following *duplication formula*.

$$\Gamma(2x) \, \Gamma\left(\frac{1}{2}\right) = 2^{2x-1} \, \Gamma(x) \, \Gamma\left(x + \frac{1}{2}\right).$$