Homework #7 Assigned on March 21, 2024
due March 28, 2024

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Problem 1 *(Three Parameters for Biholomorphism of Upper Half Plane – from Stein & Shakarchi, p.251, #15).* **(a)** Suppose Φ is a biholomorphic self-map of the open upper half plane $\mathbb{H}$ whose continuous extension to the closed upper half plane $\bar{\mathbb{H}}$ fixes three distinct points on the real axis. Prove that Φ is the identity map.

(b) Suppose (x_1, x_2, x_3) and (y_1, y_2, y_3) are two triples of distinct points on the real axis with

$$x_1 < x_2 < x_2 \quad \text{and} \quad y_1 < y_2 < y_3.$$

Prove that there exits a unique biholomorphic self-map of the open upper half plane $\mathbb{H}$ whose continuous extension to the closed upper half plane $\bar{\mathbb{H}}$ satisfies $\Phi(x_j) = y_j$ for $j = 1, 2, 3$. Prove the same conclusion if $y_3 < y_1 < y_2$ or $y_2 < y_3 < y_1$ (instead of $y_1 < y_2 < y_3$) with the same order $x_1 < x_2 < x_3$ for the triple (x_1, x_2, x_3).

Hint: By using a biholomorphic map between $\mathbb{H}$ and the open unit disk $\mathbb{D}$ and using the explicit expression of any biholomorphism of $\mathbb{D}$ as a linear fractional transformation, verify that any biholomorphism of $\mathbb{H}$ takes the form

$$e^{i\theta}\,\frac{z-\beta}{z-\overline{\beta}}$$

for some $\theta \in \mathbb{R}$ and some $\beta \in \mathbb{H}$.

Problem 2 *(Pseudo-Hyperbolic Distance and Schwarz-Pick Lemma – from Stein & Shakarchi, p.251, #13).* The *pseudo-hyperbolic distance* between two points z, w of the open unit disk $\mathbb{D}$ is defined by

$$\rho(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

(a) Prove that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$\rho(f(z), f(w)) \leq \rho(z,w) \quad \text{for all} \quad z, w \in \mathbb{D}.$$

Moreover, prove that if f is an automorphism of $\mathbb{D}$ (*i.e.,* a biholomorphic self-map of $\mathbb{D}$, also called a biholomorphism of $\mathbb{D}$), then f preserves the pseudo-hyperbolic distance

$$\rho(f(z), f(w)) = \rho(z, w) \quad \text{for all} \quad z, w \in \mathbb{D}.$$

Hint: Consider the automorphism $\psi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ and apply the Schwarz lemma to $\psi_{f(w)} \circ f \circ \psi_w^{-1}$.

(b) Prove that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$\frac{|f'(z)}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} \quad \text{for all} \quad z \in \mathbb{D}.$$

This result is called the *Schwarz-Pick lemma.*

Hint: Take limit in Part**(a)** as $z \to w$ after division by $z - w$.

Problem 3 *(Poincaré Metric and Hyperbolic Distance – from Stein & Shakarchi, p.256, #3).* For complex numbers $w \in \mathbb{C}$ and $z \in \mathbb{D}$ we define the *hyperbolic length* of w at z by

$$\|w\|_z = \frac{|w|}{1 - |z|^2},$$

where $|w|$ and $|z|$ denote the usual absolute values. This length is sometimes referred to as the *Poincaré metric*, and as a Riemannian metric it is written as

$$ds^2 = \frac{|dz|^2}{(1 - |z|^2)^2}.$$

The idea is to think of w as a vector lying in the tangent space at z. Observe that for a fixed w, its hyperbolic length grows to infinity as z approaches the boundary of the disk. We pass from the infinitesimal hyperbolic length of tangent vectors to the global hyperbolic distance between two points by integration.

(a) Given two complex numbers z_1 and z_2 in the disk, we define the *hyperbolic distance* between them by

$$d(z_1, z_2) = \inf_\gamma \int_{t=0}^{1} \left\|\gamma'(t)\right\|_{\gamma(t)} \, dt,$$

where the infimum is taken over all smooth curves $\gamma : [0, 1] \to \mathbb{D}$ joining z_1 to z_2. Use the Schwarz-Pick lemma to prove that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$d(f(z_1), f(z_2)) \leq d(z_1, z_2) \quad \text{for any} \quad z_1, z_2 \in \mathbb{D}.$$

In other words, holomorphic functions are distance-decreasing in the hyperbolic metric.

(b) Prove that any biholomorphism of the open unit disk $\mathbb{D}$ preserves the hyperbolic distance, namely

$$d(\varphi(z_1), \varphi(z_2)) = d(z_1, z_2) \quad \text{for any} \quad z_1, z_2 \in \mathbb{D}$$

for any biholomorphism φ of $\mathbb{D}$. Conversely, if $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic and preserves the hyperbolic distance, then φ is a biholomorphism of $\mathbb{D}$. Note that for the converse, one needs to prove surjectivity.

(c) Given two points $z_1, z_2 \in \mathbb{D}$, show that there exists an automorphism φ such that $\varphi(z_1) = 0$ and $\varphi(z_2)$ for some s on the line segment $[0, 1)$ on the real line.

(d) Prove that the hyperbolic distance between 0 and $s \in [0, 1)$ is

$$d(0, s) = \frac{1}{2} \log \frac{1+s}{1-s}.$$

(e) Find a formula for the hyperbolic distance between any two points in the unit disk.

Problem 4 *(Koebe-Bieberbach $\frac{1}{4}$ Theorem for Normalized Univalent Functions on Open Unit Disk – from Stein & Shakarchi, p.108, #1).* The Koebe-Bieberbach $\frac{1}{4}$ theorem for normalized univalent functions on the open unit disk

$$\mathbb{D} = \left\{ z \in \mathbb{C} \,\middle|\, |z| < 1 \right\}$$

states that if $f : \mathbb{D} \to \mathbb{C}$ is an *injective* holomorphic map with $f(0) = 0$ and $f'(0) = 1$, then the image of f contains $D_r(0) = \left\{ z \in \mathbb{C} \,\middle|\, |z| < r \right\}$ with $r = \frac{1}{4}$.

Prove the Koebe-Bieberbach $\frac{1}{4}$ theorem for normalized univalent functions on the open unit disk by following the four steps outlined below.

(a) (*Grönwall's Area Theorem*). If

$$h(z) = \frac{1}{z} + \sum_{n=0}^{\infty} c_n z^n$$

is holomorphic and injective on $\{0 < |z| < 1\}$, then

$$\sum_{n=1}^{\infty} n|c_n|^2 \leq 1.$$

Hint: The injectivity of h implies that for $0 < \rho < 1$ the image of the *clockwise* circle $\{|z| = \rho\}$ is mapped by h to a *counterclockwise* simple closed curve C_ρ in $\mathbb{C}$ (because of the behavior of h near $z = 0$). Compute the area of the domain Ω_ρ in $\mathbb{C}$ enclosed by C_ρ by

$$\int_{w\in\Omega_\rho} \frac{i}{2}\, dw \wedge d\bar{w} = \int_{w\in C_\rho} \frac{i}{2}\, w\, d\bar{w} = -\int_{|z|=\rho} \frac{i}{2}\, h(z)\, d\overline{h(z)}.$$

Use the fact that the area of Ω_ρ is nonnegative and pass to limit as $\rho \to 1$.

(b) (*Second Coefficient Estimate of Bieberbach's Conjecture*). Let $f(z)$ be a univalent (*i.e.,* injective holomorphic) function on $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Let

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n$$

be the power series expansion of $f(z)$ on $\mathbb{D}$. Prove that $|a_2| \leq 2$ by applying **(a)** to $\frac{1}{g(z)}$ where $g(z)$ is a univalent function on $\mathbb{D}$ with $g(0) = 0$ and $g'(0) = 1$ and $g(z)^2 = f(z^2)$.

Hint: On $0 < |z| < 1$ one has the expansion $\frac{1}{g(z)} = \frac{1}{z} - \frac{1}{2}a_2 z + \cdots$.

Remark. The statement $|a_n| \leq n$ for all $n \geq 2$ is known as the *Bieberbach conjecture* which was confirmed by Louis de Branges in 1985.

(c) If $h(z) = \frac{1}{z} + \sum_{n=0}^{\infty} c_n z^n$ is holomorphic and injective on $\{0 < |z| < 1\}$ and avoids the values z_1 and z_2, prove that $|z_1 - z_2| \leq 4$.

Hint: For each of the two cases of $j = 1$ and $j = 2$, apply the second coefficient estimate of Bieberbach's conjecture to the power series expansion of $\frac{1}{h(z)-z_j}$ on $\mathbb{D}$.

(d) Complete the proof of the the Koebe-Bieberbach $\frac{1}{4}$ theorem for normalized univalent functions on the open unit disk.

Hint: If f avoids w, then $\frac{1}{f}$ avoids 0 and $\frac{1}{w}$.

Problem 5 *(Construction of Biholomorphic Map from Linear Fractional Transformation, Root and Logarithmic Maps and Solving Problem of Electrostatic Potential).* Let $a > 0$ and let Ω be the domain obtained from $\mathbb{C}$ by deleting the two intervals $(-\infty, -a]$ and $[a, \infty)$. Let $b > 0$ and let S be the open horizontal strip $\{0 < \operatorname{Im} z < b\}$ in $\mathbb{C}$.

(a) By using linear fractional transformations, the square-root map, and the logarithmic map, construct explicitly a biholomorphic map from Ω onto S such that the boundary piece $(-\infty, -a]$ of Ω corresponds to the boundary piece $\{\operatorname{Im} z = 0\}$ of S and the boundary piece $[a, \infty)$ of Ω corresponds to the boundary piece $\{\operatorname{Im} z = b\}$ of S.

(b) Use the biholomorphic map in Part **(a)** to find an electrostatic potential $0 \leq V \leq 1$ on Ω such that the value of V approaches 0 on $(-\infty, -a]$ and the value of V approaches 1 on $[a, \infty)$.

Hint: For **(a)**, consider both Ω and S as open subsets of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Use a linear fractional transformation

$$z \mapsto \frac{z+a}{z-a}$$

which maps the complement $[a, \infty) \cup \{\infty\} \cup (-\infty, -a]$ of Ω in $\mathbb{C} \cup \{\infty\}$ onto the closed curve-segment $[0, \infty) \cup \{\infty\}$ in $\mathbb{C} \cup \{\infty\}$. Follow it by a square-root map. Then use a linear fractional transformation which sends $\{-1, 1\}$ to $\{0, \infty\}$. Finally use the logarithmic map.