Evaluation of Definite Integrals by Residues without Using Branches of Holomorphic Functions

For the application of the residue theorem to the evaluation of definite integrals, we group together here a number of standard types of definite integrals which can be evaluated in such a way in a routine manner. We do here the case of the integrals of rational functions of the sine and cosine functions over interval of length 2π. Then we do the case of the integrals of rational functions over the real line, allowing the inclusion of the sine and cosine functions as a factor in the integrand. Later we will do the case of integrals whose evaluation by the method of residues requires the use of branches of holomorphic functions.

Integral of Rational Function of Sine and Cosine Functions over Interval of Length 2π. A definite integral of the form

$$\int_{\theta=0}^{2\pi} R\left(\cos\theta, \sin\theta\right) d\theta,$$

where $R\left(x,y\right)$ is a rational function (with the value of $R\left(\cos\theta,\sin\theta\right)$ assumed finite at every point $\theta \in [0, 2\pi]$) can be evaluated by using the substitution

$$\begin{aligned} z &= e^{i\theta}, \\ \cos\theta &= \frac{z+\frac{1}{z}}{2}, \\ \sin\theta &= \frac{z-\frac{1}{z}}{2i}, \\ d\theta &= \frac{dz}{iz} \end{aligned}$$

to convert the integral into a contour integral

$$\int_{|z|=1} f(z)dz$$

of a meromorphic function $f(z)$ of z over the contour $|z| = 1$, which can then be evaluated by using residues to yield

$$\int_{\theta=0}^{2\pi} R\left(\cos\theta, \sin\theta\right) d\theta = \int_{|z|=1} f(z)dz = 2\pi i \sum_{|a|<1} \mathrm{Res}_a f.$$

Note that $\mathrm{Res}_a f$ is 0 unless a is a pole of the meromorphic function f and there are only a finite number of poles of f inside the unit disk $|z| < 1$ while there is no pole of f on the unit circle $|z| = 1$ under the assumption that the value of $R(\cos\theta, \sin\theta)$ is finite at every point $\theta \in [0, 2\pi]$.

Here is a simple concrete example of the evaluation of this type of definite integral. To evaluate

$$I = \int_{\theta=0}^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} \quad \text{for} \quad 0 < a < 1,$$

the use of the substitution

$$\begin{aligned} z &= e^{i\theta}, \\ \cos\theta &= \frac{z + \frac{1}{z}}{2}, \\ d\theta &= \frac{dz}{iz} \end{aligned}$$

yields

$$\begin{aligned} I &= \int_{|z|=1} \frac{\frac{dz}{iz}}{1 - a\left(z + \frac{1}{z}\right) + a^2} \\ &= \int_{|z|=1} \frac{-idz}{(1 + a^2)z - az^2 - a} \\ &= \int_{|z|=1} \frac{idz}{az^2 - (1 + a^2)z + a} \end{aligned}$$

The denominator $az^2 - (1 + a^2)z + a$ can be factored as

$$az^2 - (1 + a^2)z + a = (az - 1)(z - a).$$

Only the root $z = a$ of the two roots $a, \frac{1}{a}$ is inside the unit circle. Hence

$$\begin{aligned} I &= i\left(2\pi i \mathrm{Res}_a \frac{1}{az^2-(1+a^2)z+a}\right) \\ &= i\left(2\pi i \lim_{z\to a} \frac{z-a}{az^2-(1+a^2)z+a}\right) \\ &= i\left(2\pi i \lim_{z\to a} \frac{z-a}{(az-1)(z-a)}\right) \\ &= i\left(2\pi i \lim_{z\to a} \frac{1}{az-1}\right) \\ &= \frac{2\pi}{1-a^2}. \end{aligned}$$

Integral of Rational Function over Real Line with Degree of Denominator At Least 2 *More Than That of Numerator*. A definite integral of the form

$$\int_{x=-\infty}^{\infty} \frac{P(x)dx}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials with $\deg Q(x) \geq \deg P(x) + 2$ and no zero of $Q(x)$ for real x, can be computed by using the integral of the meromorphic function $\frac{P(z)}{Q(z)}$ over the contour of the boundary of the upper half disk of radius R centered at the origin as $R \to \infty$. The formula is

$$\int_{x=-\infty}^{\infty} \frac{P(x)dx}{Q(x)} = 2\pi i \sum_{\mathrm{Im}\, a>0} \mathrm{Res}_a \frac{P(z)}{Q(z)},$$

where $\mathrm{Res}_a \frac{P(z)}{Q(z)}$ is the residue of $\frac{P(z)}{Q(z)}$ at a. Note that $\mathrm{Res}_a \frac{P(z)}{Q(z)}$ is zero unless a is a zero of $Q(z)$ and there are only a finite number of such points a. The condition of $\deg Q \geq 2 + \deg Q$ is needed to conclude that

$$\lim_{R\to\infty} \int_{C_R}^{\infty} \frac{P(z)dz}{Q(z)} = 0$$

with $C_R = \left\{Re^{i\theta} \,\middle|\, 0 \leq \theta \leq \pi\right\}$. Another formula is

$$\int_{x=-\infty}^{\infty} \frac{P(x)dx}{Q(x)} = 2\pi i \sum_{\mathrm{Im}\, a<0} \mathrm{Res}_a \frac{P(z)}{Q(z)}$$

when the contour of the boundary of the *lower* half disk of radius R centered at the origin is used.

Here is a simple concrete example of the evaluation of this type of definite integral.

$$\begin{aligned}
\int_{x=-\infty}^{\infty} \frac{dx}{(x^2+1)^3} &= 2\pi i \mathrm{Res}_{z=i} \frac{1}{(z^2+1)^3} \\
&= 2\pi i \frac{1}{2!} \left.\frac{d^2}{dz^2} \frac{(z-i)^3}{(z^2+1)^3}\right|_{z=i} \\
&= 2\pi i \frac{1}{2!} \left.\frac{d^2}{dz^2} \frac{1}{(z+i)^3}\right|_{z=i} \\
&= 2\pi i \frac{1}{2!} \left.\frac{(-3)(-4)}{(z+i)^5}\right|_{z=i} \\
&= 2\pi i \frac{1}{2!} \frac{(-3)(-4)}{(2i)^5} \\
&= \frac{3\pi}{8}.
\end{aligned}$$

Integral of Product of Either Sine or Cosine Function and Rational Function with Real Coefficients over Real Line with Degree of Denominator At Least 1 *More Than That of Numerator.* Another type of definite integrals which can be routinely computed by residues is

$$\int_{x=-\infty}^{\infty} \frac{P(x)\cos x dx}{Q(x)} \quad \text{or} \quad \int_{x=-\infty}^{\infty} \frac{P(x)\sin x dx}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials of real coefficients with $\deg Q(x) \geq \deg P(x) + 1$ and no zero of $Q(x)$ for real x except simple ones at the zeros of respectively $\cos x$ and $\sin x$.

The meromorphic function used is

$$\frac{P(z)e^{iz}}{Q(z)}$$

over the contour of the boundary of the upper half disk of radius R centered at the origin as $R \to \infty$. Another way is to use the meromorphic function

$$\frac{P(z)e^{-iz}}{Q(z)}$$

over the contour of the boundary of the *lower* half disk of radius R centered at the origin as $R \to \infty$. The reason for the difference in the sign of the exponent of e^{iz} in the case of the upper half disk and the case of the lower half disk is that $e^{iz} = e * i(x+iy) = e^{-y}e^{ix}$ is bounded by 1 on the upper half plane $\{y > 0\}$ while $e^{-iz} = e * -i(x+iy) = e^{y}e^{ix}$ is bounded by 1 on the lower half plane $\{y < 0\}$. We now make the random choice of using the boundary of the upper half disk as the contour of integration instead of the lower half disk.

When there is a simple zero x_0 of $Q(x)$ (where $\sin x$, respectively $\cos x$, by assumption must vanish), since there is simple pole $z = x_0$ of

$$\frac{P(z)e^{-iz}}{Q(z)}$$

on the contour of integration, we have to modify the contour to make it the boundary of the upper half disk of radius R minus the disk of radius $\varepsilon > 0$ centered at x_0 and do this for every simple zero x_0 of $Q(x)$ with $|x_0| < R$ (with the choice of $R > 0$ avoiding any x_0 and $-x_0$).

In general, the additional integral over any half circle

$$C_{\varepsilon,\alpha}(z_0) = \left\{ z_0 + \varepsilon e^{i\theta} \,\middle|\, \alpha \leq \theta \leq \alpha + \pi \right\}$$

of radius $\varepsilon > 0$ (where $\alpha \in \mathbb{R}$) at a simple pole z_0 of a meromorphic function

$$f(z) = \frac{c_{-1}}{z - z_0} + \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

yields πi times the residue c_{-1} of $f(z)$ at $z = z_0$ as limit when $\varepsilon \to 0$, because

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_{C_{\varepsilon,\alpha}(z_0)} f(z)dz &= \lim_{\varepsilon\to 0} \int_{C_{\varepsilon,\alpha}(z_0)} c_{-1} \frac{dz}{z - z_0} \\
&= \lim_{\varepsilon\to 0} \int_{\theta=\alpha}^{\alpha+\pi} c_{-1} \frac{d\left(z_0 + \varepsilon e^{i\theta}\right)}{(z_0 + \varepsilon e^{i\theta}) - z_0} \\
&= \lim_{\varepsilon\to 0} \int_{\theta=\alpha}^{\alpha+\pi} c_{-1} \frac{\varepsilon i e^{i\theta} d\theta}{\varepsilon e^{i\theta}} \\
&= \pi i c_{-1}.
\end{aligned}$$

The integral

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{C_{\varepsilon,\alpha}(z_0)} f(z)dz = \frac{1}{2} c_{-1}$$

is known as the *half residue* of the meromorphic function $f(z)$ at its simple pole $z = z_0$ which is independent of the choice of $\alpha \in \mathbb{R}$.

We need to show that the integral

$$\lim_{R\to\infty}\int_{C_R}\frac{P(z)e^{iz}dz}{Q(z)}=0,$$

where

$$C_R=\{Re^{i\theta}\,\Big|\,0\le\theta\le\pi\}$$

is the circular part of the contour of integration which is the boundary of the upper half disk of radius R centered at the origin. Since $|e^{iz}|\le 1$ on the upper half plane, if the degree of $Q(z)$ is at least 2 more than the degree of $P(z)$, it is easy to see that

$$\lim_{R\to\infty}\int_{C_R}\frac{P(z)e^{iz}dz}{Q(z)}=0,$$

because the contribution of

$$\left|\frac{P(z)e^{iz}}{Q(z)}\right|$$

is of the order $O\left(\frac{1}{R^2}\right)$ while the contribution of the length of the circle is πR. However, in the case when the degree of $Q(z)$ is precisely only 1 more than the degree of $P(z)$, we need to use the following argument of integration by parts to arrive at the desired conclusion. Using e^{iz} as the factor to be integrated, we obtain

$$\int_{C_R}\frac{P(z)e^{iz}dz}{Q(z)}=\left[\frac{P(z)e^{iz}}{iQ(z)}\right]_{z=-R}^{z=R}-\int_{C_R}\left(\frac{d}{dz}\frac{P(z)}{Q(z)}\right)\frac{e^{iz}}{i}dz$$

whose limit as $R\to\infty$ is zero, because the degree of the denominator of the rational function

$$\frac{d}{dz}\frac{P(z)}{Q(z)}=\frac{P'(z)Q(z)-P(z)Q'(z)}{Q(z)^2}$$

is now at least 2 more than the degree of its numerator, while

$$\left|\left[\frac{P(z)e^{iz}}{iQ(z)}\right]_{z=-R}^{z=R}\right|\le\left|\frac{P(-R)}{Q(-R)}\right|+\left|\frac{P(R)}{Q(R)}\right|$$

is of the order $O\left(\frac{1}{R}\right)$.

The above computation with $R \to \infty$ and $\varepsilon \to 0$ yields the result

$$\int_{x=-\infty}^{\infty} \frac{P(x)e^{ix}dx}{Q(x)} = 2\pi i \sum_{\mathrm{Im}\, a>0} \mathrm{Res}_a \frac{P(z)e^{iz}}{Q(z)} + \pi i \sum_{\mathrm{Im}\, a=0} \mathrm{Res}_a \frac{P(z)e^{iz}}{Q(z)}.$$

By taking the real part and the imaginary part of both sides, we get the following two formulas.

$$\int_{x=-\infty}^{\infty} \frac{P(x)\cos x\, dx}{Q(x)} = \mathrm{Re}\left(2\pi i \sum_{\mathrm{Im}\, a>0} \mathrm{Res}_a \frac{P(z)e^{iz}}{Q(z)} + \pi i \sum_{\mathrm{Im}\, a=0} \mathrm{Res}_a \frac{P(z)e^{iz}}{Q(z)}\right),$$

$$\int_{x=-\infty}^{\infty} \frac{P(x)\sin x\, dx}{Q(x)} = \mathrm{Im}\left(2\pi i \sum_{\mathrm{Im}\, a>0} \mathrm{Res}_a \frac{P(z)e^{iz}}{Q(z)} + \pi i \sum_{\mathrm{Im}\, a=0} \mathrm{Res}_a \frac{P(z)e^{iz}}{Q(z)}\right).$$

Here is a simple concrete example of the evaluation of this type of definite integral without the use of half residues. To evaluate

$$I = \int_{x=-\infty}^{\infty} \frac{\cos x\, dx}{x^2+1},$$

use the meromorphic function

$$f(z) = \frac{e^{iz}}{z^2+1}$$

so that

$$\begin{aligned} I &= \mathrm{Re}\left(2\pi i\, \mathrm{Res}_{z=i} \frac{e^{iz}}{z^2+1}\right) \\ &= \mathrm{Re}\left(2\pi i \lim_{z\to i} \frac{(z-i)e^{iz}}{z^2+1}\right) \\ &= \mathrm{Re}\left(2\pi i \lim_{z\to i} \frac{e^{iz}}{z+i}\right) \\ &= \mathrm{Re}\left(2\pi i\, \frac{e^{-1}}{2i}\right) \\ &= \frac{\pi}{e}. \end{aligned}$$

Here is a simple concrete example of the evaluation of this type of definite integral with the use of half residues. To evaluate

$$I = \int_{x=-\infty}^{\infty} \frac{\sin x\, dx}{x},$$

use the meromorphic function

$$f(z) = \frac{e^{iz}}{z}$$

so that

$$\begin{aligned} I &= \mathrm{Im}\left(\pi i \, \mathrm{Res}_{z=0} \frac{e^{iz}}{z}\right) \\ &= \mathrm{Im}\left(\pi i \lim_{z \to 0} \frac{(z-i)e^{iz}}{z}\right) \\ &= \mathrm{Im}\left(\pi i \lim_{z \to 0} e^{iz}\right) \\ &= \pi. \end{aligned}$$