Homework #6 Assigned on February 29, 2024
due March 7, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 113

Problem 1 *(Summation of Alternate Infinite Sum with Odd Index).* Verify that

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{\pi^3}{32}.$$

Hint: Rewrite the infinite series as

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{1}{2^4}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)^3}.$$

Problem 2 *(Explicit Summation of Infinite Sum of Value of Product of Sine Function and Rational Function at Integer Point Using Cotangent Function and Residue).* Prove that if $-\pi < a < \pi$, and ξ is a real number which is not an integer,

$$\sum_{n=-\infty}^{\infty}(-1)^n\frac{n\sin na}{\xi^2-n^2}=\frac{\pi\sin a\xi}{\sin\pi\xi}.$$

Hint: Apply the residue theorem to the meromorphic function

$$\frac{z\sin az}{\xi^2-z^2}\,\pi\operatorname{cosec}\pi z$$

on an appropriate sequence of increasing open subsets in $\mathbb{C}$ whose union is $\mathbb{C}$.

Problem 3 *(Wallis's Product Formula for π — from Stein & Shakarchi, p.154, #6).* Prove Wallis's product formula

$$\frac{\pi}{2}=\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdots\frac{2m\cdot 2m}{(2m-1)\cdot(2m+1)}\cdots.$$

Hint: Use the infinite product formula for $\sin z$ at $z=\frac{\pi}{2}$.

Problem 4. Verify that

$$\frac{1}{e^z-1}=\frac{1}{z}-\frac{1}{2}+2z\sum_{n=1}^{\infty}\frac{1}{z^2+4n^2\pi^2}.$$

Hint: Use the following theorem (given in the posted lecture notes of Lecture 10) on the expansion of meromorphic function as infinite sum of its principal parts.

Theorem. Let $f(z)$ be a meromorphic function on $\mathbb{C}$ whose poles are $\{a_n\}_{1\leq n<\infty}$ simple with $0<|a_1|\leq|a_2|\leq\cdots$ so that the residue of $f(z)$ at a_n is b_n. Suppose that there is a sequence of closed contours C_n such that the enclosure of C_n includes $a_1,\cdots,a_n$ but no other poles. Assume that the distance R_n from C_n to the origin goes to infinity as $n\to\infty$ and the length C_n is of the order $O(R_n)$. Assume that for some nonnegative integer p one has $f(z)=o(R_n^{p+1})$ on C_n as $n\to\infty$, where $o(\cdot)$ is the Landau symbol to mean higher order. Then

$$f(z)=\sum_{\nu=0}^{p}\frac{z^\nu}{\nu!}f^{(\nu)}(0)+\sum_{n=1}^{\infty}b_n\left(\frac{1}{z-a_n}+\frac{1}{a_n}+\frac{z}{a_n^2}+\cdots+\frac{z^p}{a_n^{p+1}}\right).$$

Problem 5. Show that, however small $\rho>0$ is, all the zeroes of the function

$$1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\cdots+\frac{1}{n!z^n}$$

on $\mathbb{C}-\{0\}$ lie in the circle $|z|<\rho$ if n is sufficiently large.

Hint: At any $z\in\mathbb{C}-\{0\}$ the function

$$1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\cdots+\frac{1}{n!z^n}$$

approaches $e^{\frac{1}{z}}$ as $n\to\infty$. Use Rouché's theorem.

Problem 6 *(Examples of Application of Argument Principle).* **(a)** Verify that there are no roots of the quartic equation

$$z^4+z^3+4z^2+2z+3=0$$

in the open first quadrant by showing that there is zero accumulated change of the argument of the holomorphic function $f(z)=z^4+z^3+4z^2+2z+3$ along the boundary of the intersection

$$\Omega_R=\left\{\,z\in\mathbb{C}\,\middle|\,\operatorname{Re}z>0\ \text{ and }\ \operatorname{Im}z>0\ \text{ and }\ |z|<R\,\right\}$$

of the open first quadrant and the open disk of radius R when $R>0$ is sufficiently large.

(b) By using the fact that all the coefficients of $z^4+z^3+4z^2+2z+3$ are real, show that there are no roots of

$$z^4+z^3+4z^2+2z+3=0$$

in the open fourth quadrant and that there are precisely two roots in each of the two other open quadrants.

Hint: There are no *real* roots of

$$z^4 + z^3 + 4z^2 + 2z + 3 = 0,$$

because the polynomial $x^4 - x^3 + 4x^2 - 2x + 3$ in x obtained by setting $z = -x$ can be rewritten as the sum $x^2(x^2 - x + 4) + (-2x + 3)$ of two positive terms for $0 < x < 1$ and as the sum $x^3(x - 1) + (2x(2x - 1) + 3)$ of two positive terms for $x > 1$.