Characterization of Poles and Essential Singularities and the Residue Theorem

Let $a \in \mathbb{C}$ and $R > 0$. Consider a holomorphic function $f(z)$ on the punctured open disk $\{0 < |z - a| < R\}$ of radius R centered at a. We introduced the definitions of *removable singularity*, *pole*, and *essential singularity* for the isolated singularity of $f(z)$ at a. We would like to give alternative characterizations for poles and essential singularities. We will then formulate the residue theorem which will be used in the computation of definite integrals by methods of residues. Let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n$$

be the Laurent series expansion of f on the punctured open disk $\{0 < |z-a| < R\}$ with

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)dz}{(z-a)^{n+1}}$$

for any $0 < r < R$ and for all $n \in \mathbb{Z}$, where for any $0 < r_1 < r_2 < R$ the convergence is absolute and uniform convergence on the closed annulus $r_1 \leq |z - a| \leq r_2$.

Characterization of Poles. Recall that the isolated singularity at $z = a$ is a *pole* if $c_n = 0$ for $n < -k$ and $c_{-k} \neq 0$ for some $k > 0$. The number k is called the *order* of the pole of $f(z)$ at $z = a$.

Theorem (Characterization of Poles). Let $a \in \mathbb{C}$ be an isolated singularity of a $f(z)$, which is holomorphic on $\{0 < |z - a| < R\}$ for some $R > 0$. The isolated singularity a is a pole of $f(z)$ if and only if $\lim_{z \to a} |f(z)| = \infty$. Moreover, the order k is the positive integer such that

$$\lim_{z \to a} |(z - a)^k f(z)|$$

is a positive number.

Proof. The "only if" part is verified as follows. Suppose $c_n = 0$ for $n < -k$ and $c_{-k} \neq 0$ for some $k > 0$. Since $f(z)$ can be written

$$\frac{g(z)}{(z-a)^k},$$

where

$$g(z)=\sum_{n=0}^{\infty} c_{n-k}(z-a)^n,$$

and $g(a)=c_{-k}\neq 0$, it follows that

$$\lim_{z\to a}|f(z)|=\lim_{z\to a}\frac{|g(z)|}{|z-a|^k}=\infty$$

and

$$\lim_{z\to a}|(z-a)^k f(z)|=|g(a)|=|c_{-k}|>0.$$

For the verification of the "if" part, we suppose $\lim_{z\to a}|f(z)|=\infty$. Then there exists some $0<R_0<R$ such that $|f(z)|\geq 1$ for $0<|z|<R_0$. Let

$$h(z)=\frac{1}{f(z)}$$

on $\{0<|z|<R_0\}$. The holomorphic function $h(z)$ is bounded by 1 in absolute value on $\{0<|z|<R_0\}$ and is therefore a removable singularity and can be expressed as a convergent power series

$$h(z)=\sum_{n=0}^{\infty} d_n(z-a)^n$$

on $\{0<|z|<R_0\}$. Since $\lim_{z\to a}|f(z)|=\infty$, it follows that $\lim_{z\to a}|h(z)|=0$ and there exists some positive integer k such that $d_k\neq 0$ and $d_n=0$ for $0\leq n<k$. Let

$$q(z)=\sum_{n=0}^{\infty} d_{n+k}(z-a)^n$$

so that $h(z)=(z-a)^k q(z)$. Since the function $q(z)$ is holomorphic on $\{|z|<R_0\}$ with $q(a)=d_k\neq 0$, there exists some $0<R_1<R_0$ such that $q(z)$ is nowhere zero on $\{|z|<R_1\}$. We can express the holomorphic function $\frac{1}{q(z)}$ as a convergent power series

$$\frac{1}{q(z)}=\sum_{n=0}^{\infty} e_n(z-a)^n$$

on $\{|z| < R_1\}$ with $e_0 \neq 0$. From

$$f(z) = \frac{1}{h(z)} = \frac{1}{(z-a)^k q(z)} = \frac{1}{(z-a)^k} \sum_{n=0}^{\infty} e_n (z-a)^n = \sum_{n=-k}^{\infty} e_{n+k} (z-a)^n$$

and the uniqueness of the coefficients of a Laurent series it follows that $c_n = e_{n+k}$ which is 0 for $n < -k$ and nonzero for $n = -k$. It means that $z = a$ is a pole of order k for $f(z)$. Q.E.D.

Characterization of Essential Singularity. Recall that the isolated singularity $z = a$ of $f(z)$ is an *essential singularity* if $c_n \neq 0$ for an infinite number of negative integers n.

Theorem (Characterization of Essential Singularity). Let $a \in \mathbb{C}$ be an isolated singularity of a $f(z)$, which is holomorphic on $\{0 < |z-a| < R\}$ for some $R > 0$. The isolated singularity a is an essential singularity of $f(z)$ if and only if for any $0 < r < R$ the image of $\{0 < |z-a| < r\}$ under the map $z \mapsto f(z)$ is dense in $\mathbb{C}$ (in the sense that any nonempty open subset of $\mathbb{C}$ contains some point of the image).

This characterization of essential singularity is also known as the *theorem of Casorati-Weierstrass.*

Proof. The verification of the "if" part is as follows. If the isolated singularity of $f(z)$ at $z = a$ is a removable singularity so that $f(a)$ can be defined to make $f(z)$ holomorphic on $\{|z| < R\}$, then there exists some $0 < r < R$ such that $|f(z)| < |f(a)| + 1$ for $0 < |z| < r$ and the image of $\{0 < |z-a| < r\}$ under the map $z \mapsto f(z)$ cannot be dense in $\mathbb{C}$. If the isolated singularity of $f(z)$ at $z = a$ is a pole so that $\lim_{z \to a} |f(z)| = \infty$, then there exists some $0 < r < R$ such that $|f(z)| > 1$ for $0 < |z| < r$ and the image of $\{0 < |z| < r\}$ under the map $z \mapsto f(z)$ is cannot be dense in $\mathbb{C}$. Thus, if for any $0 < r < R$ the image of $\{0 < |z-a| < r\}$ under the map $z \mapsto f(z)$ is dense in $\mathbb{C}$, then the isolated singularity of $f(z)$ at $z = a$ must be an essential singularity.

For the verification of the "only if" part, we assume that the isolated singularity of $f(z)$ at $z = a$ is an essential singularity. To prove that for any $0 < r < R$ the image of $\{0 < |z-a| < r\}$ under the map $z \mapsto f(z)$ is dense in $\mathbb{C}$, we assume the contrary and we are going to derive contradiction. There exists some nonempty open subset U of $\mathbb{C}$ which is disjoint from the image

image of $\{0 < |z| < r\}$ under the map $z \mapsto f(z)$ for some $r > 0$. Without loss of generality we can assume that $U = \{|z - b| < \rho\}$ for some $\rho > 0$. Let $g(z) = \frac{1}{f(z)-b}$ on $\{0 < |z-a| < r\}$. Since $|g(z)| \leq \frac{1}{\rho}$ on $\{0 < |z-a| > r\}$, the point $z = a$ is a removable singularity for $g(z)$ and as a result $g(z)$ defines a non identically zero holomorphic function on $\{0 < |z - a| < r\}$. Let k be the vanishing order of $g(z)$. Then $f(z) = \frac{1}{g(z)} + b$ is either holomorphic on $\{|z - a| < r\}$ when $g(a) \neq 0$ with $k = 0$ or having a pole of order k at $z = a$ when $k > 0$. This contradicts the assumption that $z = a$ is an essential singularity for $f(z)$. Q.E.D.

Recall that for an isolated singularity $z = a$ of a holomorphic function $f(z)$ defined on $\{0 < |z - a| < R\}$ for some $R > 0$, the *residue* of $f(z)$ at $z = a$, denoted by $\mathrm{Res}_a f$, is defined as

$$\mathrm{Res}_a f = \frac{1}{2\pi i} \int_{|z|=r} f(z) dz$$

for any $0 < r < R$. It is also given by

$$\mathrm{Res}_a f = c_{-1},$$

where

$$f(z) = \sum_{n=-k}^{\infty} c_n (z-a)^n$$

is the Laurent series expansion of $f(z)$ on $\{0 < |z - a| < R\}$. When a is a pole of $f(z)$ of order $\leq k$ at $z = a$,

$$\mathrm{Res}_a f = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left((z-a)^k f(z) \right) \Big|_{z=a}.$$

To facilitate the computation of definite integrals by the method of residues, we introduce the following theorem on residues.

Residue Theorem. Suppose Ω is a bounded open subset of $\mathbb{C}$ with piecewise smooth boundary $\partial\Omega$ and U is an open neighborhood of the topological closure $\bar{\Omega}$ of Ω in $\mathbb{C}$. Suppose $a_1, \cdots, a_p$ are distinct points in Ω and $f(z)$ is a holomorphic function on $U - \{a_1, \cdots, a_p\}$. Then

$$\int_{\partial\Omega} f(z) dz = 2\pi i \sum_{j=1}^{p} \mathrm{Res}_{a_j} f.$$

Proof. This is the direct consequence of apply the theorem of Cauchy-Goursat to the holomorphic function on the domain

$$\Omega - \bigcup_{j=1}^{p} \overline{\mathbb{D}}(a_j, r_j),$$

where $\overline{\mathbb{D}}(a_j, r_j)$ is $\{z \in \mathbb{C} \,\big|\, |z - a_j| \leq r_j\}$ and each $r_j > 0$ is chosen so small that $\overline{\mathbb{D}}(a_j, r_j)$ is contained in Ω. Q.E.D.

Meromorphic Functions. In order to be able to more easily describe an important class of functions in complex analysis, we now introduce the notion of a *meromorphic function*. Suppose U is an open subset of $\mathbb{C}$ and E is a discrete subset of U (in the sense that every point of E admits a deleted open neighborhood in $U - E$). A holomorphic function $f(z)$ on $U - E$ is called a *meromorphic function* on U if each a in E is a pole of $f(z)$.

Remark. The name *holommorphic* combines two Greek words *ὸλοζ* (meaning "whole" or "complete") and *μορφή* (meaning "shape" or "form"). The name *meromorphic* combines two Greek words *μέροζ* (meaning "part") and *μορφή* (meaning "shape" or "form").