**Homework #5 Assigned on February 22, 2024**
**due February 29, 2024**

**Please submit the PDF file of your homework to the CANVAS website for Math 113**

**Problem 1** *(Applications of Cauchy's Inequality and Maximum Modulus Principle – from Stein & Shakarchi, p.105, #15).* Use the Cauchy inequalities or the maximum modulus principle to solve the following problems.

**(a)** Prove that if $f(z)$ is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

for all $R > 0$ and for some $k \geq 0$ and some constants $A, B > 0$ then $f(z)$ is a polynomial of degree $\leq k$.

**(b)** Let $0 < \theta < \varphi < 2\pi$. Show that if $f(z)$ is holomorphic in the open unit disk $\mathbb{D}$, is bounded, and converges uniformly to zero in the sector

$$\{\theta < \arg z < \varphi\}$$

as $|z| \to 1$, then $f \equiv 0$.

*Hint:* For some appropriate large positive integer $\ell$ the holomorphic function

$$F(z) = \prod_{k=0}^{\ell-1} f\left(e^{\frac{2\pi i k}{\ell}} z\right)$$

on $\mathbb{D}$ satisfies

$$\lim_{|z|\to 1-} |F(z)| = 0.$$

Apply the maximum modulus principle to $F(z)$.

**(c)** Let $w_1, \cdots, w_n$ be points on the unit circle in the complex plane. Prove that there exists a point $z$ on the unit circle such that the product of the distances from $z$ to the point $w_j$, $1 \leq j \leq n$, is at least 1. Conclude that there exists a point $w$ on the unit circle such that the product of the distances from $w$ to the points $w_j$, $1 \leq j \leq n$, is exactly equal to 1.

*Hint:* Consider the holomorphic function $f(z) = \prod_{j=1}^{m}(z - w_j)$ and its value at $z = 0$ and apply the maximum modulus principle to $f(z)$ on the unit disk.

**(d)** Show that if the real part of an entire function $f$ is bounded, then $f$ is constant.

**Problem 2** *(Mean Square Convergence for Holomorphic Functions – from Stein & Shakarchi, p.107, #20).* This problem shows how the *mean square convergence* dominates the uniform convergence of holomorphic functions. If $U$ is an open subset of $\mathbb{C}$ we use the notation

$$\|f\|_{L^2(U)} = \left(\int_U |f(z)|^2 dxdy\right)^{\frac{1}{2}}$$

for the *mean square norm*, and

$$\|f\|_{L^\infty(U)} = \sup_{z \in U} |f(z)|$$

for the sup norm.

**(a)** If $f$ is holomorphic in a neighborhood of the disk

$$\overline{D_r(z_0)} = \{z \in \mathbb{C} \,\big|\, |z - z_0| \leq r\},$$

show that for any $0 < s < r$ there exists a constant $C > 0$ (which depends on $s$ and $r$) such that

$$\|f\|_{L^\infty(D_s(z_0))} \leq C\|f\|_{L^2(D_r(z_0))}.$$

**(b)** Prove that if $\{f_n\}$ is a Cauchy sequence of holomorphic functions in the mean square norm $\|\cdot\|_{L^2(U)}$, then the sequence $\{f_n\}$ converges uniformly every compact subset of $U$ to a holomorphic function.

*Hint:* Use the mean value property of holomorphic functions.

**Problem 3** *(Application of Identity Theorem for Holomorphic Functions – from Stein & Shakarchi, p.67, #13).* Suppose $f(z)$ is a holomorphic function defined everywhere in $\mathbb{C}$ such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that $f$ is a polynomial.

*Hint:* Use $c_n n! = f^{(n)}(z_0)$ and the fact from the Baire category theorem that as a complete metric space, $\mathbb{C}$ cannot be the countable union of closed subsets without any interior points.

**Problem 4** *(Application of Schwarz Reflection Principle for Unit Disk – from Stein & Shakarchi, p.67, #15).* Suppose $f$ is a non-vanishing continuous function on the topological closure $\overline{\mathbb{D}}$ of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$ such that $f$ is holomorphic on $\mathbb{D}$. Prove that if

$$|f(z)| = 1 \quad \text{for} \quad |z| = 1,$$

then $f(z)$ is constant.

*Hint:* Extend $f$ to all of $\mathbb{C}$ by

$$f(z) = \frac{1}{\overline{f\left(\frac{1}{\bar{z}}\right)}}$$

whenever $|z| > 1$ and use the argument from the Schwarz reflection principle.

**Problem 5** *(Example of Holomorphic Function on Unit Disk Non Continuable Across Any Point on Unit Circle – from Stein & Shakarchi, p.67, #1(a)).* Let $f$ be a holomorphic function on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$. Let $C$ be the unit circle $\{z \in \mathbb{C} \,\big|\, |z| = 1\}$. A point $w$ of $C$ is said to be *regular* for $f$ if there is an open neighborhood $U$ of $w$ in $\mathbb{C}$ and there is a holomorphic function $g$ on $U$ such that $f = g$ on $\mathbb{D} \cap U$. One says that a holomorphic function $f$ defined on $\mathbb{D}$ cannot be *continued holomorphically* past the unit circle $C$ if no point of $C$ is regular for $f$. Suppose

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \quad \text{for} \quad |z| < 1.$$

Notice that the radius of convergence is 1. Show that $f$ cannot be continued holomorphically past the unit circle $C$.

*Hint:* Suppose $\theta = \frac{2\pi p}{2^k}$, where $p$ and $k$ are positive integers. Let $z = re^{i\theta}$. Then $\left|f\left(re^{i\theta}\right)\right| \to \infty$ as $r \to 1$.

**Problem 6** *(Conformality and Orientation-Preserving Property of Holomorphic Function with Nowhere Zero Derivative).* Let $\Omega$ be a connected open subset of $\mathbb{C}$ and $f(z)$ be a holomorphic function on $\Omega$ such that its complex derivative $f'(z)$ is nowhere zero on $\Omega$. Prove that $f$ is orientation preserving and is conformal in the sense that the angle between two curves in $\Omega$ at their intersection point is equal to the angle of the images of the two curves under the map defined by $f$. Here the preservation of orientation by the map defined by $f$ means that the Jacobian determinant of the map from $(x, y)$ to $(u, v)$ is strictly positive when $u + iv = f(x + iy)$.

*Hint:* For the conformality property, consider a curve $t \mapsto \varphi(t) \in \Omega$ (with $-1 < t < 1$ and nonzero derivative of $\varphi$ at $t = 0$), and compare

$$\arg\left(\left.\frac{d\varphi}{dt}\right|_{t=0}\right) \quad \text{and} \quad \arg\left(\left.\frac{d(f\circ\varphi)}{dt}\right|_{t=0}\right).$$