Homework #4 Assigned on February 15, 2024
due February 22, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 113

Problem 1. Let $f(z)$ be the meromorphic function

$$\frac{(\log z)^3}{(z^3+1)^2}$$

on $\mathbb{C}$ minus the nonnegative real axis $\{x \in \mathbb{R} \,\big|\, x \geq 0\}$, where the branch of the holomorphic function $\log z$ is defined by $\log z = \log r + i\theta$ with $z = re^{i\theta}$ and $0 < \theta < 2\pi$. For $0 < \varepsilon < 1 < R$, let $C_{\varepsilon,R}$ be the contour which is the boundary of the open annulus of radii ε and R minus the horizontal closed right half-strip of width 2ε centered at the x-axis

$$\left\{z \in \mathbb{C} \,\Big|\, \varepsilon < |z| < R\right\} - \left\{z \in \mathbb{C} \,\Big|\, -\varepsilon \leq \operatorname{Im} z \leq \varepsilon,\ \operatorname{Re} z \geq 0\right\}.$$

By taking the real part and imaginary part of the limit of the equation

$$\int_{C_{\varepsilon,R}} f(z)dz = 2\pi i \sum_{k=0}^{2} \operatorname{Res}_{z=e^{\frac{(2k-1)\pi i}{3}}} f(z)$$

as $\varepsilon \to 0$ and $R \to \infty$, evaluate explicitly the two definite integrals

$$\int_{x=0}^{\infty} \frac{\log x \, dx}{(x^3+1)^2} \quad \text{and} \int_{x=0}^{\infty} \frac{(\log x)^2 \, dx}{(x^3+1)^2}.$$

Problem 2. For $0 < \alpha < 1$, verify that

$$\int_{x=0}^{\infty} \frac{x^{\alpha-1}dx}{1+x} = \frac{\pi}{\sin(\pi\alpha)}$$

by choosing an appropriate branch of $z^{\alpha-1}$ and evaluating

$$\int_{C_{\varepsilon,R}} \frac{z^{\alpha-1}}{1+z}$$

by residue theory (for $0 < \varepsilon < 1 < R$) and passing to limit as $\varepsilon \to 0$ and $R \to \infty$, where $C_{\varepsilon,R}$ is the contour which is the boundary of the open annulus of radii ε and R minus the horizontal closed right half-strip of width 2ε centered at the x-axis

$$\left\{z \in \mathbb{C} \,\middle|\, \varepsilon < |z| < R\right\} - \left\{z \in \mathbb{C} \,\middle|\, -\varepsilon \leq \operatorname{Im} z \leq \varepsilon,\ \operatorname{Re} z \geq 0\right\}.$$

Problem 3 *(Gaussian Distribution Unchanged Under Fourier Transform – from Stein & Shakarchi, p.65, #4).* Prove that for all $\xi \in \mathbb{C}$ we have

$$e^{-\pi\xi^2} = \int_{x=-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi} dx.$$

Hint: First check it for $\xi \in \mathbb{R}$ with $\xi > 0$ by applying Cauchy's theorem to

$$f(z) = e^{-\pi z^2}$$

on the rectangle with vertices at $z = R, R + i\xi, -R + i\xi, -R$ as $R \to \infty$.

Problem 4 *(from Stein & Shakarchi, p.104, #11).* Show that if $|a| < 1$, then

$$\int_{\theta=0}^{2\pi} \log\left|1 - ae^{i\theta}\right| d\theta = 0.$$

Then, prove that the above result remains true if we assume that $|a| \leq 1$. What is the result when $|a| > 1$?

Problem 5 *(Biholomorphic Map of Bounded Domain Fixing 1-Jet at One Point – from Stein & Shakarchi, p.66, #9).* Let Ω be a bounded connected open subset of $\mathbb{C}$ and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\varphi(z_0) = z_0 \quad \text{and} \quad \varphi'(z_0) = 1,$$

then φ is the identity.

Hint: Reduce the general case to $z_0 = 0$. Write $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0, and prove that if $\varphi_k = \varphi \circ \cdots \circ \varphi$ (where φ appears k times), then $\varphi_k(z) = z + ka_n z^n + O(z^{n+1})$ as $z \to 0$. Apply the Cauchy inequalities and let $k \to \infty$ to conclude the proof. Here the standard Landau O notation $f(z) = O(g(z))$ as $z \to 0$ means that $|f(z)| \leq C|g(z)|$ for some constant C as $z \to 0$.

Problem 6 *(Injective Entire Function Must Be Polynomial of Degree At Most One – from Stein & Shakarchi, p.105, #14).* Prove that all entire functions $f(z)$ that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.

Hint: Show first that $w = 0$ is an isolated singularity for the function $w \mapsto f\left(\frac{1}{w}\right)$. Then use the trichotomy of isolated singularities to conclude that $w = 0$ must be a pole of order 1 for the function $w \mapsto f\left(\frac{1}{w}\right)$. Apply Liouville's theorem to $f(z) - az$, where a is determined from the coefficients of the Laurent series of $w \mapsto f\left(\frac{1}{w}\right)$ at $w = 0$.