Homework #3 Assigned on February 8, 2024
due February 15, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 113

Problem 1 *(Removable Isolated Singularity from Fractional Growth Order – from Stein & Shakarchi, p.105, #13).* Suppose $f(z)$ is holomorphic in a punctured open disk

$$D_r(z_0) - \{z_0\} = \{z \in \mathbb{C} \,\big|\, 0 < |z - z_0| < r\}.$$

Suppose there exist $A > 0$, $0 < \varepsilon < 1$, and $0 < r' < r$ such that

$$|f(z)| \leq \frac{A}{|z - z_0|^{1-\varepsilon}}$$

for all $z \in \mathbb{C}$ with $0 < |z - z_0| < r'$. Show that the isolated singularity of f at z_0 is removable.

Problem 2 *(Removable Singularity from Square-Integrability).* Suppose f is a holomorphic function on the deleted open unit disk $\mathbb{D} - \{0\} = \{z \in \mathbb{C} \,\big|\, 0 < |z| < 1\}$ which is square-integrable on $\mathbb{D} - \{0\}$ in the sense that

$$\int_{\mathbb{D}-\{0\}} |f(z)|^2 dx \wedge dy$$

is finite.

(a) For any $0 < r_0 < 1$, prove that $\int_{|z|=r_0} f(z)dz = 0$ by filling in the details for the following three steps (i), (ii), and (iii).

(i) $\int_{|z|=r} f(z)dz$ is independent of $0 < r \leq r_0$.

(ii) $\int_{|z|=r_0} f(z)dz = \frac{1}{r_2 - r_1} \int_{r_1 < |z| < r_2} ie^{i\theta} f\left(re^{i\theta}\right) r dr d\theta$ for $0 < r_1 < r_2 < r_0$.

(iii) Use $\int_{\frac{r_2}{2} < |z| < r_2} |f(z)|^2 dxdy \to 0$ for $0 < r_2 < r_0$ as $r_2 \to 0$ and Hölder's inequality.

Hint: Use Cauchy's theorem for (i). Average over r in (r_1, r_2) for (ii). For (iii), apply Hölder's inequality

$$\left|\int_\Omega FG\right|^2 \leq \left(\int_\Omega |F|^2\right)\left(\int_\Omega |G|^2\right)$$

to $F \equiv f(z)$, $G \equiv 1$ and $\Omega = \{\frac{r_2}{2} < |z| < r_2\}$.

(b) By applying **(a)** to $f(z)z^n$ instead of $f(z)$ for $n \in \mathbb{N} \cup \{0\}$, verify that $\int_{|z|=r} f(z)z^n dz = 0$ for $0 < r < 1$ and $n \in \mathbb{N} \cup \{0\}$.

(c) Assume as known the statement concerning Fourier series expansion that, if $g(\theta)$ is a complex-valued continuously differentiable function on $\mathbb{R}$ with period 2π (that is, $g(\theta + 2\pi) = g(\theta)$ for $\theta \in \mathbb{R}$), then

$$g(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \quad \text{on} \quad \mathbb{R},$$

where

$$c_n = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} g(\theta)e^{-in\theta} d\theta$$

for $n \in \mathbb{Z}$. By using **(b)** and the fact that $\int_{|z|=r} f(z)z^n dz$ is independent of $0 < r < 1$ for $n \in \mathbb{Z}$, prove that if $f(z)$ is a holomorphic function on $\mathbb{D} - \{0\}$ which is square-integrable on $\mathbb{D} - \{0\}$, then $f(z)$ is equal to a convergent power series on $\mathbb{D} - \{0\}$ and hence can be extended to a holomorphic function on $\mathbb{D}$.

Hint: For fixed $0 < r < 1$ consider the Fourier series expansion of the complex-valued function $\theta \mapsto f\left(re^{i\theta}\right)$ on $\mathbb{R}$ with period 2π to conclude that the coefficients of the Laurent series of $f(z)$ at $z = 0$ vanish for the negative powers of z.

Problem 3 *(Limit of Ratio of Consecutive Coefficients of Power Series for Meromorphic Function with Only One Pole at Circle of Convergence – from Stein & Shakarchi, p.67, #14).* Let $R > 1$ and $z_0 \in \mathbb{C}$ with $|z_0| = 1$. Let $h(z)$ be a holomorphic function on $\{\,|z| < R\,\}$ with $h(z_0) \neq 0$. Let m be a positive integer and

$$f(z) = \frac{h(z)}{(z - z_0)^m}.$$

Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f on $\{\,|z|<1\,\}$, then

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=z_0.$$

Hint: $f(z)$ is of the form

$$\sum_{k=1}^{m}\frac{A_k}{(z-z_0)^k}+g(z)$$

with $A_1,\cdots,A_m\in\mathbb{C}$, where $A_m=h(z_0)\neq 0$ and $g(z)$ is a power series $\sum_{n=0}^{\infty}b_nz^n$ with radius of convergence at least equal to R so that for any $|z_0|<r<R$ there exists a positive number B such that

$$|b_n|\leq\frac{B}{r^n}$$

for all nonnegative integer n. Express a_n in terms of b_n and $A_1,\cdots,A_m$.

Problem 4 *(from Stein & Shakarchi, p.104, #7).* Prove that

$$\int_{\theta=0}^{2\pi}\frac{d\theta}{(a+\cos\theta)^2}=\frac{2\pi a}{(a^2-1)^{\frac{3}{2}}}$$

whenever $a>1$ by first converting it to an integral of the form

$$\int_{|z|=1}R(z)\,dz$$

and then using the Residue Theorem, where $R(z)$ is a rational function (*i.e.,* quotient of two polynomials) in the complex variable z.

Problem 5 *(from Stein & Shakarchi, p.104, #6).* Show that

$$\int_{x=-\infty}^{\infty}\frac{dx}{(1+x^2)^{n+1}}=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\,\pi$$

for $n\in\mathbb{N}$ by using

$$\int_{C_R}F(z)dz=2\pi i\,\mathrm{Res}_{z=i}F$$

(from the Residue Theorem) and passing to limit as $R\to\infty$, where

$$F(z)=\frac{1}{(1+z^2)^{n+1}}$$

and C_R is the boundary of the open upper half-disk

$$\left\{\,z\in\mathbb{C}\,\middle|\,|z|<R,\ \mathrm{Im}\,z>0\,\right\}$$

of radius $R>0$ centered at the origin.

Problem 6 *(from Stein & Shakarchi, p.64, #3).* Let a and b be positive numbers. Evaluate

$$\int_{x=0}^{\infty} e^{-ax} \cos bx \, dx \quad \text{and} \quad \int_{x=0}^{\infty} e^{-ax} \sin bx \, dx$$

by integrating e^{-Az}, with $A = \sqrt{a^2 + b^2}$, over an appropriate sector whose angle is ω with $\cos \omega = \frac{a}{A}$.