Homework #2 Assigned on February 1, 2024
due February 8, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 113

Problem 1 *(from Stein & Shakarchi, p.28, #16).* Determine the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$ when:

(a) $\qquad a_n = (\log n)^2$

(b) $\qquad a_n = n!$

(c) $\qquad a_n = \dfrac{n^2}{4^n + 3n}$

(d) $\qquad a_n = \dfrac{(n!)^3}{(3n)!}$

Hint: Use the following Stirling's formula which states that

$$n! \sim c\, n^{n+\frac{1}{2}} e^{-n} \quad \text{for some} \quad c > 0,$$

where $\sim$ means that the limit of the quotient of the two sides is 1 as $n \to \infty$.

(e) Find the radius of convergence of the *hypergeometric series*

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\,\beta(\beta+1)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)}\; z^n.$$

Here $\alpha, \beta \in \mathbb{C}$ and $\gamma \neq 0, -1, -2, \cdots$.

(f) Find the radius of convergence of the *Bessel function* of order r:

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n},$$

where r is an integer.

Problem 2 *(from Stein & Shakarchi, p.29, #17).* Show that if $\{a_n\}_{n=0}^{\infty}$ is a sequence of nonzero complex numbers such that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

then

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L.$$

In particular, this shows that when applicable, the *ratio test* can be used to calculate the radius of convergence of a power series.

Problem 3 *(from Stein & Shakarchi, p.29, #18).* Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.
Hint: Write $z = z_0 + (z - z_0)$ and use the binomial expansion

$$z^n = (z_0 + (z - z_0))^n = \sum_{k=0}^{n} \binom{n}{k} {z_0}^k (z - z_0)^{n-k},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Problem 4 *(from Stein & Shakarchi, p.29, #19).* Prove the following:

(a) The power series $\sum_{n=1}^{\infty} nz^n$ does not converge at any point of the unit circle.

(b) The power series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges at every point of the unit circle.

(c) The power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges *conditionally* at every point of the unit circle except at $z = 1$ where it does not converge.

Hint: For the proof of (c), apply to $a_n = \frac{1}{n}$ and $b_n = z^n$ the following Abel's formula of *summation by parts.*

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n,$$

where $a_1, \cdots, a_N, b_1, \cdots, b_N$ are complex numbers and $B_k = \sum_{n=1}^{k} b_n$ (with the convention that $B_0 = 0$).

Remark. The formula of summation by parts is the discrete analogue of the formula of integration by parts.

Problem 5 *(Fresnel Integrals as Examples of Computation of Definite Integrals by Methods of Complex Analysis – from Stein & Shakarchi, p.64, #1).* **(a)** Verify

$$\int_{x=-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

by squaring the integral and convert it by transforming the integral over $\mathbb{R}^2$ in terms of Cartesian coordinates (x, y) to an integral over $\mathbb{R}^2$ in terms of polar coordinates (r, θ).

(b) For $R > 0$ let

$$\Omega_R = \left\{ re^{i\theta} \,\middle|\, 0 < r < R,\, 0 < \theta < \frac{\pi}{4} \right\}$$

whose boundary is the union of its three parts: $[0, R] \subset \mathbb{R}$,

$$C_R = \left\{ Re^{i\theta} \,\middle|\, 0 \leq \theta \leq \frac{\pi}{4} \right\},$$

and

$$L_R = \left\{ re^{\frac{i\pi}{4}} \,\middle|\, 0 \leq r \leq R \right\}.$$

Apply the theorem of Cauchy-Goursat to the entire function e^{iz^2} and the domain Ω_R to derive

$$\int_{x=0}^{R} \left(\cos(x^2) + i \sin(x^2)\right) dx + \int_{\theta=0}^{\frac{\pi}{4}} e^{-R^2 \sin(2\theta) + iR^2 \cos(2\theta)} d\left(Re^{i\theta}\right) - \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \int_{r=0}^{R} e^{-r^2} dr = 0.$$

(c) By taking the real and imaginary parts of the equation in **(b)** and letting $R \to \infty$, verify that

$$\int_{x=0}^{R} \cos(x^2) dx = \int_{x=0}^{R} \sin(x^2) dx = \sqrt{\frac{\pi}{2}}.$$

Hint: Prove

$$\lim_{R \to \infty} \int_{\theta=0}^{\frac{\pi}{4}} e^{-R^2 \sin(2\theta)} d\theta = 0$$

by using the fact that there exists some $\alpha, \beta > 0$ such that $\sin \theta \geq \alpha\theta$ for $0 \leq \theta \leq \beta$.