Complex Differentiation and Cauchy-Riemann Equations

Review of Differentiability in Calculus of Real Variable. For a real-valued function $f(x)$ of a real variable x defined on an open interval (α, β) containing a, the differentiability of $f(x)$ at $x = a$ means that the difference quotient

$$\frac{f(x) - f(a)}{x - a}$$

admits a limit L as $x \to a$. More precisely, given any $\varepsilon > 0$ there exists some $\delta > 0$ (which depends on ε) such that

$$\left|\frac{f(x) - f(a)}{x - a} - L\right| < \varepsilon$$

for x satisfying $0 < |x - a| < \delta$. In other words, for any positive number ε the difference quotient

$$\frac{f(x) - f(a)}{x - a}$$

is ε-close to L if x is δ-close to a for some ε-dependent positive δ. Another description is that the difference quotient can be made arbitrarily close to L by making the variable x sufficiently close to a. The definition deliberately excludes the testing of the inequality for $x = a$ because the difference quotient is undefined at $x = a$. The limit L (if $f(x)$ is differentiable at $x = a$) is denoted by $f'(a)$ and is known as the derivative of $f(x)$ at $x = a$.

Alternative Definition of Differentiability in Calculus of Real Variable. An alternative definition of differentiability of $f(x)$ at $x = a$ is that the function $f(x)$ can be approximated by a polynomial of degree ≤ 1 to an order > 1 at $x = a$. That is, there exists a polynomial $A'x + B' = A(x - a) + B$ of degree ≤ 1 such that

$$f(x) - (A(x - a) + B) = E(x)$$

where the "error term" $E(x)$ vanishes to an order > 1 in the sense that

$$\frac{|E(x)|}{|x - a|} \to 0$$

as $x \to a$. The reason for the terminology is that $x - a$ vanishes at a to order 1 and the fact that

$$\lim_{x \to a} \frac{|E(x)|}{|x - a|} = 0$$

means that the vanishing order of $E(x)$ at $x = a$ is higher than the vanishing order of $x = a$. So we say that the "error term" $E(x)$ vanishes to an order > 1. In particular, $\lim_{x\to a} E(x) = 0$.

By taking the limit as $x \to a$ in the equation

$$f(x) - (A(x-a) + B) = E(x)$$

yields $f(a) - B = 0$ or $f(a) = B$ so that we can rewrite the equation

$$f(x) - (A(x-a) + B) = E(x)$$

as

$$\frac{f(x) - f(a)}{x - a} - A = \frac{E(x)}{x - a}$$

for $x \neq a$. By taking the limit as $x \to a$, we conclude that the limit of the difference quotient

$$\frac{f(x) - f(a)}{x - a}$$

as $x \to a$ is A. Thus $A = f'(a)$. Conversely, when

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a),$$

the polynomial $f(a) + f'(a)(x-a)$ of degree ≤ 1 approximates $f(x)$ at $x = a$ to an order > 1. Note that the approximating polynomial $f(a) + f'(a)(x-a)$ is the Taylor polynomial of degree 1 for $f(x)$ at $x = a$.

The geometric interpretation for the alternative definition of differentiability of $f(x)$ at $x = a$ is that the tangent line $y = f(a) + f'(a)(x-a)$ approximates the graph $y = f(x)$ to an order > 1 at $x = a$.

Differentiability of Real-Valued Function of Two Real Variables. An advantage of the alternative definition of differentiability is that it can extended straightforwardly to the case of a real-valued function $f(x, y)$ of two real variables x, y at the point $(x, y) = (a, b)$. Namely, a real-valued function $f(x, y)$ of two real variables x, y defined on an open neighborhood U of (a, b) in $\mathbb{R}^2$ is differentiable at the point $(x, y) = (a, b)$ if there exists a polynomial

$A'x + B'y + C' = A(x-a) + B(y-b) + C$ of degree ≤ 1 which approximates $f(x)$ to an order > 1 at $(x,y) = (a,b)$ in the sense that

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - (A(x-a) + B(y-b) + C)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

That is, for any $\varepsilon > 0$ there exists some $\delta > 0$ (which depends on ε) such that

$$\left| \frac{f(x,y) - (A(x-a) + B(y-b) + C)}{\sqrt{(x-a)^2 + (y-b)^2}} \right| < \varepsilon$$

for x satisfying $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. We can rewrite it in the form

$$f(x,y) = A(x-a) + B(y-b) + C + E(x,y)$$

with

$$\lim_{(x,y)\to(a,b)} \frac{E(x)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

In particular,

$$\lim_{(x,y)\to(a,b)} E(x,y) = 0$$

so that by taking limit in

$$f(x,y) = A(x-a) + B(y-b) + C + E(x,y)$$

as $(x,y) \to (a,b)$, we conclude that $C = f(a,b)$ and

$$f(x,y) = f(a,b) + A(x-a) + B(y-b) + E(x,y).$$

By setting $y = b$ in

$$f(x,y) = f(a,b) + A(x-a) + B(y-b) + E(x,y),$$

we get

$$f(x,b) = f(a,b) + A(x-a) + E(x,b).$$

By taking limit as $x \to a$ we get

$$\lim_{x\to a} \left| \frac{f(x,b) - f(a,b)}{x-a} - A \right| = \lim_{x\to a} \left| \frac{E(x,b)}{x-a} \right| = 0$$

so that the partial derivative $\frac{\partial f}{\partial x}(a,b)$ of $f(x,y)$ with respect to x at $(x,y)=(a,b)$ exists and is equal to A. Likewise, we conclude from the differentiability of the real-valued function $f(x,y)$ of two real variables x,y at $(x,y)=(a,b)$ that $\frac{\partial f}{\partial y}(a,b)$ of $f(x,y)$ with respect to y at $(x,y)=(a,b)$ exists and is equal to B. The polynomial $A(x-a)+B(y-b)+C$ which approximates $f(x,y)$ to an order >1 is equal to the Taylor polynomial

$$f(a,b)+(x-a)\frac{\partial f}{\partial x}(a,b)+(y-b)\frac{\partial f}{\partial y}(a,b)$$

of degree 1 at $(x,y)=(a,b)$.

The geometric interpretation for differentiability of $f(x,y)$ at $(x,y)=(a,b)$ is that the tangent plane

$$z=f(a,b)+(x-a)\frac{\partial f}{\partial x}(a,b)+(y-b)\frac{\partial f}{\partial y}(a,b)$$

approximates the graph $z=f(x,y)$ to an order >1 at $(x,y)=(a,b)$.

Let $x=x(t)$ and $y=y(t)$ be two real-valued differentiable functions of a single variable $t=t_0$ with $x(t_0)=a$ and $y(t_0)=b$. Then the composite function $t\mapsto f(x(t),y(t))$ is differentiable at $t=t_0$ whose derivative at $t=t_0$ is given by the chain rule

$$\left(\frac{d}{dt}f(x(t),y(t))\right)_{t=t_0}=\left(\frac{\partial f}{\partial x}\right)_{(x,y)=(a,b)}x'(t_0)+\left(\frac{\partial f}{\partial y}\right)_{(x,y)=(a,b)}y'(t_0).$$

In particular, for any given angle φ the directional derivative of $f(x,y)$ along the line

$$\begin{cases}x=a+t\cos\varphi\\ y=b+t\sin\varphi\end{cases}$$

at $t=0$ is equal to

$$\left(\frac{\partial f}{\partial x}\right)_{(x,y)=(a,b)}\cos\varphi+\left(\frac{\partial f}{\partial y}\right)_{(x,y)=(a,b)}\sin\varphi.$$

Differentiability of Real-Valued Function of Two Real Variables Stronger Than Existence of Both Partial Derivatives. Consider the simple example of

$$f(x,y)=\frac{xy}{x^2+y^2}$$

for $(x,y) \neq (0,0)$ with $f(0,0)=0$. In polar coordinates (r,θ),

$$f(x,y) = \sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$$

for $(x,y) \neq 0$ with $f(0,0)=0$. It is clear that from definition both partial derivatives of f (with respect to x and with respect to y) exist and are 0. On the other hand, for fixed φ the restriction of $f(x,y)$ to the line

$$t \mapsto (t\cos\varphi, t\sin\varphi)$$

is not continuous when $\varphi = \frac{\pi}{4}$, because the value of $f(x,y)$ at $(t\cos\varphi, t\sin\varphi)$ is equal to $\sin(2\varphi) = \sin\frac{\pi}{2} = 1$ for $t \neq 0$ and does not approach 0 as $t \to 0$. As a matter of fact, the function

$$f(x,y) = \frac{xy}{x^2+y^2}$$

for $(x,y) \neq (0,0)$ with $f(0,0)=0$ is not even continuous at $(x,y)=(0,0)$.

Differentiability in Calculus of Complex Variable. Let $w = f(z)$ be a complex-valued function of a complex variable $z = x+iy$. Write $w = u+iv$. Let $c = a+ib$. The function $w = f(z)$ is said be *complex-differentiable* (also known as *holomorphic*) at $z = c$ if the limit of the difference quotient

$$\lim_{z\to c}\frac{f(z)-f(c)}{z-c}$$

exists, which we denote by $f'(c)$. That is, given any $\varepsilon > 0$ there exists some $\delta > 0$ (which depends on ε) such that

$$\left|\frac{f(z)-f(c)}{z-c} - f'(c)\right| < \varepsilon$$

for z satisfying $0 < |z-c| < \delta$. We can rewrite it as

$$f(z) = f(c) + f'(c)(z-c) + E(z)$$

with

$$\lim_{z\to c}\frac{E(z)}{z-c} = 0.$$

Let $f'(c) = A + iB$ and $E(z) = E_1(x,y) + iE_2(x,y)$. Then

$$u(x,y)+iv(x,y) = u(a,b)+iv(a,b)+(A+iB)((x-a)+i(y-b))+E_1(x,y)+iE_2(x,y).$$

Equating the real and imaginary parts of both sides, we get

$$u(x,y) = u(a,b) + A(x-a) - B(y-b) + E_1(x,y)$$

and

$$v(x,y) = v(a,b) + B(x-a) + A(y-b)) + E_2(x,y).$$

Since

$$\lim_{(x,y)\to(a,b)} \frac{E_j(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

for $j = 1, 2$, this means that both $u(x,y)$ and $v(x,y)$ as real-valued functions of two real variables x, y are differentiable at $(x,y) = (a,b)$ with

$$\frac{\partial u}{\partial x}(a,b) = A \quad \text{and} \quad \frac{\partial u}{\partial y}(a,b) = -B$$

and

$$\frac{\partial v}{\partial x}(a,b) = B \quad \text{and} \quad \frac{\partial v}{\partial y}(a,b) = -A.$$

In particular,

$$\frac{\partial u}{\partial x}(a,b) = \frac{\partial v}{\partial y}(a,b) \quad \text{and} \quad \frac{\partial u}{\partial y}(a,b) = -\frac{\partial v}{\partial x}(a,b).$$

We can also reverse the argument so that we obtain the following theorem.

Theorem (Cauchy-Riemann Equations). Let $w = u + iv$ and $z = x + iy$ and $c = a + ib$. Let $w = f(z)$ be a complex-valued function of a complex variable z on an open neighborhood U of c in $\mathbb{C}$. The complex-differentiability of $w = f(z)$ at $z = c$ if and only if the real part $u(x,y)$ and the imaginary part $v(x,y)$ of $f(z)$ are differentiable at $(x,y) = (a,b)$ as real-valued functions of the two real variables x, y and the following Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}(a,b) = \frac{\partial v}{\partial y}(a,b) \quad \text{and} \quad \frac{\partial u}{\partial y}(a,b) = -\frac{\partial v}{\partial x}(a,b)$$

are satisfied. In such a case,

$$f'(c) = \frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}.$$

Remark. The existence of both partial derivatives and the holding of the Cauchy-Riemann equations are not sufficient to guarantee complex-differentiability *without the assumption of differentiability of both the real and imaginary parts as real-valued functions of two real variables.* The example is

$$f(z) = \frac{xy}{x^2+y^2}$$

for $(x, y) \neq (0, 0)$ with $f(0, 0) = 0$. All four partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

vanish at $(x, y) = (0, 0)$, but $f(z)$ is not complex differentiable at $z = 0$, because $u(x, y)$ is not even continuous at $(x, y) = (0, 0)$.

Another Way to Derive Cauchy-Riemann Equations. The above derivation of the Cauchy-Riemann equations is to separate the real part and the imaginary part of the dependent variable $w = u+iv$. Another way to derive the Cauchy-Riemann equations is to isolate the real part and the imaginary part of the dependent variable $z = x + iy$. Suppose $f(z)$ is complex-differentiable at $z = c = a + ib$ so that

$$\lim_{z\to c} \frac{f(z) - f(c)}{z - c} = f'(c).$$

When we pass to limit as $z \to c$, we can restrict ourselves to the horizontal line $y = b$ to get

$$\lim_{x\to a} \frac{f(x + ib) - f(a + ib)}{x - a} = f'(c)$$

so that

$$\frac{\partial f}{\partial x}(c) = f'(c).$$

Likewise, when we pass to limit as $z \to c$, we can restrict ourselves to the vertical line $x = a$ to get

$$\lim_{y\to b} \frac{f(a + iy) - f(a + ib)}{i(y - b)} = f'(c)$$

so that

$$\frac{1}{i}\frac{\partial f}{\partial y}(c) = f'(c).$$

The key difference between the two restrictions (to the horizontal line and the vertical line) is that in the latter case there is an additional factor of i in the denominator. By putting the two equations together, we end up with

$$\frac{\partial f}{\partial x}(c) = \frac{1}{i}\frac{\partial f}{\partial y}(c),$$

which is another way of writing the Cauchy-Riemann equations, because by taking the real and imaginary parts of the equation

$$\frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b) = \frac{1}{i}\left(\frac{\partial u}{\partial y}(a,b) + i\frac{\partial v}{\partial y}(a,b)\right),$$

we end up with

$$\frac{\partial u}{\partial x}(a,b) = \frac{\partial v}{\partial y}(a,b) \quad \text{and} \quad \frac{\partial u}{\partial y}(a,b) = -\frac{\partial v}{\partial x}(a,b).$$

This second way of deriving the Cauchy-Riemann equations by writing

$$f'(c) = \frac{\partial f}{\partial x}(c) = \frac{1}{i}\frac{\partial f}{\partial y}(c)$$

makes the Cauchy-Riemann equations more transparent and easier to remember.

Another way to rewrite the Cauchy-Riemann equations

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$$

is to write it as

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = 0$$

and introduce the notation

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

so that the Cauchy-Riemann equations now read as

$$\frac{\partial f}{\partial \bar{z}}.$$

The reason for using the factor $\frac{1}{2}$ and for the notation

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

is that

$$\frac{\partial}{\partial \bar{z}}\, f(z) = 1 \quad \text{when} \quad f(z) \equiv \bar{z}.$$

We will later also have the occasion of using the complex-conjugate

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

of the operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Linear-Algebraic Interpretation of Cauchy-Riemann Equations in Terms of $\mathbb{C}$*-Linear Transformation.* A $\mathbb{R}$-linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ can be represented by a 2×2 matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

When $\mathbb{C}$ is identified with $\mathbb{R}^2$, the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ as a map $T : \mathbb{C} \to \mathbb{C}$ may not be $\mathbb{C}$-linear. It is $\mathbb{C}$-linear if $T(iv) = iT(v)$, which means that the map T commutes the map $\mathbb{C} \to \mathbb{C}$ defined by multiplication by i. Denote by $J : \mathbb{R}^2 \to \mathbb{R}^2$ which is the map $\mathbb{C} \to \mathbb{C}$ defined by multiplication by i when $\mathbb{R}^2$ is identified with $\mathbb{C}$. Then J sends $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and sends $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, because multiplication by i sends the unit vector in the real axis to the unit vector in the imaginary axis and sends the unit vector in the imaginary axis to the negative of the unit vector in the real axis. Thus the 2×2 matrix which represents J is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Commutativity $TJ = JT$ means

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

which means

$$\begin{pmatrix} \beta & -\alpha \\ \delta & -\gamma \end{pmatrix} = \begin{pmatrix} -\gamma & -\delta \\ \alpha & \beta \end{pmatrix},$$

or $\alpha = \delta$ and $\beta = -\gamma$. In summary, the $\mathbb{R}$-linear transformation of $\mathbb{R}^2$ to itself is a $\mathbb{C}$-linear when $\mathbb{R}^2$ is identified with $\mathbb{C}$ if and only if the 2×2 matrix with real entries which represents the the $\mathbb{R}$-linear transformation of $\mathbb{R}^2$ to itself satisfies the condition that the two elements in the diagonal are equal and the two elements in the antidiagonal are negative of each other.

The Jacobian matrix of a map $(x, y) \mapsto (u(x, y), v(x, y))$ is given by the 2×2 matrix

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

(where the subscript notation means partial derivative with respect to the variable in the subscript). The Jacobian matrix represents a $\mathbb{C}$-linear map of $\mathbb{C}$ to itself if and only if the Cauchy-Riemann equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

are satisfied.