Homework #1 Assigned on January 25, 2024
due February 1, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 113

Problem 1 *(from Stein & Shakarchi, p.26, #7).* The family of mappings introduced here plays an important rôle in complex analysis. These mappings sometimes called *Blaschke factors*, will reappear in various applications later.

(a) Let z, w be two complex numbers such that $\bar{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\bar{w}z}\right| < 1 \quad \text{if} \quad |z| < 1 \quad \text{and} \quad |w| < 1,$$

and also that

$$\left|\frac{w-z}{1-\bar{w}z}\right| = 1 \quad \text{if} \quad |z| = 1 \quad \text{or} \quad |w| = 1.$$

Hint: Why can one assume that z is real? It then suffices to prove that

$$(r-w)(r-\bar{w}) \leq (1-rw)(1-r\bar{w})$$

with equality for appropriate r and $|w|$.

(b) Prove that for a fixed w in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$, the mapping

$$F : z \mapsto \frac{w-z}{1-\bar{w}z}$$

satisfies the following conditions:

(i) F maps the open unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is complex-differentiable at every point of the open unit disk $\mathbb{D}$.

(ii) F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F : \mathbb{D} \to \mathbb{D}$ is bijective. [*Hint:* Calculate $F \circ F$.]

Problem 2 *(from Stein & Shakarchi, p.27, #9).* Consider the polar coordinates (r, θ) so that $x = r\cos\theta$ and $y = r\sin\theta$, which can also be written as

$$z = x + iy = r\left(\cos\theta + i\sin\theta\right).$$

Show that in polar coordinates (r, θ), the Cauchy-Riemann equations for the function $f = u + iv$ take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \text{with} \quad -\pi < \theta < \pi$$

is complex-differentiable in the region $r > 0$ and $-\pi < \theta < \pi$.

Problem 3 *(from Stein & Shakarchi, p28, #13).* Suppose that f is complex-differentiable at every point of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$. Prove that in any one of the following three cases:

(a) $\operatorname{Re} f$ is constant on $\mathbb{D}$;

(b) $\operatorname{Im} f$ is constant on $\mathbb{D}$;

(c) $|f|$ is constant on $\mathbb{D}$;

one can conclude that f is constant on $\mathbb{D}$.

Hint: In each of the three cases, use the Cauchy-Riemann equations to show that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all identically zero, where $f = u + iv$.