ELLIPTIC FUNCTIONS

To be able to solve problems with solutions in *closed form*, it is important to have at our disposal a large pool of known functions. The known functions which we have encountered so far are rather limited. For example: the rational functions (as quotients of two polynomials), algebraic functions (which satisfy a polynomials whose coefficients are themselves polynomials), the exponential function (and functions related to it such as the logarithmic function, the trigonometric functions, and the hyperbolic functions). The exponential function has one period which is $2\pi i$. Now we introduce another kind of special functions, known as elliptic functions, which are meromorphic functions on $\mathbb{C}$ with two $\mathbb{R}$-linearly independent periods.

Solution of Equation for Motion for Simple Pendulum and Computation of Period. We start out with the problem of a simple pendulum. Let m be the mass of the bob at the end of the pendulum, a be the length of the pendulum, θ be the angle of inclination which the pendulum makes with a vertical line, α be the initial angle of inclination when the pendulum is released from rest position at time zero, t be the time variable, and g be the constant of the gravity of the earth. The equation of the conservation of energy (which is the sum of the kinetic energy and the potential energy) is

$$\frac{1}{2}ma^2\left(\frac{d\theta}{dt}\right)^2 - mga\cos\theta = -mga\cos\alpha.$$

It follows that

$$\left(\frac{d\theta}{dt}\right)^2 = 2\,\frac{g}{a}\left(\cos\theta - \cos\alpha\right) = 4\,\frac{g}{a}\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right).$$

The motivation to use the double angle formula for the cosine function is that when we express t in terms of an integral in θ we would like to end up with the square root of a quadratic expression in the denominator and in the case of a small value for θ with $\sin\theta$ approximated by θ we can express t in terms of the inverse sine function involving θ, that is, θ in terms of a sine function involving t.

In order to replace $\sin^2\frac{\alpha}{2}$ by 1 to get to an integral defining the inverse sine function as an approximation, we introduce the substitution

$$\sin\varphi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}.$$

From its differentiation with respect to t

$$\cos\varphi\,\frac{d\varphi}{dt}=\frac{1}{2}\frac{\cos\frac{\theta}{2}}{\sin\frac{\alpha}{2}}\,\frac{d\theta}{dt}$$

we have

$$\begin{aligned}\cos^2\varphi\left(\frac{d\varphi}{dt}\right)^2&=\frac{1}{4}\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\alpha}{2}}\left(\frac{d\theta}{dt}\right)^2\\&=\frac{1}{4}\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\alpha}{2}}\,4\,\frac{g}{a}\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)\\&=\frac{1}{4}\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\alpha}{2}}\,4\,\frac{g}{a}\left(\sin^2\frac{\alpha}{2}-\left(\sin\varphi\,\sin\frac{\alpha}{2}\right)^2\right)\\&=\cos^2\frac{\theta}{2}\,\frac{g}{a}\left(1-\sin^2\varphi\right)\\&=\left(1-\sin^2\frac{\theta}{2}\right)\,\frac{g}{a}\cos^2\varphi\\&=\left(1-\sin^2\frac{\alpha}{2}\sin^2\varphi\right)\,\frac{g}{a}\cos^2\varphi\end{aligned}$$

to get the differential equation

$$\left(\frac{d\varphi}{dt}\right)^2=\frac{g}{a}\left(1-\sin^2\frac{\alpha}{2}\sin^2\varphi\right)$$

which can be integrated to yield

$$t=\sqrt{\frac{a}{g}}\int\frac{d\varphi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\varphi}}.$$

More precisely,

$$t=\sqrt{\frac{a}{g}}\int_{\psi=0}^{\varphi}\frac{d\psi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\psi}}.$$

To simplify notations, we let $k=\sin\frac{\alpha}{2}$ so that

$$t=\sqrt{\frac{a}{g}}\int\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}.$$

Since the initial angle α should be strictly between 0 and $\frac{\pi}{2}$, we know that the constant $k=\sin\frac{\alpha}{2}$ is in the interval $\left(0,\frac{1}{\sqrt{2}}\right)$.

To replace the quadratic expression in sine by a polynomial, we introduce the substitution $x = \sin\varphi$ so that from

$$dx = \cos\varphi\, d\varphi = \sqrt{1-\sin^2\varphi}\, d\varphi = \sqrt{1-x^2}\, d\varphi$$

and

$$t = \sqrt{\frac{a}{g}} \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

The elliptic sine function sn (which depends on the parameter k, known as its *modulus*) is defined by the following formula for its multi-valued inverse

$$\mathrm{sn}^{-1} y = \int_{x=0}^{y} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

The final answer for the motion of a simple pendulum is now given by

$$\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\,\mathrm{sn}\left(\sqrt{\frac{g}{a}}\, t\right)$$

with the modulus k equal to $\sin\frac{\alpha}{2}$. One quarter of the period is achieved when the angle θ goes from α to 0, which means that φ goes from $\frac{\pi}{2}$ to 0 on account of the relation

$$\sin\varphi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}.$$

When we use the formula

$$t = \sqrt{\frac{a}{g}} \int \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

to compute the quarter-period, we obtain the periodic time of the pendulum as

$$\begin{aligned} & 4\sqrt{\frac{a}{g}} \int_{\varphi=0}^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \\ = {} & 4\sqrt{\frac{a}{g}} \int_{\varphi=0}^{\frac{\pi}{2}} \left(1 + \frac{1}{2}k^2\sin^2\varphi + \frac{1\cdot 3}{2\cdot 4}k^4\sin^4\varphi + \cdots\right) d\varphi \\ = {} & 2\pi\sqrt{\frac{a}{g}} \left(1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots\right), \end{aligned}$$

because
$$\int_0^{\frac{\pi}{2}} \sin^{2n} \varphi d\varphi = \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}.$$
This motivates the introduction of the constant
$$K = \int_{\varphi=0}^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \int_{x=0}^{1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$
which is the generalization of $\frac{\pi}{2}$ for the special case of $k = 0$. A period of the elliptic sine function is $4K$. As we will see later, the other period is not real so that the lattice in $\mathbb{C}$ generated by the two periods is of full rank 2 in $\mathbb{C}$.

Justification of Defining the Elliptic Sine Function by Inverting an Integral. We defined the elliptic sine function sn (with modulus 0¡k¡1)) by inverting the integral
$$\mathrm{sn}^{-1} y = \int_{x=0}^{y} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$
We need to justify its invertibility in a mathematically rigorous way. The integral
$$y \mapsto \int_{x=0}^{y} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$
when y is allowed to be in the upper half-plane $\mathbb{H}$ is actually a Schwarz-Christoffel transformations with points $-\frac{1}{k}$, -1, 1, $\frac{1}{k}$ on the real line corresponding to the four vertices K, $K + iK'$, $-K + iK'$, $-K$ of a rectangle R, where
$$K = \int_{x=0}^{1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$
and
$$iK' = \int_{x=1}^{\frac{1}{K}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$
which means that
$$K' = \int_{x=1}^{\frac{1}{K}} \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}$$
is positive. Since we now allow complex variables, instead of
$$y \mapsto \int_{x=0}^{y} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

we change the notations to

$$w \mapsto \int_{z=0}^{w} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

where z and w are complex variables.

One way to justify the inversion process in a mathematically rigorous way is to reflect the map from the rectangle R to $\mathbb{H}$ with respect to the sides of R and the sides of rectangles generated by reflections, repeatedly by the use of Schwarz reflection.

Another way is to construct a mathematical object which is on top of $\mathbb{C}$ with two points on top of one (except at $-\frac{1}{k}$, -1, 1, $\frac{1}{k}$) to make the integrand

$$\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

single-valued nonzero so that the integral is holomorphic with nonzero derivative. This mathematical object is known as a Riemann surface. The reason why we use

$$\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

as the integrand instead of

$$\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

is that the presence of dx automatically builds into the expression the use of the Jacobian determinant in the charge of the independent variable of integration.

We will do both ways of justification and start with the second way of constructing a Riemann surface.

Riemann Surface to Remove Multi-Valuedness of Function or Form. To study the indefinite integral of the 1-form

$$\int \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

with complex variable, we have to handle the problem of multi-valuedness from the square root. We can just focus on the multi-valued function

$$z \mapsto F(z) = \sqrt{(1-z^2)(1-k^2z^2)}.$$

The simplest way is to consider the graph M of the doubly-valued function

$$z \mapsto F(z) = \sqrt{(1-z^2)(1-k^2z^2)}.$$

on the Riemann sphere. We will put (biholomorphically related) local complex coordinate systems on M so that we can do complex analysis on it. The mathematical object M so constructed is a *Riemann surface*. We now present more details.

By removing the two line segments $[-\frac{1}{k}, -1]$ and $[1, \frac{1}{k}]$ from the Riemann sphere $\mathbb{C} \cup \{\infty\}$, we can select two branches of the function

$$z \mapsto F(z) = \sqrt{(1-z^2)(1-k^2z^2)},$$

with the two differing by a sign. We can look at two different Riemann spheres, each one with the two line segments $[-\frac{1}{k}, -1]$ and $[1, \frac{1}{k}]$ removed, and get one single-valued nowhere zero holomorphic function

$$z \mapsto F(z) = \sqrt{(1-z^2)(1-k^2z^2)}$$

on each $\mathbb{C} \cup \{\infty\} - \big([-\frac{1}{k}, -1] \cup [1, \frac{1}{k}]\big)$, with the two differing by a sign. To glue the two Riemann spheres minus slits

$$\mathbb{C} \cup \{\infty\} - \left([-\frac{1}{k}, -1] \cup [1, \frac{1}{k}]\right)$$

into one object, we identify the upper edge of $[-\frac{1}{k}, -1]$ on one with the lower edge of $[-\frac{1}{k}, -1]$ on the other and likewise the upper edge of $[1, \frac{1}{k}]$ on one with the lower edge of $[1, \frac{1}{k}]$ on the other. We call this the crisscross way of identification of edges of slits. The result is the same as the graph M of the doubly-valued function

$$z \mapsto F(z) = \sqrt{(1-z^2)(1-k^2z^2)}.$$

on the Riemann sphere. Of course, we have a projection map from the graph M to the Riemann sphere $\mathbb{C} \cup \{\infty\}$. There are four "branch-points" of M coming from the four zero-points $-\frac{1}{k}, -1, 1, \frac{1}{k}$ of

$$F(z) = \sqrt{(1-z^2)(1-k^2z^2)}.$$

At a point of M other than these four branch-points, a local coordinate can be defined by using the local coordinate of the Riemann sphere $\mathbb{C}\cup\{\infty\}$ via the projection π.

At any point a of the four branch points over the four roots, we need to introduce the local coordinate ζ for the Riemann surface M such that $z = a + \zeta^2$. With respect to such a local coordinate the 1-form

$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

is equal to a nowhere holomorphic factor times

$$\frac{d(a+\zeta^2)}{\sqrt{(a+\zeta^2)-a}} = 2d\zeta$$

near $z = a$, which means that the 1-form obtained by pulling back

$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

to M via π is holomorphic and nowhere zero on the compact Riemann surface M.

Visualization of Branched Double Cover of Riemann Sphere from Crisscross Gluing of Edges of Two Slits. We would like to know topologically what the Riemann surface looks like. We need to visualize the process of removing the slits $\left[-\frac{1}{k}, -1\right]$ and $\left[1, \frac{1}{k}\right]$ and identifying the sides of the slits in two copies of $\mathbb{P}_1$ in a crisscross manner. By turning one of the two $\mathbb{P}_1$ inside out, the crisscross identification can be replaced by direct identification to end up with a torus as the underlying topological space for the compact Riemann surface, as illustrated in the following sequence of figures.

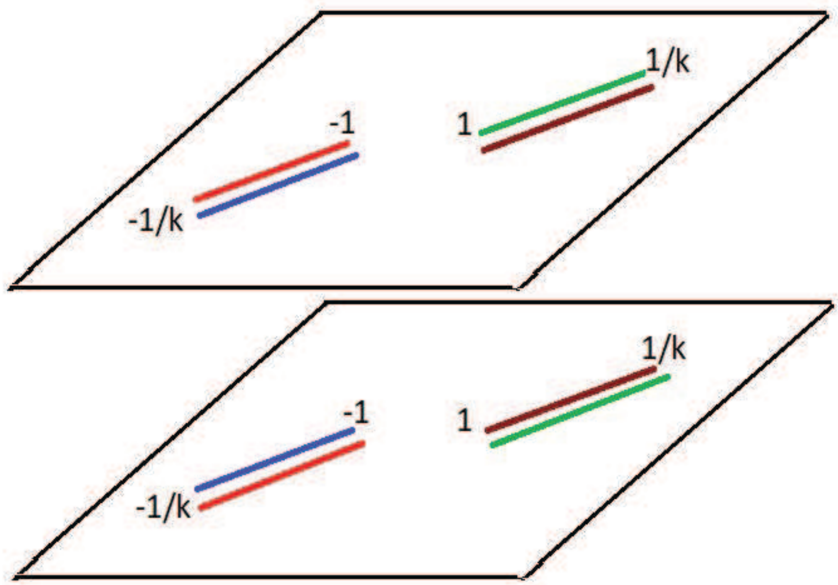


Figure 1: Two red-colored edges identified. Two blue-colored edges identified. Two green-colored edges identified. Two brown-colored edges identified.

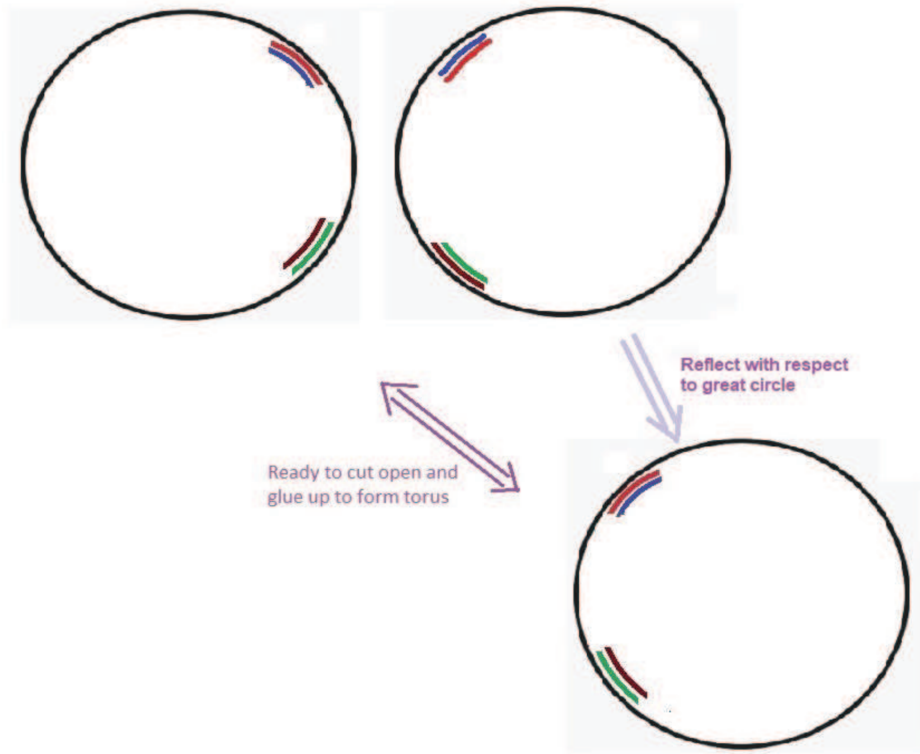


Figure 2: Reflect one sphere with respect to its great circle (same as turning it inside out) to prepare for cutting up and glueing to form torus.

To see why reflecting a sphere with respect its great circle is the same as turning it inside out, consider the sphere S in $\mathbb{R}^3$ (with coordinates x, y, z) defined by $x^2 + (y+1)^2 + z^2 = 1$. The map $(x, y, z) \mapsto (x, -y, z)$ turns S inside out and transforms S to S' which is defined by $x^2 + (y-1)^2 + z^2 = 1$. Use the x-axis as the axis of rotation to rotate S' by 180 degrees to transform S' to S'' so that S'' as a subset of $\mathbb{R}^3$ occupies the same position as S. The upper edge of the equator $z = 0$ of S now becomes the lower edge of the equator $z = 0$ of S''. The lower edge of the equator $z = 0$ of S now becomes the upper edge of the equator $z = 0$ of S''. In other words, S'' is the same as the reflection of S with respect to the equator of S.

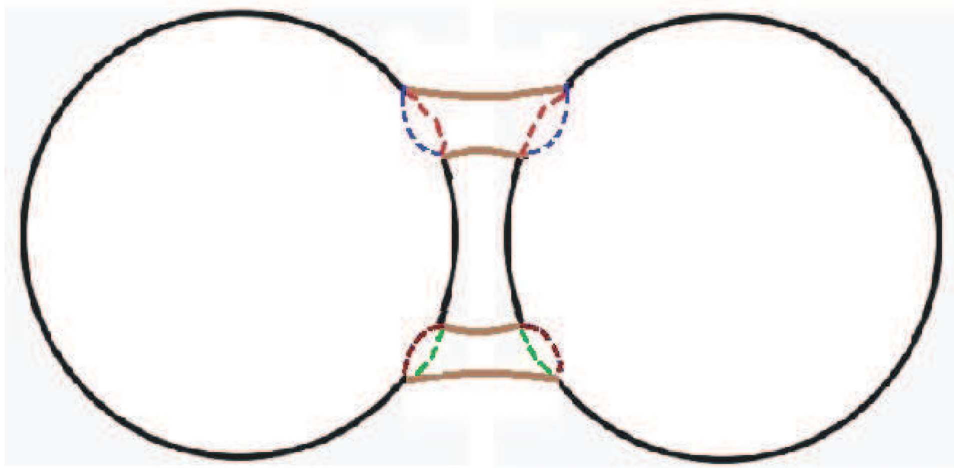
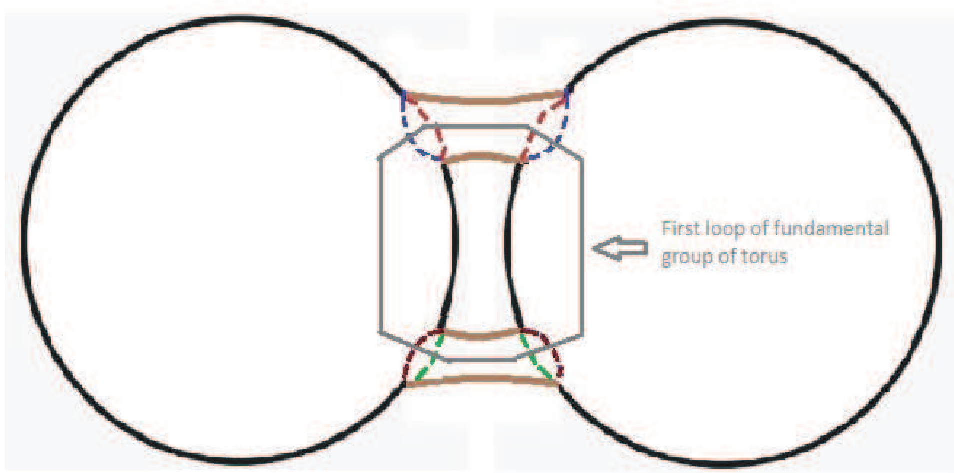
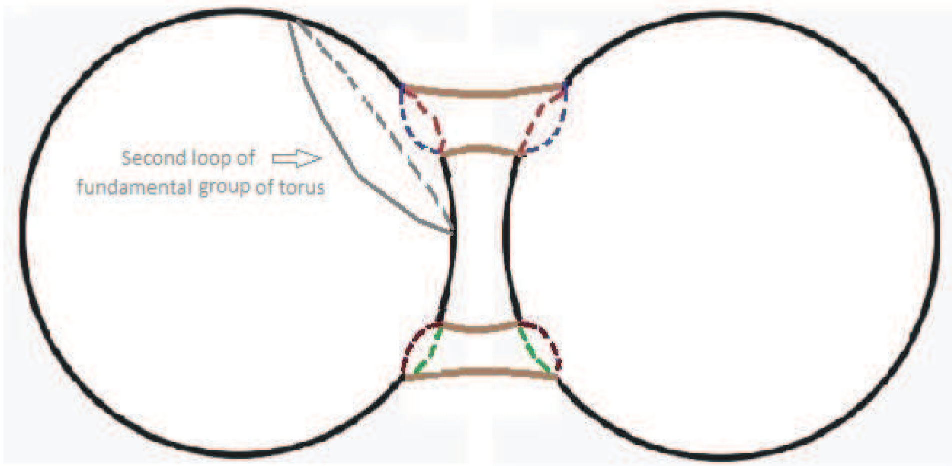


Figure 3: Torus constructed after cutting up branch-cuts of both spheres and glueing corresponding edges.

Topologically there are two loops to generate the fundamental group of the torus. They are described as follows.





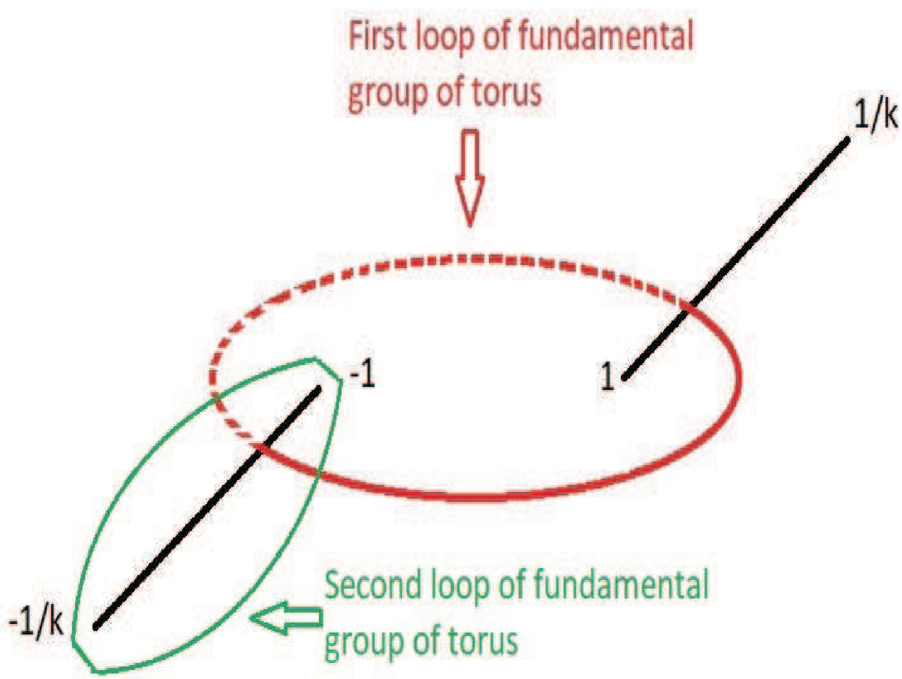


Figure 4: Both loops of the fundamental group of the torus. First goes to another sheet through one branch-cut (represented by dotted curve) and then returns to the original sheet through another branch-cut.

Another way of visualizing the Riemann surface is to focus on the graph of the square root function at the four points $\frac{-1}{k}$, -1, 1, $\frac{1}{k}$ in the following figure when we look down at the two Riemann spheres connected by the square roots at the four points. At each of the four branch-points, the construction is like a two-floor spiral parking ramp which leads back to the first floor when one tries to use it to rise one more floor from the second floor (more like the "Relativity" woodcut by M.C. Eschel).

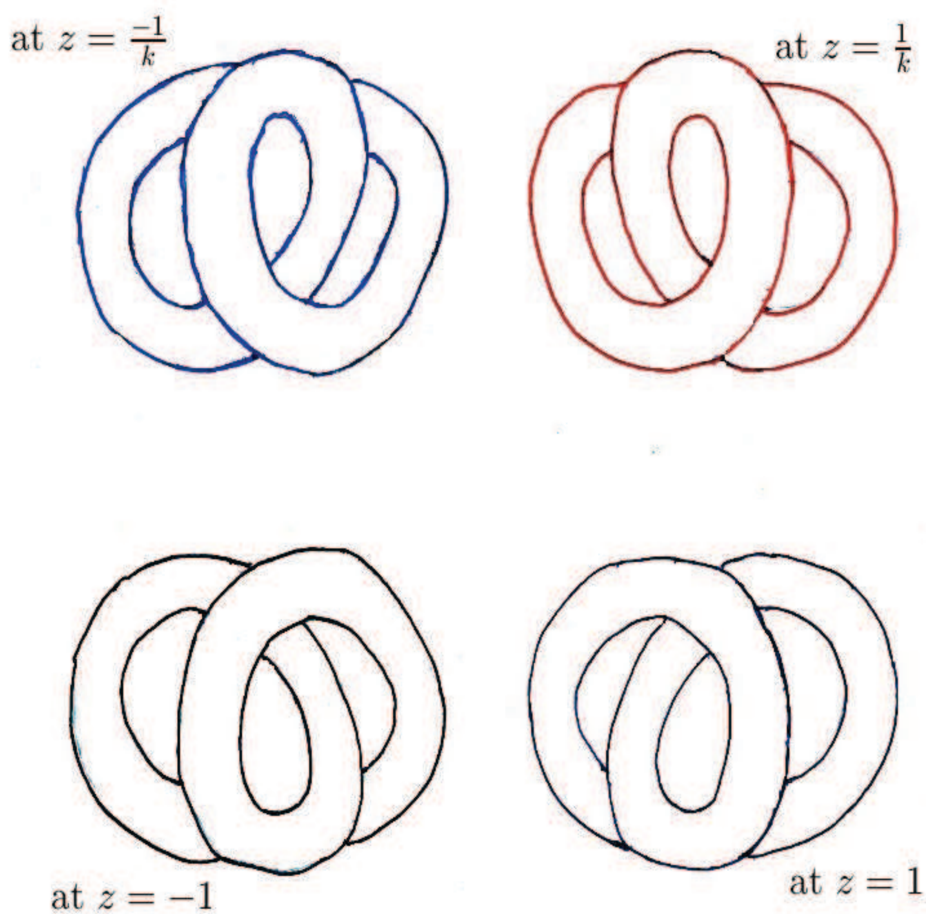


Figure 5: Visualization of the Riemann surface (as graph of double-valued function) by looking down at the two Riemann spheres connected by the square roots at $\frac{-1}{k}$, -1, 1, $\frac{1}{k}$.

The above figure, as the graph of a doubly-valued holomorphic function on the Riemann sphere, is one single connected Riemann surface M. In order to break up M into two disjoint Riemann spheres with two slits removed from each, one can either (i) remove the slit $[-\frac{1}{k}, -1]$ and the slit $[1, \frac{1}{k}]$ or (ii) remove the slit $[-1, 1]$ and the slit $[-\frac{1}{k}, \infty, \frac{1}{k}]$. From the above figure, this means two different ways of blocking (or removing the parts) between two pairs of spiral ramps, as illustrated in the two figures below.

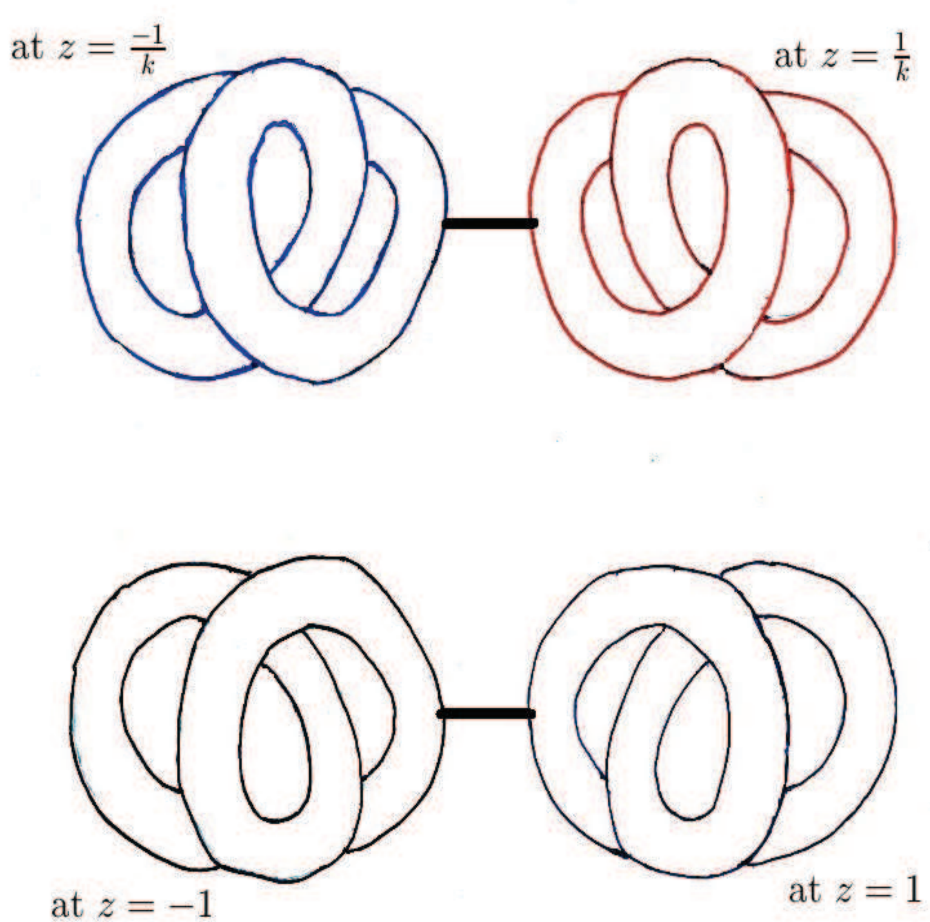


Figure 6: Break-up into two two disjoint Riemann spheres with two slits $[-1, 1]$ and $[\frac{1}{k}, \infty, \frac{-1}{k}]$ removed from each.

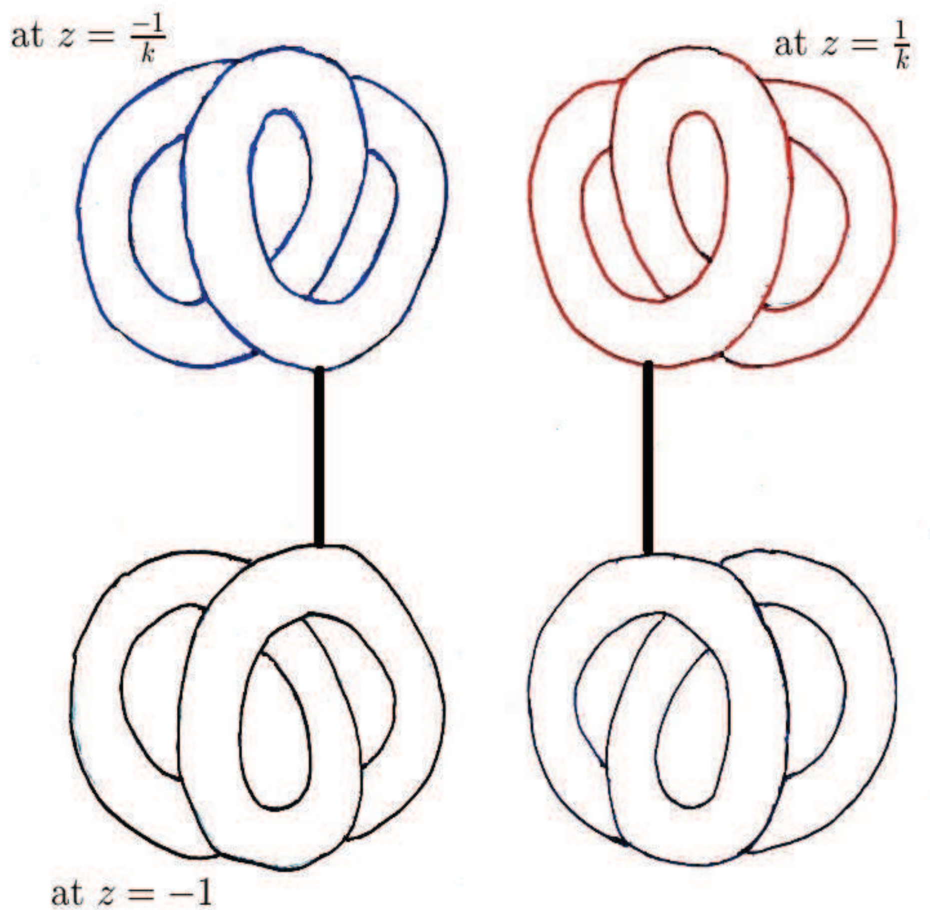


Figure 7: Break-up into two two disjoint Riemann spheres with two slits $[\frac{-1}{k}, -1]$ and $[1, \frac{1}{k}]$ removed from each.

After using either one of the two ways of breaking up, we end up with four pieces: the upper half-plane of the upper Riemann sphere, the lower half-plane of the upper Riemann sphere, the upper half-plane of the lower Riemann sphere, and the lower half-plane of the lower Riemann sphere. We will see later that, through the Schwarz-Christoffel transformation, each of these four pieces corresponds to one of the four equal sub-rectangles of the rectangle with vertices $2K, 2K + 2iK', -2K + 2iK', -2K$.

Inverting Indefinite Elliptic Integral. Now we return to the original question of inverting the indefinite integral

$$\int \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

with $k \in \mathbb{C} - \{1, -1, 0\}$. After our discussion on handling multi-valuedness, we have the setting of a compact Riemann surface M (which means a compact complex manifold of complex dimension 1) and a proper holomorphic map $\pi : M \to \mathbb{C} \cup \{\infty\}$ (which is two-to-one except over four distinct points of the Riemann surface $\mathbb{C} \cup \{\infty\}$) and a nowhere zero holomorphic 1-form φ on M, which is obtained by pulling back

$$\int \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

via π. Since the underlying topological space of M is a torus, its first homology group is generated by two loops γ_1, γ_2. The indefinite integral

$$P \mapsto \int_{P_0}^{P} \varphi$$

(with some fixed initial point P_0 and the variable point P in M) is a multi-valued holomorphic function on M. Its multi-valuedness comes from the two periods

$$\omega_1 = \int_{\gamma_1} \varphi \quad \text{and} \quad \omega_2 = \int_{\gamma_2} \varphi$$

obtained by integrating the holomorphic 1-form φ over the two loops γ_1 and γ_2. We are going to verify later that the two periods as elements of $\mathbb{C}$ are $\mathbb{R}$-linearly independent so that

$$P \mapsto \int_{P_0}^{P} \varphi$$

as a map Φ from M to the compact Riemann surface $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ is single-valued holomorphic. We will verify that this holomorphic map Φ is actually biholomorphic and the inversion of the indefinite integral

$$\int \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

is actually the composite map

$$\mathbb{C} \to \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) \xrightarrow{\Phi^{-1}} M \xrightarrow{\pi} \mathbb{P}_1$$

from $\mathbb{C}$ to $\mathbb{P}_1$ which is a doubly periodic meromorphic function with primitive periods ω_1 and ω_2.

We now verify that ω_1 and ω_2 are linearly independent over $\mathbb{R}$ and that Φ is invertible. Suppose ω_1 and ω_2 are linearly dependent over $\mathbb{R}$. Then the 2×2 matrix with the two row-vectors $(\operatorname{Re}\omega_\nu,\ \operatorname{Im}\ \omega_\nu)$, $\nu = 1, 2$, has determinant zero. We can find real numbers λ and μ not both zero such that $\lambda \operatorname{Re}\omega_\nu + \mu \operatorname{Im}\omega_\nu = 0$ for $\nu = 1, 2$. Let $\alpha = \lambda - i\,\mu$. Then

$$\operatorname{Re}(\alpha\,\omega_\nu) = \lambda \operatorname{Re}\omega_\nu + \mu \operatorname{Im}\omega_\nu = 0$$

for $\nu = 1, 2$. In other words, the integral

$$\begin{aligned}
\operatorname{Re}\int_{\gamma_\nu} \alpha\,\varphi &= \int_{\gamma_\nu} (\lambda \operatorname{Re}\varphi + \mu \operatorname{Im}\varphi) \\
&= \operatorname{Re}\int_{\gamma_\nu} \lambda\,\varphi + \operatorname{Im}\int_{\gamma_\nu} \mu\,\varphi \\
&= \lambda\operatorname{Re}\int_{\gamma_\nu} \varphi + \mu\operatorname{Im}\int_{\gamma_\nu} \varphi \\
&= \lambda \operatorname{Re}\omega_\nu + \mu \operatorname{Im}\omega_\nu
\end{aligned}$$

vanishes for $\nu = 1, 2$. The function

$$P \mapsto \operatorname{Re}\int_{P_0}^{P} \alpha\,\varphi$$

is a single-valued harmonic function on M. Since M is compact, it must be constant by the maximum principle for harmonic functions. Its local harmonic conjugate

$$P \mapsto \operatorname{Im}\int_{P_0}^{P} \alpha\,\varphi$$

must be constant. Thus

$$P \mapsto \int_{P_0}^{P} \alpha\,\varphi$$

is locally constant and its derivative $\alpha\,\varphi$ must be identically zero, which is a contradiction.

Now we verify that Φ is invertible. Since the differential of the map

$$\Phi : M \to \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$$

is φ which is nowhere zero, the map Φ is locally a biholomorphism. Since topologically Φ maps the fundamental group of M is isomorphically onto the fundamental group of

$$\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2),$$

we conclude that Φ is a biholomorphic map and can be inverted.

Elliptic Functions Analogous to the Trigonometric Functions Cosine, Tangent, Cotangent, Secant, Cosecant. Define locally near w=0,

$$\operatorname{cn} w = \sqrt{1 - \operatorname{sn}^2 w}, \quad \operatorname{cn} 0 = 1$$

and

$$\operatorname{dn} w = \sqrt{1 - k^2 \operatorname{sn}^2 w}, \quad \operatorname{dn} 0 = 1.$$

The chain rule yields

$$\left(\frac{d}{dw} \operatorname{cn} w\right)^2 = \left(1 - \operatorname{cn}^2 w\right)\left(1 - k^2 + k^2 \operatorname{cn}^2 w\right).$$

$$\left(\frac{d}{dw} \operatorname{dn} w\right)^2 = \left(1 - \operatorname{dn}^2 w\right)\left(\operatorname{dn}^2 w - 1 + k^2\right).$$

We can globally define $\operatorname{cn} w$ as the inverse of the following indefinite integral

$$w(P) = \int_1^P \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 + k^2 x^2)}}$$

and globally define $\operatorname{dn} w$ as the inverse of the following indefinite integral

$$w(P) = \int_1^P \frac{dx}{\sqrt{(1 - x^2)(x^2 - 1 + k^2)}}$$

so that $\operatorname{cn} w$ and $\operatorname{dn} w$ are single-valued doubly periodic functions which are meromorphic on $\mathbb{C}$. Since the alternative definitions of $\operatorname{cn} w$ and $\operatorname{dn} w$ and their respective original definitions agree in some open neighborhood of the origin, they must agree everywhere. From $\operatorname{sn} w$ and $\operatorname{cn} w$ we can also alternatively define $\operatorname{dn} w$ as

$$\frac{(\operatorname{sn} w)'}{\operatorname{cn} w}$$

to guarantee that $\operatorname{dn} w$ is a single-valued doubly periodic function. As in the case of trigonometric functions we have the following formulae for differentiation

$$\begin{aligned}(\operatorname{sn} w)' &= \operatorname{cn} w \operatorname{dn} w \\ (\operatorname{cn} w)' &= -\operatorname{sn} w \operatorname{dn} w \\ (\operatorname{dn} w)' &= -k^2 \operatorname{sn} w \operatorname{cn} w\end{aligned}$$

and the algebraic relations

$$\begin{aligned}\operatorname{sn}^2 w + \operatorname{cn}^2 w &= 1 \\ \operatorname{dn}^2 w + k^2 \operatorname{sn}^2 w &= 1.\end{aligned}$$

One has also elliptic functions corresponding to the tangent, cotangent, secant and cosecant functions formed by taking reciprocals and quotients. The six trigonometric functions are obtained by choosing two functions from 1, $\sin w$, $\cos w$ and forming their quotients. So there are $3 \cdot 2$ such functions. By choosing two functions from 1, $\operatorname{sn} w$, $\operatorname{cn} w$, $\operatorname{dn} w$ and forming their quotient, we get $4 \cdot 3 = 12$ such functions. One can follow Jacobi and call them tangent amplitude, etc. or call them modular tangent function, etc with the notation $\operatorname{tn} w$. A better system is to use the notations

$$\begin{aligned}&\operatorname{ns} w = \tfrac{1}{\operatorname{sn} w}, \quad \operatorname{nc} w = \tfrac{1}{\operatorname{cn} w}, \quad \operatorname{nd} w = \tfrac{1}{\operatorname{dn} w} \\ &\operatorname{sc} w = \tfrac{\operatorname{sn} w}{\operatorname{cn} w}, \quad \operatorname{sd} w = \tfrac{\operatorname{sn} w}{\operatorname{dn} w}, \quad \operatorname{cd} w = \tfrac{\operatorname{cn} w}{\operatorname{dn} w} \\ &\operatorname{cs} w = \tfrac{\operatorname{cn} w}{\operatorname{sn} w}, \quad \operatorname{ds} w = \tfrac{\operatorname{dn} w}{\operatorname{sn} w}, \quad \operatorname{dc} w = \tfrac{\operatorname{dn} w}{\operatorname{cn} w}\end{aligned}$$

to denote the nine functions in addition to the functions $\operatorname{sn} w$, $\operatorname{cn} w$, and $\operatorname{dn} w$. All these twelve elliptic functions are known as the *Jacobi Elliptic Functions*. We now investigate their periodicity properties. We deal with only the modular sine function as the others can be dealt with in a completely analogous manner.

Justification of Inverting Indefinite Integral from Viewpoint of Schwarz-Christoffel Transformation. The indefinite integral

$$\int \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}}$$

is a Schwarz-Christoffel transformation which maps the open upper half-plane $\mathbb{H}$ to the rectangle R_0 with vertices K, $K+iK'$, $-K+iK'$, $-K$. We can use Schwarz reflections with respect to the sides of the rectangle R_0 and those reflected from it to extend the inverse of the map from the rectangle to the Riemann surface M. We now describe how these Schwarz reflections work.

Let $A=-\frac{1}{k}$, $B=-1$, $C=1$, $D=\frac{1}{k}$ in the z-plane. We add ∞ to the z-plane to form the z-Riemann-sphere, because we would like to look at the upper half plane in the z-plane as a domain in the z-Riemann-sphere.

Slits $[A,B]$ and $[C,D]$ are taken. Two copies of the z-Riemann-sphere are used, one upper-copy z-Riemann-sphere and one lower-copy z-Riemann-sphere. The upper-copy z-Riemann-sphere and lower-copy z-Riemann-sphere are joined along the edges of $[A,B]$ and $[C,D]$ in the following crisscross manner.

The upper edge of $[A,B]$ on the upper-copy z-Riemann-sphere is identified with the lower edge of $[A,B]$ in the lower-copy z-Riemann-sphere. The lower edge of $[A,B]$ on the upper-copy z-Riemann-sphere is identified with the upper edge of $[A,B]$ in the lower-copy z-Riemann-sphere. The upper edge of $[A,B]$ on the upper-copy z-Riemann-sphere is identified with the lower edge of $[A,B]$ in the lower-copy z-Riemann-sphere. The lower edge of $[C,D]$ on the upper-copy z-Riemann-sphere is identified with the upper edge of $[C,D]$ in the lower-copy z-Riemann-sphere. The upper edge of $[C,D]$ on the upper-copy z-Riemann-sphere is identified with the lower edge of $[C,D]$ in the lower-copy z-Riemann-sphere.

The integrand on $[B,C]$ in the upper-copy z-Riemann-sphere is equal to the negative of the integrand on $[B,C]$ in the lower-copy z-Riemann-sphere. The integrand on the segment $[D,\infty,A]$ in the upper-copy z-Riemann-sphere is equal to the negative of the integrand on the segment $[D,\infty,A]$ (of the Riemann sphere) in the lower-copy z-Riemann-sphere.

The elliptic sine function $z=\operatorname{sn} w$ maps the w-plane to the Riemann surface M which is obtained by the crisscross gluing of the upper-copy z-Riemann-sphere and lower-copy z-Riemann-sphere and then projects to the z-Riemann-sphere from the upper-copy z-Riemann-sphere and lower-copy z-Riemann-sphere.

Start with $B = -1$ which is the same point on either the upper-copy z-Riemann-sphere or lower-copy z-Riemann-sphere. Assume that the integrand is positive from B to C on the upper-copy z-Riemann-sphere and negative from B to C on the lower-copy z-Riemann-sphere. When we go from B to C on the upper-copy z-Riemann-sphere, the corresponding point in the w-plane goes from $-K$ to K. We follow this by going from C back to B on the lower-copy z-Riemann-sphere. The corresponding point in the w-plane continues to go from K to $3K$, because the difference in the sign of the integrand is compensated by the difference in the orientation of the interval of integration. The effect of going around the loop γ circling A and B is the same as adding $4K$, not only for the point B but also for any other point in M. This yields a period of $4K$ for the loop γ.

Start with $C = 1$ which is the same point on either the upper-copy z-Riemann-sphere or lower-copy z-Riemann-sphere. We go along the upper edge of the slit $[C, D]$ on the upper-copy z-Riemann-sphere to $D = \frac{1}{k}$ and then along the lower edge of the slit $[C, D]$ on the same upper-copy z-Riemann-sphere back to C and denote by γ' the loop which we just went along. The corresponding point in the w-plane goes from K to $K+iK'$ (for the movement from C to D on the upper edge of the slit of the upper-copy z-Riemann-sphere) and then from $K + 2iK'$ (for the movement from D to C on the lower edge of the slit of the upper-copy z-Riemann-sphere). The integrand at a point on the upper edge of $[C, D]$ equals to the negative of the integrand at the same point on the lower edge of $[C, D]$, but the sense of the orientation of the path of integration from C to D is the reverse of the sense of the orientation of the path of integration from D back to C. The effect of going around the loop γ' circling $[C, D]$ in the upper-copy z-Riemann-sphere is the same as adding $2iK'$, not only for the point C but also for any other point in M. This yields a period of $2iK'$ for the loop γ'.

The relation between z and w can be interpreted in terms of Schwarz reflection. Recall that R_0 is the rectangle in the w-plane with vertices $K, K+iK', -K+iK', -K$. Let R_1 be the translation of R_0 by $-iK'$ and R_2 be the translation of R_1 by $-iK'$. The point w_0 on R_0 comes from the integral with upper limit z_0 in the upper half of the upper-copy z-Riemann-sphere. The reflection $w_1 = \bar{w}_0$ (in R_1) of w_0 with respect to the real axis in the w-plane (*i.e.,* side $\partial R_0 \cap \partial R_1$ common to R_0 and R_1) corresponds to the point in the upper-copy z-Riemann-sphere which is the reflection $z_1 = \bar{z}_0$ of

z_0 with respect to $[B, C]$ on the real axis. The line joining w_0 to w_1 goes vertically down and is perpendicular to the $\partial R_0 \cap \partial R_1$. Correspondingly the line joining z_0 to z_1 also goes vertically down and is perpendicular to the real axis. We now reflect w_1 (in R_1) with respect to the the side $\partial R_1 \cap \partial R_2$ common to R_1 and R_2 to get w_2 in R_2. The side $\partial R_1 \cap \partial R_2$ in the w-coordinate system corresponds to $[D, \infty, A]$ in the z-coordinate system. The line segment $[w_1, w_2]$ from w_1 to w_2 corresponds to a curve $C_{1,2}$ in the Riemann surface M with end-points z_1 and z_2 which is perpendicular to $[B, C]$ in the upper-copy z-Riemann-sphere and perpendicular to $[D, \infty, A]$ also in the upper-copy z-Riemann-sphere. Projected down to the original z-Riemann-sphere, the image of z_2 should be the complex conjugate of z_1. This means that z_2 is actually z_0 and $[z_0, z_1] \cup C_{1,2}$ is a loop around the slit $[C, D]$ in the upper-copy z-Riemann-sphere so that $w_2 = w_1 - 2iK'$.

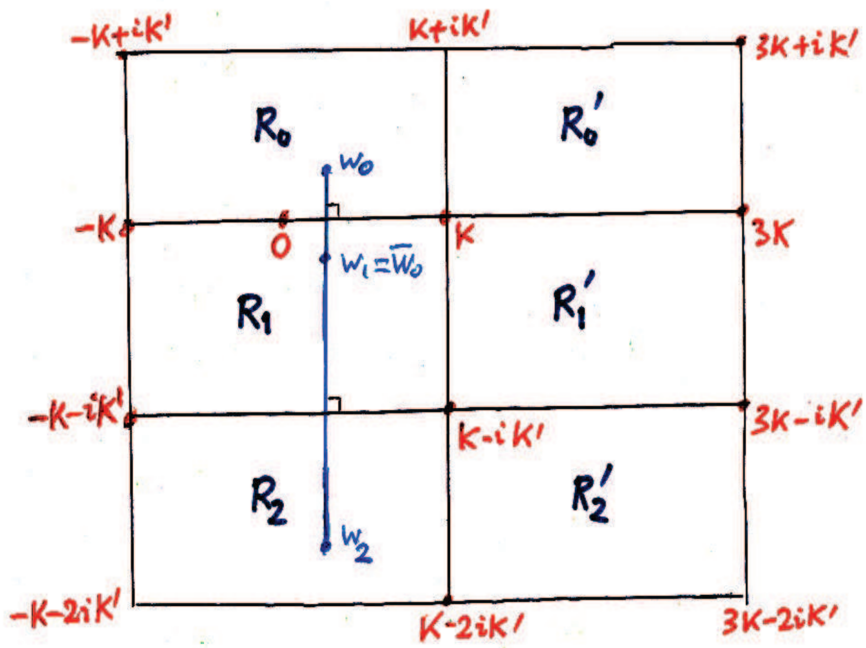


Figure 8: Schwarz reflections in the w coordinate system.

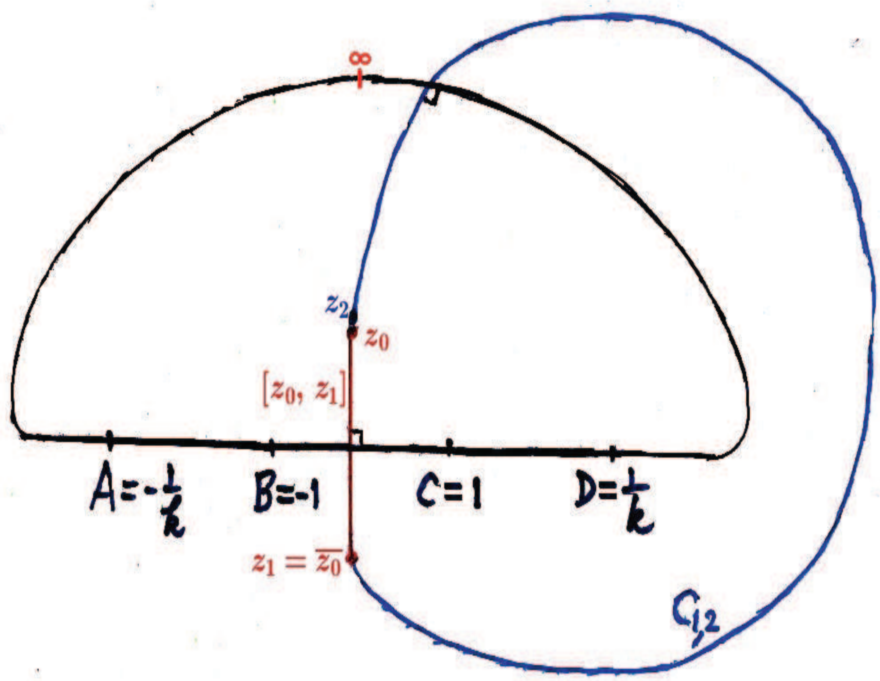


Figure 9: Corresponding Schwarz reflections in the z coordinate system.

For tiling in the horizontal direction, we argue in a similar manner, which is illustrated in the following figure. In this figure, black color is used for the upper-copy z-Riemann sphere and red color is used for the lower-copy z-Riemann sphere. When the slits $[A, B]$ and $[C, D]$ are removed to separate the two copies of z-Riemann sphere, the four parts in z coordinates (upper half of upper-copy z-Riemann sphere, lower half of upper-copy z-Riemann sphere, upper half of lower-copy z-Riemann sphere, lower half of lower-copy z-Riemann sphere) correspond to the four rectangles in w coordinates (R_0, R_1, R'_1, R'_0). The points z_j (for $1 \leq j \leq z$) in z coordinates correspond to the points w_j (for $1 \leq j \leq 4$) in w coordinates via the Schwarz-Christoffel transformation

$$w = \int \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.$$

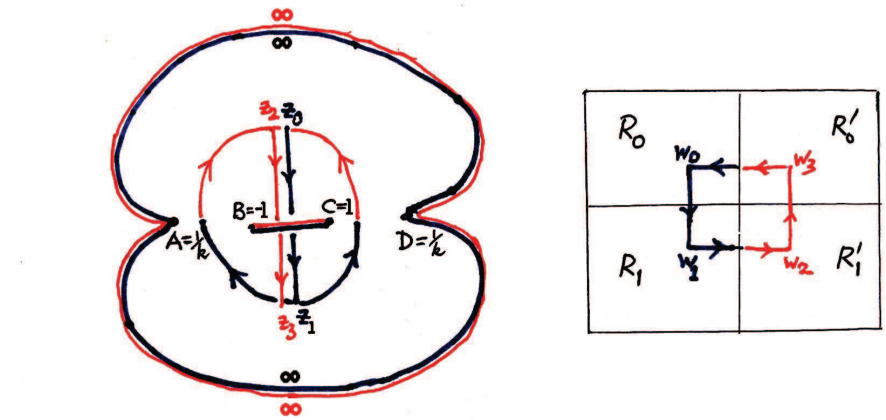


Figure 10: Under Schwarz-Christoffel transformation, two Riemann spheres with crisscross gluing of edges of two slits correspond to the union of four rectangles.

Weierstrass's Approach of Using Schwarz-Christoffel's Transformation with Three Points on Real Axis to Correspond to Three Points of the Rectangle. When we discussed Schwarz-Christoffel transformations to map the upper half-plane to a polygon of n sides, we considered the case of using $n-1$ points on the real axis to correspond to $n-1$ vertices of the polygon with the remaining vertex of the polygon corresponding to the point at infinity. Note that as a subset of the Riemann sphere, the real line $\mathbb{R}$ has only one point at infinity instead of two. The difference between the use of n points on $\mathbb{R}$ and the use of $n-1$ points on $\mathbb{R}$ is the question whether the point ∞ of the Riemann sphere is a zero of the denominator in the defining integral for the Schwarz-Christoffel transformation. One case can be transformed into the other by a linear fractional transformation.

Effects of Linear Fractional Transformation on Elliptic Integrals. A linear fractional transformation transforms

$$\frac{dz}{\sqrt{F_4(z)}}$$

to

$$\frac{dz}{\sqrt{G_3(z)}}$$

where $F_4(z)$ is a polynomial of degree 4 and $G_3(z)$ is a polynomial of degree 3. Here are more details of the argument with a linear fractional transformation. Let us write $F_4(z)$ as $\prod_{\nu=1}^4 (z - a_\nu)$ with all four a_ν distinct. We transform the integral by the Möbius transformation

$$z = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}.$$

Then

$$dz = \frac{(\alpha\delta - \gamma\beta)d\zeta}{(\gamma\zeta + \delta)^2}$$

and

$$\int \frac{dz}{\sqrt{F_4(z)}} = (\alpha\delta - \gamma\beta) \int \frac{d\zeta}{\sqrt{G_3(\zeta)}}$$

with

$$G_3(\zeta) = \prod_{\nu=1}^4 ((\alpha z\eta + \beta) - a_\nu(\gamma\zeta + \delta)).$$

We can choose $\alpha, \beta, \gamma, \delta$ so that $\alpha - a_0\gamma = 0$ and $\beta - a_0\delta \neq 0$. Then the degree of $Q_3(z\eta)$ is 3 in ζ. Geometrically this means that the Möbius transformation

$$z = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}$$

maps the point $\zeta = \infty$ to the point $z = a_0$ and with respect to the ζ coordinate our integral comes from a polynomial of degree 3. The inequality $\beta - a_0\delta \neq 0$ simply follows from the linear independence of (α, γ) and (β, δ) which is a consequence of the fact that we really do have a Möbius transformation.

What is the advantage of using $G_3(z)$ instead of $F_4(z)$? The Riemann surface M which is a double cover with 4 branch points over ± 1 and $\pm\frac{1}{k}$ cannot be represented as the graph of $w^2 = (1 - z^2)(1 - k^2z^2)$ either in $\mathbb{P}_1 \times \mathbb{P}_1$ or in $\mathbb{P}_2$ precisely, because over the point $z = -\infty$ there will be only one point (which is the intersection of two local nonsingular pieces) instead of two distinct points as required by M. Since the point over ∞ for the case of $G_3(z)$ corresponds to a point over one of the four finite roots for the case of $F_4(z)$, we can use the graph of

$$w^2 = G_3(z)$$

in $\mathbb{P}_2$ directly (without further separation of two nonsingular intersecting pieces) as the compact Riemann surface. The form of the integrand

$$\frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}$$

for an elliptic integral is known as the *Weierstrass normal form*, whereas he form of the integrand

$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

for an elliptic integral is known as the *Legendre normal form.* The Legendre normal form comes from the solution of the motion of a simple pendulum and the Weierstrass normal form is more natural when it comes to describing the Riemann surface as a complex plane curve.

The Weierstrass $\wp$-function $\wp(w)$ is defined as the inverse of the indefinite integral

$$z \mapsto w = \int_{\zeta=\infty}^{z} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}},$$

where g_2, g_3 are complex numbers satisfying

$$g_2^3 - 27g_3^2 \neq 0$$

so that the three roots of the cubic polynomial

$$4\zeta^3 - g_2\zeta - g_3$$

in the variable ζ are distinct.

Principal Part of Weierstrass $\wp$ Function at the Origin. The choice of ∞ (which is a branch-point) as the initial point of integration means that 0 is a double pole of the inverse function. The reason is that the initial point of integration is ∞ in z-coordinate which is mapped by the indefinite integral to $w = 0$. The inverse function $w \mapsto z$ is 2-to-1 in a punctured disk of small radius in w coordinate but the point $w = 0$ itself is mapped to the single point $z = \infty$. Moreover, the inverse function $w \mapsto z$ is an even function near $w = 0$ for the following reason. Take a curve C from $z = \infty$ in the Riemann surface M to a nearby point z^* such that C is projected down by π to part of a great circle $\pi(C)$ in the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The inverse image

$\pi^{-1}(\pi(C))$ consists of C and another curve C' in M so that the value of the integrand as a holomorphic 1-form on M at a point of C is the negative of its value at a point of C'. The integral w^* of the integrand along C is the negative $-w^*$ of its integral along C'. This means that the inverse function at w^* and at $-w^*$ has the same value z^*, implying that the inverse function $w \mapsto z$ is even.

The principal part of the even meromorphic function $\wp(w)$ at 0 must be $\frac{C}{w^2}$ for some complex constant C. Being the inverse function of the indefinite integral

$$z \mapsto w = \int_{\zeta=\infty}^{z} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}},$$

by the fundamental theorem of calculus the Weierstrass $\wp$ function satisfies the differential equation

$$\wp'(w)^2 = 4\wp(w)^3 - g_2\wp(w) - g_3.$$

By comparing the principal parts of both sides, we conclude that $\frac{4C^2}{w^6} = \frac{4C^3}{w^6}$ and $C = 1$.

Construction of the Weierstrass function $\wp(w)$ by Explicit Infinite Series. Instead of going through the complicated arguments given above to justify the process of inverting the indefinite integral to obtain a doubly periodic meromorphic function, Weierstrass introduced an alternative way of defining his function $\wp(w)$ by an explicit infinite series. The function $\wp(w)$ is doubly periodic and its principal part at $w = 0$ is $\frac{1}{w^2}$. By using this information, one can define $\wp(w)$ as an explicit infinite series as follows. Most books of complex analysis choose to introduce elliptic functions only from the approach of Weierstrass. Such an approach avoids completely the discussion of the justification of inverting an indefinite integral and is cleaner and much easier to present. Here we choose to start out with the discussion of the Riemann surface constructed as the graph of a doubly-valued holomorphic function on the Riemann sphere and the use of the topology of a torus or the Schwarz-Christoffel transformation for a rectangle, because it is a good opportunity to link different topics in complex and to put together several perspectives. Moreover, by this approach one gets a good understanding of the two simplest kinds of Riemann surface, namely the Riemann sphere and the torus.

Let L be any lattice of points generated by two $\mathbb{R}$-linearly independent complex numbers ω_1 and ω_2 over the integers. We have in our mind the lattice of the periods of the Weierstrass $\wp$ function under consideration. The only principal part of the Weierstrass $\wp$ function $\wp(w)$ in a fundamental parallelogram is $\frac{1}{w^2}$ at the origin. Here a fundamental parallelogram is a parallelogram whose vertices are w_0, $w_0+\omega_1$, $w_0+\omega_2$, $w_0+\omega_1+\omega_2$ with $w_0 \in \mathbb{C}$. The complex number w_0 is usually chosen so that the boundary of a fundamental parallelogram does not contain any pole or any zero of the doubly meromorphic function on $\mathbb{C}$ under consideration. If we simply write down the sum of all the principal parts for all the poles of $\wp(w)$, we would have

$$\sum_{\ell\in L}\frac{1}{(w-\ell)^2}$$

which unfortunately will not converge. So we modify each term $\frac{1}{(w-\ell)^2}$ (except the case of $\ell=0$) by subtract from $\frac{1}{(w-\ell)^2}$ it Taylor polynomial at $w=0$ of degree 0 (which simply is its value at $w=0$) to define

$$\wp(w)=\frac{1}{w^2}+\sum_{\ell\in L-\{0\}}\left(\frac{1}{(w-\ell)^2}-\frac{1}{\ell^2}\right).$$

We have to show that every element of L is a period for

$$\wp(w)=\frac{1}{w^2}+\sum_{\ell\in L-\{0\}}\left(\frac{1}{(w-\ell)^2}-\frac{1}{\ell^2}\right),$$

which is not obvious because the term $\frac{1}{w^2}$ is treated differently from the other terms

$$\frac{1}{(w-\ell)^2}-\frac{1}{\ell^2}$$

with $\ell\neq 0$. The difference is from $\frac{1}{\ell^2}$. By differentiating both sides, we get rid of $\frac{1}{\ell^2}$ to get

$$\wp'(w)=\sum_{\ell\in L-0}\frac{-2}{(w-\ell)^3}$$

which clearly has ω_1 and ω_2 as periods. By integrating $\wp'(w+\omega_1)-\wp'(w)=0$ with respect to w, we get $\wp(w+\omega_1)-\wp(w)=$ constant. Plucking in the value $w=-\frac{1}{2}\omega_1$ and using the fact that $\wp(w)$ is an even function, we conclude that the constant is zero and ω_1 is a period for $\wp(w)$. Likewise ω_2 is a period for $\wp(w)$. This technique of using the parity of a function (*i.e.,* even or odd function) and the evaluation at the half-period to determine the constant of integration is very useful and we will see it used again later in the context of the Weierstrass σ function and the Weierstrass ζ function.

Introduce the notation

$$s_n = \sum_{\ell \in L - 0} \frac{1}{\ell^n}$$

for $n \geq 3$. Note that s_n is zero when n is odd. In order to show that $\wp(w)$ is the inverse of the indefinite elliptic integral

$$w = \int_{\zeta=\infty}^{z} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}},$$

we are going to prove that

$$\wp'(w)^2 = 4\wp(w)^3 - g_2\wp(w) - g_3,$$

where

$$g_2 = 60 s_4 = 60 \sum_{\ell \in L - 0} \frac{1}{\ell^4}\ , \ g_3 = 140 s_6 = 140 \sum_{\ell \in L - 0} \frac{1}{\ell^6}.$$

For the verification, all we have to do is to show that the principal parts of

$$\wp'(w)^2 - \left(4\wp(w)^3 - g_2\wp(w) - g_3\right)$$

at $w = 0$ cancel out. From

$$\frac{1}{(w-\ell)^2} = \frac{1}{\ell^2\left(1 - \frac{w}{\ell}\right)^2} = \frac{1}{\ell^2} + 2\frac{w}{\ell^3} + 3\frac{w^2}{\ell^4} + \cdots \text{ for } |w| < |\ell|,$$

it follows that

$$\begin{aligned}
\wp(w) &= w^{-2} + 3s_4w^2 + 5s_6w^4 + \cdots \\
\wp'(w) &= -2w^{-3} + 6s_4w + 20s_6w^3 + \cdots \\
\wp'(w)^2 &= 4w^{-6} - 24s_4w^{-2} - 80s_6 + \cdots \\
\wp(w)^3 &= w^{-6} + 9s_4w^{-2} + 15s_6 + \cdots \\
\wp'(w)^2 - 4\wp(w)^3 + 60s_4\wp(w) &= -140s_6 + \cdots .
\end{aligned}$$

The left-hand side of the last equation is an elliptic function without any poles and therefore must be constant. Thus we have the differential equation

$$\wp'(w)^2 = 4\wp(w)^3 - g_2\wp(w) - g_3,$$

where

$$g_2 = 60 s_4 = 60 \sum_{\ell\in L-0} \frac{1}{\ell^4}\ ,\ g_3 = 140 s_6 = 140 \sum_{\ell\in L-0} \frac{1}{\ell^6}.$$

General Elliptic Functions as Doubly Periodic Meromorphic Functions and Their Three Properties on Zero-Sets and Pole-Sets. The Jacobian elliptic functions $\operatorname{sn} w$, $\operatorname{cn} w$, $\operatorname{dn} w$ and the Weierstrass function $\wp(w)$ are all doubly periodic meromorphic functions on $\mathbb{C}$. In general we can define an *elliptic function* as a doubly periodic meromorphic function on $\mathbb{C}$. Equivalently, an elliptic function is a meromorphic function on a torus which is $\mathbb{C}$ modulo a lattice of rank 2. Let ω_1, ω_2 be two primitive periods in the lattice of all periods. We now verify the following three fundamental properties of an elliptic function (under the assumption that the function has no pole and no zero on the boundary of the parallelogram):

(i) The sum of the residues of the function inside a fundamental parallelogram is zero.

(ii) The number of zeroes of the function equals the number of poles inside a fundamental parallelogram.

(iii) Inside a fundamental parallelogram the sum of the coordinates of the zeroes equals the sum of the coordinates of the poles modulo a period.

To prove (i) we integrate $f(w)dw$ along the boundary of the fundamental parallelogram. By the residue theorem the integral is simply $2\pi i$ times the sum of the residues of f inside the parallelogram. On the other hand the integral is zero, because by the periodicity of the integral over $[a, a+\omega_1]$ equals the integral over $[a+\omega_2, a+\omega_1+\omega_2]$ and the integral over $[a, a+\omega_2]$ equals the integral over $[a+\omega_1, a+\omega_1+\omega_2]$. Property (ii) follows from integrating

$$\frac{1}{2\pi i}\frac{f'(w)}{f(w)}dw$$

over the boundary of the fundamental parallelogram and from the argument principle. The proof of Property (iii) is slightly more complicated. One integrates

$$\frac{1}{2\pi i}\frac{wf(w)}{f(w)}dw$$

over the boundary of the fundamental parallelogram, but in this case the integral may not be zero, because

$$\frac{1}{2\pi i}\int_{[a+\omega_1,a+\omega_1+\omega_2]}\frac{wf'(w)}{f(w)}dw-\frac{1}{2\pi i}\int_{[a,a+\omega_2]}\frac{wf'(w)}{f(w)}dw=\omega_1\frac{1}{2\pi i}\int_{[a,a+\omega_2]}\frac{f'(w)}{f(w)}dw.$$

However,

$$\frac{1}{2\pi i}\int_{[a,a+\omega_2]}\frac{f'(w)}{f(w)}dw$$

equals $\frac{1}{2\pi i}$ times the difference of the value of $\log f(w)$ at $a+\omega_2$ and at a when w runs along $[a, a+\omega_2]$. Since $f(w)$ has the same value at a as at $a+\omega_2$, the difference of the value of $\log f(w)$ at $a+\omega_2$ and at a when z runs along $[a, a+\omega_2]$ must be $2\pi i$ times an integer. Therefore

$$\frac{1}{2\pi i}\int_{[a,a+\omega_2]}\frac{f'(w)}{f(w)}dw$$

is an integer and

$$\frac{1}{2\pi i}\int_{[a+\omega_1,a+\omega_1+\omega_2]}\frac{wf'(w)}{f(w)}dw-\frac{1}{2\pi i}\int_{[a,a+\omega_2]}\frac{wf'(w)}{f(w)}dw$$

is a period of f. Likewise

$$\frac{1}{2\pi i}\int_{[a,a+\omega_1]}\frac{wf'(w)}{f(w)}dw-\frac{1}{2\pi i}\int_{[a+\omega_2,a+\omega_1+\omega_2]}\frac{wf'(w)}{f(w)}dw$$

is also a period of f. This concludes the proof of (iii).

The questions arise whether Condition (i) is also sufficient for a collection of principal parts on $\mathbb{C}/(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)$ to be achieved by a meromorphic function on $\mathbb{C}/(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)$ and whether Conditions (ii) and (iii) are also sufficient for two sets of points in $\mathbb{C}/(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)$ to be the zero-set and pole-set of a a meromorphic function on $\mathbb{C}/(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)$. The answers to both questions are in the affirmative, but we will not discuss them at this point. Their analogues in the setting of a general compact Riemann surface (instead of $\mathbb{C}/(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)$) are respectively the theorem of Riemann-Roch and the theorem of Abel for compact Riemann surfaces.

One problem in the last homework assignment is to prove the addition theorem

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v + \mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1 - k^2\mathrm{sn}^2 u\,\mathrm{sn}^2 v},$$

which is reduced to the addition theorem for the trigonometric sine function

$$\sin(u+v) = \sin u\,\cos v + \cos u\,\sin v$$

when the modulus k becomes 0. The proof given in the hint for the problem involves some rather ingenious intermediate identities obtained by applying the rules of differentiation for products and quotients of functions. In contrast, the addition theorem for the Weierstrass $\wp$ function can be derived rather elegantly from the intersection of a cubic equation with a line, with the use of the three fundamental properties for general elliptic functions given above, when $w \mapsto (\wp(w),\, \wp'(w))$ is considered as a parametrization of a variable point (x, y) in the curve satisfying the cubic equation $y^2 = 4x^3 - g_2 x - g_3$ in the same way that $w \mapsto (\cos w,\, -\sin w) = (\cos w, \cos' w)$ is considered as parametrization of a variable point (x, y) in the circle $x^2 + y^2 = 1$.

Addition Theorem for Weierstrass $\wp$ Function and Proof by Intersection of Cubic Equation with a Line. Let $x = \wp(w)$ and $y = \wp'(w)$ and, for some complex numbers $a \neq 0$ and b to be determined later, we consider the doubly periodic function $y + ax + b$. This doubly periodic function has a pole of order 3 at the origin and no other poles inside a fundamental parallelogram. So the sum of its three zeroes must be zero modulo a period. We are free to choose a and b. We can choose a and b so that the doubly periodic function $y + ax + b$ vanishes at w_1 and w_2. Then $y + ax + b$ must also vanish at $-(w_1 + w_2)$. On the other hand we have the equation

$$y^2 = 4x^3 - g_2 x - g_3.$$

So by solving the two equation

$$y + ax + b = 0$$

and

$$y^2 = 4x^3 - g_2 x - g_3,$$

we would get the values of x and y at $-(w_1 + w_2)$. Since one equation is a linear equation and the second one is a cubic equation, we expect to get

3 solutions for (x, y). The other two solutions are the values of (x, y) at w_1 and w_2. Knowing these two solutions makes getting the third solution very easy, because one can use the fact that for a cubic equation with unit leading coefficient the sum of the three roots is the negative of the second coefficient. We now use the method explained above to get our addition theorem. From

$$\begin{aligned} \wp'(w_1) + a\wp(w_1) + b = 0 \\ \wp'(w_2) + a\wp(w_2) + b = 0 \end{aligned}$$

we get

$$a = -\frac{\wp'(w_1) - \wp'(w_2)}{\wp(w_1) - \wp(w_2)}.$$

(As we see later we do not need to solve for b.) From the equation

$$(ax + b)^2 = 4x^3 - g_2 x - g_3$$

we obtain, on account of the sum of the three roots of a monic cubic equation being equal to the negative of the coefficient of the second-degree term and the evenness of $\wp(w)$),

$$\wp(w_1) + \wp(w_2) + \wp(w_1 + w_2) = \frac{a^2}{4} = \frac{1}{4}\left(\frac{\wp'(w_1) - \wp'(w_2)}{\wp(w_1) - \wp(w_2)}\right)^2.$$

Thus we have the addition formula

$$\wp(w_1 + w_2) = -\wp(w_1) - \wp(w_2) + \frac{1}{4}\left(\frac{\wp'(w_1) - \wp'(w_2)}{\wp(w_1) - \wp(w_2)}\right)^2.$$

Any Elliptic Function Expressible as Rational Function of Weierstrass $\wp$ Function and its Derivative. We would like to remark that any doubly periodic function $f(w)$ can be expressed as a rational function of $\wp(w)$ and $\wp'(w)$. First consider the case when $f(w)$ is even. By replacing $f(w)$ by $f(w)\wp(w)^p$ for a suitable integer p we can assume without loss of generality that the origin is neither a zero nor a pole of $f(w)$. The zero-set of $f(w)$ is $\{a_1, \cdots, a_k, -a_1, \cdots, -a_k\}$ together with all congruent points and its pole-set is $\{b_1, \cdots, b_k, -b_1, \cdots, -b_k\}$ together with all congruent points. When none of $a_1, \cdots, a_k, b_1, \cdots, b_k$ is zero, the function $f(w)$ has the same zeroes and poles as the function

$$\frac{[\wp(w) - \wp(a_1)][\wp(w) - \wp(a_2)] \cdots [\wp(w) - \wp(a_k)]}{[\wp(w) - \wp(b_1)][\wp(w) - \wp(b_2)] \cdots [\wp(w) - \wp(b_k)]}$$

and the two can only differ by a constant factor. When one of $a_1, \cdots, a_k$ is zero, *e.g.,* $a_1 = 0$, the above expression has to be replaced by

$$\frac{1}{\wp(w)} \frac{[\wp(w)-\wp(a_2)][\wp(w)-\wp(a_3)]\cdots[\wp(w)-\wp(a_k)]}{[\wp(w)-\wp(b_1)][\wp(w)-\wp(b_2)]\cdots[\wp(w)-\wp(b_k)]}.$$

When one of $b_1, \cdots, b_k$ is zero, *e.g.,* $b_1 = 0$, the above expression has to be replaced by

$$\wp(w) \frac{[\wp(w)-\wp(a_1)][\wp(w)-\wp(a_2)]\cdots[\wp(w)-\wp(a_k)]}{[\wp(w)-\wp(b_2)][\wp(w)-\wp(b_3)]\cdots[\wp(w)-\wp(b_k)]}.$$

So in this case $f(w)$ is a rational function of $\wp(w)$. For the general case we can write $f(w)$ as the sum of an odd function $f_1(w) = \frac{1}{2}(f(w) - f(-w))$ and an even function $f_1(w) = \frac{1}{2}(f(w) - f(-w))$. Since the doubly periodic function $\frac{f_1(w)}{\wp'(w)}$ is even, we conclude that $f(w)$ is a rational function of $\wp(w)$ and $\wp'(w)$.

Factorization of Elliptic Function Analogous to Factorization of Rational Function, as the Motivation to Introduce Theta Functions. A rational function $g(w)$ is a meromorphic function on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and can be factored by the fundamental theorem of algebra into

$$g(w) = C \frac{(w-a_1)(w-a_2)\cdots(w-a_k)}{(w-b_1)(w-b_2)\cdots(w-b_\ell)}$$

for $a_1, \cdots, a_k, b_1, \cdots, b_\ell \in \mathbb{C}$, where C is a nonzero complex constant. Each of the factors $w - a_j$, $w - b_j$ is a meromorphic function on the Riemann sphere, with a single zero and is the translate of the coordinate function w.

We ask whether there is an analogous factorization for elliptic functions $f(w)$ which are meromorphic functions on a torus $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$. From the fundamental properties of elliptic functions we know that there exists no elliptic function with a single zero in a fundamental parallelogram, because it means that there is precisely one single pole in the fundamental parallelogram and the sum of the residues for points in the fundamental parallelogram cannot be zero. So we look for an appropriate substitute for the coordinate function w in the setting of elliptic functions. The logarithmic derivative of w is $\frac{1}{w}$, whose derivative is $-\frac{1}{w^2}$. The principal part of the Weierstrass $\wp(w)$ function at $w = 0$ is $\frac{1}{w^2}$. With this motivation, we define the Weierstrass σ

function $\sigma(w)$ by integrating $-\wp$ twice and exponentiating (with appropriate constants of integration in each term and factor to guarantee convergence of the final infinite product). More precisely, let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $L^* = L - \{0\}$. Then

$$\sigma(w) = w \prod_{\ell \in L^*} \left(1 - \frac{w}{\ell}\right) e^{\frac{w}{\ell} + \frac{1}{2}\left(\frac{w}{\ell}\right)^2}$$

so that $\wp(w) = -(\log \sigma)''(w)$ or equivalently when we define the Weierstrass ζ function $\zeta(w)$ by

$$\zeta(w) = \frac{\sigma'(w)}{\sigma(w)} = \frac{1}{w} + \sum_{\ell \in L^*} \left(\frac{1}{w-\ell} + \frac{1}{\ell} + \frac{w}{\ell^2}\right)$$

as the logarithmic derivative of σ, we have $\wp(w)$ as the negative of the derivative of $\zeta(w)$.

The Weierstrass σ function $\sigma(w)$ is entire and has a single zero in a fundamental parallelogram and can be used in the factorization of elliptic functions as the analogue of the coordinate function w for the factorization of rational functions. If $a_1, \cdots, a_k$ are the zeroes and $b_1, \cdots, b_k$ are the poles (with multiplicities counted) of an elliptic function f in a fundamental parallelogram, we seek to factor f into

$$f(w) = C\,\frac{\sigma(w - (a_1 + 2\pi i\ell))\sigma(w - a_2)\cdots\sigma(w - a_k)}{\sigma(w - b_1)\sigma(w - b_2)\cdots\sigma(w - b_k)}$$

for some nonzero complex constant C, where $ell \in L$ such that

$$(a_1 + 2\pi i\ell) + a_2 + a_3 + \cdots + a_k = b_1 + b_2 + \cdots + b_k.$$

Note that the condition

$$(a_1 + 2\pi i\ell) + a_2 + a_3 + \cdots + a_k = b_1 + b_2 + \cdots + b_k$$

is needed to apply Property (iii) of the three fundamental properties of elliptic functions to conclude that

$$\frac{\sigma(w - (a_1 + 2\pi i\ell))\sigma(w - a_2)\cdots\sigma(w - a_k)}{\sigma(w - b_1)\sigma(w - b_2)\cdots\sigma(w - b_k)}$$

is periodic with respect to the period lattice L, from the use of the "periodicity factor" of σ as explained below.

The Weierstrass σ function and the Weierstrass ζ function are not doubly periodic functions, but we know precisely the effect of translating them by an element of ℓ of L by the following argument which is the same argument we used to conclude the double periodicity of $\wp(w)$ from the double periodicity of $\wp'(w)$ by evaluating at $-\frac{\omega_1}{2}$ and $-\frac{\omega_2}{2}$ and using the even function property of $\wp(w)$. Both functions $\zeta(w)$ and $\sigma(w)$ are odd functions, as can be easily verified by changing w to $-w$ and at the same time ℓ to $-\ell$ in their definitions. For fixed $\ell \in L$, there exists some constant η_ℓ such that

$$\zeta(w+\ell) - \zeta(w) \equiv \eta_\ell.$$

By evaluating at $w = -\frac{\omega_1}{2}$ and $-\frac{\omega_2}{2}$, we obtain

$$\eta_{\omega_1} = 2\zeta\left(\frac{\omega_1}{2}\right) \quad \text{and} \quad \eta_{\omega_2} = 2\zeta\left(\frac{\omega_2}{2}\right).$$

Likewise, by arguing with the odd function property of $\sigma(w)$ and evaluation at $-\frac{\omega_1}{2}$ and $-\frac{\omega_2}{2}$, one can conclude that

$$\sigma(w+\omega_1) = -\, e^{\eta_{\omega_1}\left(w+\frac{\omega_1}{2}\right)}\,\sigma(w) \quad \text{and} \quad \sigma(w+\omega_2) = -\, e^{\eta_{\omega_2}\left(w+\frac{\omega_2}{2}\right)}\,\sigma(w).$$

For translation by an element of the period matrix L, there is a *periodicity factor* which is the exponential of a polynomial of degree ≤ 1. This motivates the following definition of a theta function.

Definition of Theta Function. Let ω_1, ω_2 be two complex numbers which are linearly independent over $\mathbb{R}$ and let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. A theta function $F(w)$, with respect the lattice L, is an entire function such that for any $\ell \in L$ there exist some complex constants a_ℓ and b_ℓ for which the equation

$$F(w+\ell) = e^{a_\ell w + b_\ell} F(w)$$

holds for all $w \in \mathbb{C}$.

Trivial Theta Functions. A theta function without zero is the exponential of a quadratic polynomial. In one direction, it is clear that the exponential of a quadratic polynomial admits a periodicity factor equal to the exponential of a polynomial of degree ≤ 1. On the other hand, one can use Liouville's theorem to show that the second derivative of the logarithm of a nowhere zero theta function is constant. A nowhere zero theta function is called a *trivial theta function*.

By Property (ii) and Property (iii) of the three fundamental properties of elliptic functions, we have the following result on the factorization of any elliptic function as the quotient of the products of translates of a theta function which has a single zero in a fundamental parallelogram, by arguing that the quotient is an elliptic function with the same zeroes and poles as the given elliptic function.

Theorem (Factorization of Elliptic Function as Quotient of Products of Translates of Theta Function). Let L be a lattice in $\mathbb{C}$ of rank 2. Let $F(w)$ be an elliptic function with respect to L whose zeroes are $a_1, \cdots, a_k$ (modulo L) and whose poles are $b_1, \cdots, b_k$ (modulo L), with multiplicities counted. Let $\theta(w)$ be a theta function, with respect to L, which has a single zero in a fundamental parallelogram of L. Assume that $a_1, \cdots, a_k$ and $b_1, \cdots, b_k$ are chosen such that $\sum_{j=1}^{\ell} a_j = \sum_{j=1}^{\ell} b_j$. Then there exists a complex constant C such that

$$F(w) = C\,\frac{\theta(w-a_1)\theta(w-a_2)\cdots\theta(w-a_k)}{\theta(w-b_1)\theta(w-b_2)\cdots\theta(w-b_k)}$$

for $w \in \mathbb{C}$.

Example. Here is an example of the application of the theorem on factorization of elliptic functions in terms of theta functions. The identity

$$\wp(z) - \wp(u) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma^2(u)\sigma^2(v)}$$

holds for $u, v \in \mathbb{C}$, when one compares the zeroes and the poles in z of both sides for fixed u and considersthe principal parts of both sides at $z = 0$. The details of the verification of this example and similar statements are left to a homework problem.

Theta Functions in the Setting of Jacobian Elliptic Functions. The above discussion of theta functions is from the approach of Weiertrass in the context the Weierstrass $\wp$ function and its indefinite integral. We now discuss, in the context of Jacobian elliptic functions, theta functions as entire functions admitting periodicity factors which are the exponent of polynomials of degree ≤ 1 and having single zero in a fundamental parallelogram.

First of all, the approach of Jacobi, by multiplying a theta function by an appropriate trivial theta function, one can get to the situation that one of the two periodicity factor is identically 1. The lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ can be normalized to $L = \mathbb{Z}\pi + \mathbb{Z}\pi\tau$ with $\operatorname{Im}\tau > 0$ by a change of coordinate in $\mathbb{C}$ and a change of basis in L.

In Jacobi's approach, one seeks to write down the simplest and most natural theta function for the lattice $L = \mathbb{Z}\pi + \mathbb{Z}\pi\tau$ so that π is a period (*i.e.,* the periodicity factor is 1). Since π is a period of $f(w)$, we can write f as a function of e^{2iw}. So one way to write down a Jacobian theta function is express it as a series in e^{2iw} and determine the coefficients so that we get a linear function of w in the exponent of the only periodicity factor. Let us try to use the method of undetermined coefficients to get explicitly one such function and we try to get one of the simplest possible. We write

$$f(w) = \sum_{n=-\infty}^{\infty} c_n e^{2niw}$$

and use the periodicity factor equation

$$f(w + \pi\tau) = e^{-2iw+a} f(w)$$

(for some complex constant a) to solve for the undetermined coefficients c_n. The choice of $-2iw$ in the exponent of the periodicity factor it to enable us to write down easily recurrent equations for the coefficients c_n, because both sides of the equation are series in e^{2iw}. It is clear that we should try $-e^{-2iw-\pi\tau i}$ as the periodicity factor. In terms of the coefficients c_n the periodicity factor equation now becomes

$$\sum_{n=-\infty}^{\infty} c_n e^{2ni(w+\pi\tau)} = -e^{-2iw-\pi\tau i} \sum_{n=-\infty}^{\infty} c_n e^{2niw}$$

$$= \sum_{n=-\infty}^{\infty} -e^{-\pi\tau i} c_n e^{2i(n-1)w} = \sum_{n=-\infty}^{\infty} -e^{-\pi\tau i} c_{n+1} e^{2niw}.$$

Thus we have the recurrent formula $c_{n+1} = -e^{(2n+1)\pi\tau i} c_n$. We choose the simplest initial condition $c_0 = 1$. Let $q = e^{\pi\tau i}$. Since the sum of $1, 3, \cdots, 2n-$

1 is n^2, it follows that $c_n = (-1)^n q^{n^2}$. This way we arrive this way at the function

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niw}$$

and we denote it by $\vartheta(w)$ (also by $\vartheta(w|\tau)$ to emphasize its dependence on τ). It is the simplest Jacobi theta function. The periodicity factor equation is

$$\vartheta(w + \pi\tau) = -e^{-2iw - i\pi\tau}\vartheta(w).$$

We still have to answer the equation whether there is only one zero for our new theta function $\vartheta(w)$ in a fundamental parallelogram of $L = \mathbb{Z}\pi + \mathbb{Z}\pi\tau$.

For a general elliptic function f for the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with periodicity factors

$$f(w + \omega_\nu) = e^{\eta_\nu w + b_\nu} f(w)$$

for $\nu = 1, 2$, we can use the argument principle to find out the number of zeroes k of f in a parallelogram by integrating $d \log f$ over the boundary of a fundamental parallelogram and the answer is as follows.

$$\begin{aligned} 2\pi i k &= \int_{w_0}^{w_0+\omega_2} (d \log f(w + \omega_1) - d \log f(w)) dw \\ &\quad - \int_{w_0}^{w_0+\omega_1} (d \log f(w + \omega_2) - d \log f(w)) dw \\ &= \omega_2\eta_1 - \omega_1\eta_2. \end{aligned}$$

This formula is also known as *Legendre's formula* for computing the number of zeroes of a theta function inside a fundamental parallelogram.

For our case with $f = \vartheta$, from

$$\eta_1 = 0, \quad \omega_1 = \pi, \quad \eta_2 = -2i, \quad \text{and} \quad \omega_2 = \pi\tau$$

we get $\omega_2\eta_1 - \omega_1\eta_2 = 2\pi i$ and $k = 1$. Having found out that our theta function ϑ has one single zero in the fundamental parallelogram for $L = \mathbb{Z}\pi + \mathbb{Z}\pi\tau$, we still have to locate the single zero. We can rewrite

$$\vartheta(w + \pi\tau) = -e^{-2iw - i\pi\tau}\vartheta(w)$$

as

$$\vartheta_4(w) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nw$$

by grouping together the term with positive index n and the term with index $-n$. If the cosine function there becomes the sine function, we can more easily locate its zero. The sine function differs from the cosine function by translation by $\frac{\pi}{2}$. So we translate ϑ by the half-periods $\frac{\pi}{2}$, $\frac{\pi\tau}{2}$, $\frac{\pi+\pi\tau}{2}$ to get three other Jacobian theta functions. After we multiply the first one of the four Jacobian theta functions by a trivial theta function to make its series expansion more consistent with the other three, we end up with the following list of the four Jacobian theta functions.

$$\vartheta_1(w) = -i\,\exp\left(iw + \frac{1}{4}\pi i\tau\right)\vartheta\left(w + \frac{1}{2}\pi\tau\right) = 2\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\sin(2n+1)w,$$

$$\vartheta_2(w) = \vartheta_1\left(w + \frac{\pi}{2}\right) = 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}\cos(2n+1)w,$$

$$\vartheta_3(w) = \vartheta\left(w + \frac{\pi}{2}\right) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos\ 2nw,$$

$$\vartheta_4(w) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nw.$$

The last one $\vartheta_4(w)$ is actually our simplest, most natural theta function $\vartheta(w)$. The ordering is from the location of their zeroes, respectively 0, $\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi+\pi\tau}{2}$, enumerated in the counterclockwise sense.

The periodicity factors of the four Jacobian theta functions are given in the following table

	$\vartheta_1(w)$	$\vartheta_2(w)$	$\vartheta_3(w)$	$\vartheta_4(w)$
π	-1	-1	1	1
$\pi\tau$	$-N$	N	N	$-N$

Here

$$N = \frac{1}{q}\,e^{-2iw} = e^{-2iw-\pi i\tau}.$$

Note that the periodicity factors of π for $\vartheta_1(w)$ and $\vartheta_2(w)$ are not 1. They are -1. We could easily have made them 1, but we want to stick with the historical notations.

For lack of time, our discussion of elliptic functions and theta functions has to stop at this point. However, to complete the picture in the Jacobian approach, we would like to say a little bit more without justifying by detailed proofs.

Analogous to (and in the limiting case reducible to) the trigonometric identity

$$\cos^2\theta + \sin^2\theta = 1,$$

we have the following quadratic algebraic relations among the four Jacobian theta functions

$$\vartheta_2^2(w)\vartheta_4^2(0) - \vartheta_4^2(w)\vartheta_2^2(0) = -\vartheta_1^2(w)\vartheta_3^2(0),$$

$$\vartheta_3^2(w)\vartheta_4^2(0) - \vartheta_4^2(w)\vartheta_3^2(0) = -\vartheta_1^2(w)\vartheta_2^2(0),$$

$$\vartheta_1^2(w)\vartheta_4^2(0) - \vartheta_3^2(w)\vartheta_2^2(0) = -\vartheta_2^2(w)\vartheta_3^2(0),$$

$$\vartheta_4^2(w)\vartheta_4^2(0) - \vartheta_3^2(w)\vartheta_3^2(0) = -\vartheta_2^2(w)\vartheta_2^2(0).$$

The values $\vartheta_2(0)$, $\vartheta_3(0)$, $\vartheta_4(0)$ (and also $\vartheta_1'(0)$ oftentimes) are known as Jacobian *theta constants*. The above quadratic relations are proved by comparing zero-sets and pole-sets and evaluation at one point. That is the reason why the Jacobian theta functions appear in them.

There is the question of how the elliptic sine and cosine functions are represented as quotients of products of translates of the Jacobian theta functions. Such representations can be obtained by first-order differential equations involving the Jacobian theta functions by comparing pole-sets and zero-sets and evaluation at one point. For the evaluation of the derivatives of Jacobian theta functions the following deep identity

$$\vartheta_1'(0) = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0)$$

among the Jacobian theta constants is needed. This identity corresponds to $\sin' = 1$. Let

$$k = \frac{\vartheta_2^2(0)}{\vartheta_3^2(0)}.$$

The Jacobian elliptic functions are expressed in terms of the Jacobian theta functions as follows.

$$\begin{aligned}
\operatorname{sn}(w,k) &= \frac{\vartheta_3(0)}{\vartheta_2(0)} \frac{\vartheta_1\left(\frac{w}{\vartheta_3^2(0)}\right)}{\vartheta_4\left(\frac{w}{\vartheta_3^2(0)}\right)}, \\
\operatorname{cn}(w,k) &= \frac{\vartheta_4(0)}{\vartheta_2(0)} \frac{\vartheta_2\left(\frac{w}{\vartheta_3^2(0)}\right)}{\vartheta_4\left(\frac{w}{\vartheta_3^2(0)}\right)}, \\
\operatorname{dn}(w,k) &= \frac{\vartheta_4(0)}{\vartheta_3(0)} \frac{\vartheta_3\left(\frac{w}{\vartheta_3^2(0)}\right)}{\vartheta_4\left(\frac{w}{\vartheta_3^2(0)}\right)}.
\end{aligned}$$