

Schwarz-Christoffel Transformations

Schwarz-Christoffel transformations are used to map the open upper half-plane $\mathbb{H} = \{x + iy \mid y > 0\}$ to a given polygon which may be bounded or unbounded. An unbounded polygon simply means a domain in \mathbb{C} defined by a finite number of line-segments and rays. Let us start out with an n -sided polygon Ω with vertices w_1, \dots, w_{n-1} . We do not put down the n -th vertex w_n to allow the possibility that w_n is ∞ , in which case the n -sided polygon Ω is unbounded. We would like to study the problem by constructing a holomorphic map $w = f(z)$ from \mathbb{H} to Ω so that $n - 1$ real points $x_1 < x_2 < \dots < x_{n-1}$ are mapped respectively to the prescribed $n - 1$ vertices w_1, w_2, \dots, w_{n-1} of the n -sided polygon Ω . The important requirement is that the holomorphic map $w = f(z)$ should map the straight line segment $[x_j, x_{j+1}] \subset \mathbb{R}$ to the line-segment in \mathbb{C} joining w_j to w_{j+1} . Let us recall how a holomorphic map transforms the direction of the tangent to a curve in \mathbb{C} .

We now recall the mapping behavior of the holomorphic map $w = f(z)$ at a point z_0 where its derivative $f'(z_0)$ is nonzero. Take a smooth curve $t \mapsto z(t)$ passing through the point z_0 so that $z(0) = z_0$. The image of the curve $t \mapsto w(t) := f(z(t))$ passes through the point $w_0 := f(z_0)$. By the chain rule and the Cauchy-Riemann equation we have

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= \frac{\partial f}{\partial x} \frac{dx}{dt} + i \frac{\partial f}{\partial x} \frac{dy}{dt} \\ &= \frac{\partial f}{\partial x} \left(\frac{dx}{dt} + i \frac{dy}{dt} \right) \\ &= \frac{\partial f}{\partial x} \frac{dz}{dt} = f'(z) \frac{dz}{dt}. \end{aligned}$$

This means that the angle made between the tangent to the curve $t \mapsto z(t)$ at z_0 and the real axis in the z variable is equal to the angle made between the tangent to the curve $t \mapsto w(t)$ at w_0 and the real axis in the w variable plus the angle of the polar representation of $f'(z_0)$. So, if the angle of $f'(z)$ (which we will from now on call the argument of $f'(z)$ and denote by $\arg f'(z)$) stays unchanged as z varies along $[x_j, x_{j+1}]$ from x_j to x_{j+1} , then the line $[x_j, x_{j+1}]$ will be mapped to part of a straight line, because the curve

(x_j, x_{j+1}) is mapped to its image curve C in \mathbb{C} so that the tangent direction at every point of C is the same as the tangent direction of (x_j, x_{j+1}) plus the constant value of $\arg f'(z)$ for $z \in (x_j, x_{j+1})$.

When z goes along \mathbb{R} from the left of x_j to the right of x_j , we want the image to turn an angle of magnitude θ_j when it follows the sides of the n -sided polygon going through the vertex w_j in the counterclockwise sense. To achieve this requirement, we would have to require that the constant value of $\arg f'(z)$ at z on \mathbb{R} to the left of x_j should increase by the amount θ_j to get to the constant value of $\arg f'(z)$ at z on \mathbb{R} to the right of x_j .

We now look for a holomorphic function $g(z)$ to serve as $f'(z)$ which would have this property of its argument being constant along \mathbb{R} to the left of x_j and then increase by the amount θ_j to a constant value along \mathbb{R} to the right of x_j . When $\theta_j = -\pi$, one simple function to fit the requirement is $g(z) = z - x_j$ which satisfies the property that $\arg g(z) = \arg(z - x_j) \equiv \pi$ for $z \in \mathbb{R}$ and $z < x_j$ and $\arg g(z) = \arg(z - x_j) \equiv 0$ for $z \in \mathbb{R}$ and $z > x_j$. To assign a numerical value to the angle $\arg g(z) = \arg(z - x_j)$ would require a choice of the range of the values, which is the same as choosing one branch of the function $\log(z - x_j)$ and then taking its imaginary part $\arg(z - x_j)$. When we say that $\arg g(z) = \arg(z - x_j) \equiv \pi$ for $z \in \mathbb{R}$ and $z < x_j$ and $\arg g(z) = \arg(z - x_j) \equiv 0$ for $z \in \mathbb{R}$ and $z > x_j$, we make the choice of choosing the cut $\{x = x_j, y \leq 0\}$ to define a branch of $\log(z - x_j)$. That particular cut is chosen to define $\log(z - x_j)$, because we would like the function $\log(z - x_j)$ to be defined on the upper half-plane \mathbb{H} .

We now consider the general value $k_j\pi$ for the amount θ_j of jump instead of the special value $\theta_j = -\pi$. This we can do by multiplying $\arg(z - x_j)$ by the constant $-k_j$, which is the same as using $(z - x_j)^{-k_j}$ instead of $z - x_j$. This takes care of one single jump of $\arg g(z)$ by the amount $k_j\pi$ at the point $z = x_j$. How about all the other jumps at all the other real points among x_1, x_2, \dots, x_{n-1} ? An easy solution is to add the individual jumps together. When the jump of $\arg g(z)$ is $k_j\pi$ at $z = x_j$, we should use just take

$$\arg g(z) = \sum_{j=1}^{n-1} (-k_j) \arg(z - x_j).$$

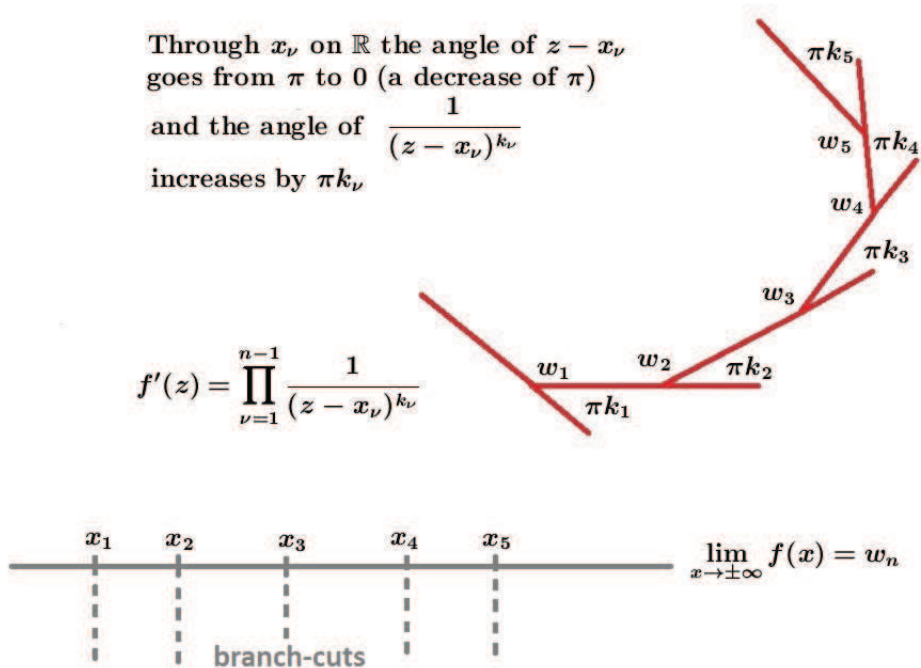
In other words, we should just choose

$$g(z) = \prod_{j=1}^{n-1} (z - x_j)^{-k_j}.$$

Recall that $g(z)$ is actually a stand-in for $f'(z)$. So we should use

$$f'(z) = \prod_{j=1}^{n-1} (z - x_j)^{-k_j}$$

in order for the image $w = f(z)$ to achieve the correct turning of corners along the n -sided polygon Ω at the vertices w_1, \dots, w_{n-1} in the counterclockwise sense.



We should do more than just achieving the correct turning of corners along the n -sided polygon. We should make sure that the ray $(-\infty, x_1]$ is mapped to the side of Ω just before w_1 in the counterclockwise sense. In order to get the initial matching of $(-\infty, x_1]$ with the side of Ω just before w_1 , we can multiply $f'(z)$ by a nonzero complex constant A , resulting in the entire image of $f(z)$ being rotated by the argument $\arg A$ of A . So far as matching the orientation is concerned, the absolute value $|A|$ of the complex number A plays no rôle. The absolute value $|A|$ of the complex number A , however, plays the rôle of magnification. Since only the derivative $f'(z)$ is

specified, to get back to the holomorphic function $f(z)$ there is the question of the constant of integration. If we use a definite integral with x_1 as the lower limit, we should use w_1 as the constant of integration, because we would like x_1 to be mapped to w_1 . Thus we have the following formula for the Schwarz-Christoffel transformation

$$f(z) = A \int_{z_0}^z \prod_{j=1}^{n-1} (\zeta - x_j)^{-k_j} d\zeta + B,$$

where, for example, z_0 and B can be chosen respectively to be x_1 and w_1 .

So far we have been talking about fitting the angles and one initial direction and one initial point. How about fitting all the vertices so that the n vertices $w_1, w_2, \dots, w_{n-1}, w_n$ (with w_n possibly equal to ∞) are precisely the images of $x_1, x_2, \dots, x_{n-1}, \infty$ under $w = f(z)$ respectively? Of course, once all the angles fit together with one initial direction and one initial point, the only remaining problem is the lengths of the sides. There are precisely n real numbers (including the possibility of ∞) for the lengths of the sides of the n -sided polygon Ω . On the other hand, we have precisely n real degrees of freedom, namely $|A|, x_1, \dots, x_{n-1}$, to do the job.

One simple way of getting the value for B is to choose $z_0 = x_1$. Then $B = w_1$. Since the line segment $[x_1, x_2]$ is mapped to the side $[w_1, w_2]$, of the n -sided polygon Ω , for $1 \leq \nu \leq n - 2$ we have the $n - 2$ equations

$$(*)_\nu \quad A \int_{t=x_\nu}^{x_{\nu+1}} \prod_{j=1}^{n-1} (t - x_j)^{-k_j} dt = w_{\nu+1} - w_\nu.$$

For $t \in \mathbb{R}$ the complex number $(t - x_j)^{-k_j}$ is given by

$$(\ddagger) \quad (t - x_j)^{-k_j} = \begin{cases} |t - x_j|^{-k_j} & \text{for } t > x_j \\ |t - x_j|^{-k_j} e^{-ik_j\pi} & \text{for } t < x_j \end{cases}$$

according to the choice of the branch for the function $(\zeta - x_j)^{-k_j}$ defined for $\text{Im } \zeta \geq 0$. The equation $(*)_\nu$ can now be rewritten as

$$A \left(\prod_{j=\nu+1}^{n-1} e^{-ik_j\pi} \right) \int_{x_\nu}^{x_{\nu+1}} \prod_{j=1}^{n-1} |t - x_j|^{-k_j} dt = w_{\nu+1} - w_\nu,$$

because $t < x_j$ for $\nu + 1 \leq j \leq n - 1$ when $x_\nu < t < x_{\nu+1}$. This gives right away the value for $\arg A$, for example, from the equation $(*)_1$, namely

$$\arg A = \arg(w_2 - w_1) + \sum_{j=2}^{n-1} k_j \pi.$$

At this point we are left with the n unknowns $|A|, x_1, \dots, x_{n-1}$. By taking the absolute values of both sides of $(*)_\nu$, we get

$$(\dagger)_\nu \quad \int_{x_\nu}^{x_{\nu+1}} \prod_{j=1}^{n-1} |t - x_j|^{-k_j} dt = \frac{|w_{\nu+1} - w_\nu|}{|A|}$$

for $1 \leq \nu \leq n - 2$. Since we have n unknowns $|A|, x_1, \dots, x_{n-1}$ to determine, the $n - 2$ equations $(*)_1, \dots, (*)_{n-2}$ are not enough. We still need two more equations. One piece of information we have not yet used, namely the vertex w_n to worry about. This vertex w_n should be reached by the limit $f(t)$ both

(i) by letting $x_{n-1} < t < \infty$ go to $+\infty$ and

(ii) by letting $-\infty < t < x_1$ go to $-\infty$.

So we have the two equations

$$A \int_{t=x_{n-1}}^{+\infty} \prod_{j=1}^{n-1} (t - x_j)^{-k_j} dt = w_n - w_{n-1},$$

$$A \int_{t=-\infty}^{x_1} \prod_{j=1}^{n-1} (t - x_j)^{-k_j} dt = w_1 - w_n.$$

Using (\ddagger) , we can rewrite these two equations as

$$A \int_{t=x_{n-1}}^{+\infty} \prod_{j=1}^{n-1} |t - x_j|^{-k_j} dt = w_n - w_{n-1},$$

$$A \left(\prod_{j=1}^{n-1} e^{-ik_j \pi} \right) \int_{t=-\infty}^{x_1} \prod_{j=1}^{n-1} (t - x_j)^{-k_j} dt = w_1 - w_n.$$

Again we can take the absolute value of both sides of the two equations and get

$$(\dagger)_{n-1} \quad \int_{t=x_{n-1}}^{+\infty} \prod_{j=1}^{n-1} |t - x_j|^{-k_j} dt = \frac{|w_n - w_{n-1}|}{|A|},$$

$$(\dagger)_n \quad \int_{t=-\infty}^{x_1} \prod_{j=1}^{n-1} |t - x_j|^{-k_j} dt = \frac{|w_1 - w_n|}{|A|}.$$

We now use the n equations $(\dagger)_\nu$ for $1 \leq \nu \leq n$ to solve for the n unknowns $|A|, x_1, \dots, x_{n-1}$. These n equations are integral equations with the $n - 1$ unknowns x_1, \dots, x_{n-1} both in the integrands as well as in the upper limits of the integrals. In general, it is very difficult to solve these n equations simultaneously. However, the solvability of the equations can be proved by using the Riemann mapping theorem to map the polygon with vertices w_1, \dots, w_n biholomorphically onto the upper half-plane \mathbb{H} and then using the continuous extension of a biholomorphic map between two domains with piecewise smooth boundaries.

Justification of Schwarz-Christoffel Transformation from Riemann Mapping Theorem and Continuous Extension of Biholomorphic Maps to Boundary. First, use the Riemann mapping theorem f to map the open upper half-plane \mathbb{H} biholomorphically onto the n -gon with vertices w_1, \dots, w_n and then use the continuity of f to the boundary so that we have n points x_1, \dots, x_n on the real line corresponding to w_1, \dots, w_n . The upper half-plane is considered as a domain in the Riemann sphere so that after applying a biholomorphism of \mathbb{H} we can assume without loss of generality that none of the n vertices w_1, \dots, w_n of the n -gon corresponds to the boundary point ∞ of \mathbb{H} . We claim that the logarithmic derivative $\frac{f''}{f'}$ of the derivative f' of f is a meromorphic function on the Riemann sphere (*i.e.*, a rational function on \mathbb{C}) such that its only poles are simple at x_1, \dots, x_n with residues $-k_1, \dots, -k_n$ respectively and it vanishes at ∞ . The technique to prove this is by applying Schwarz reflection with respect to \mathbb{R} to the composite of f with a straightening fractional power map in the w -plane centered at w_ν (for $1 \leq \nu \leq n$) and applying Schwarz reflection to the restriction of f to the outside of a large circle with respect to \mathbb{R} .

Let $q_\nu = 1 - k_\nu$ so that πq_ν is the interior angle at the vertex w_ν . Use the straightening fractional power map

$$g_\nu(w) = (w - w_\nu)^{\frac{1}{q_\nu}}$$

which straightens the two sides $[w_{k-1}, w_\nu]$ and $[w_\nu, w_{k+1}]$ making an angle πq_ν into a straight line L_k segment passing through 0. The branch of $g_\nu(w)$ is chosen to be defined on the polygon. We now compose the straightening fractional power map g_ν with f to define

$$h_\nu(z) = (g_\nu \circ f)(z) = (f(z) - w_\nu)^{\frac{1}{q_\nu}}$$

so that the line segment $[x_{\nu-1}, x_{\nu+1}]$ in the z -plane is mapped to the line segment L_ν in the w -plane. Express $f(z)$ in terms of h_ν to get

$$f(z) = h_\nu(z)^{q_\nu} + w_\nu$$

so that

$$f'(z) = q_\nu h_\nu(z)^{-k_\nu} h'_\nu(z)$$

and its logarithmic derivative is

$$\frac{f''(z)}{f'(z)} = -k_\nu \frac{h'_\nu(z)}{h_\nu(z)} + \frac{h''_\nu(z)}{h'_\nu(z)}.$$

The function h_ν obtained by Schwarz reflection is holomorphic on $\mathbb{C} - ((-\infty, x_{\nu-1}] \cup [x_{\nu+1}, \infty))$ with nonzero derivative on $[x_{\nu-1}, x_{\nu+1}]$ and with only one zero at x_ν . Hence

$$\frac{f''(z)}{f'(z)} = \frac{-k_\nu}{z - x_\nu} + E_\nu(z)$$

with $E_\nu(z)$ holomorphic on $\mathbb{C} - ((-\infty, x_{\nu-1}] \cup [x_{\nu+1}, \infty))$. This means that

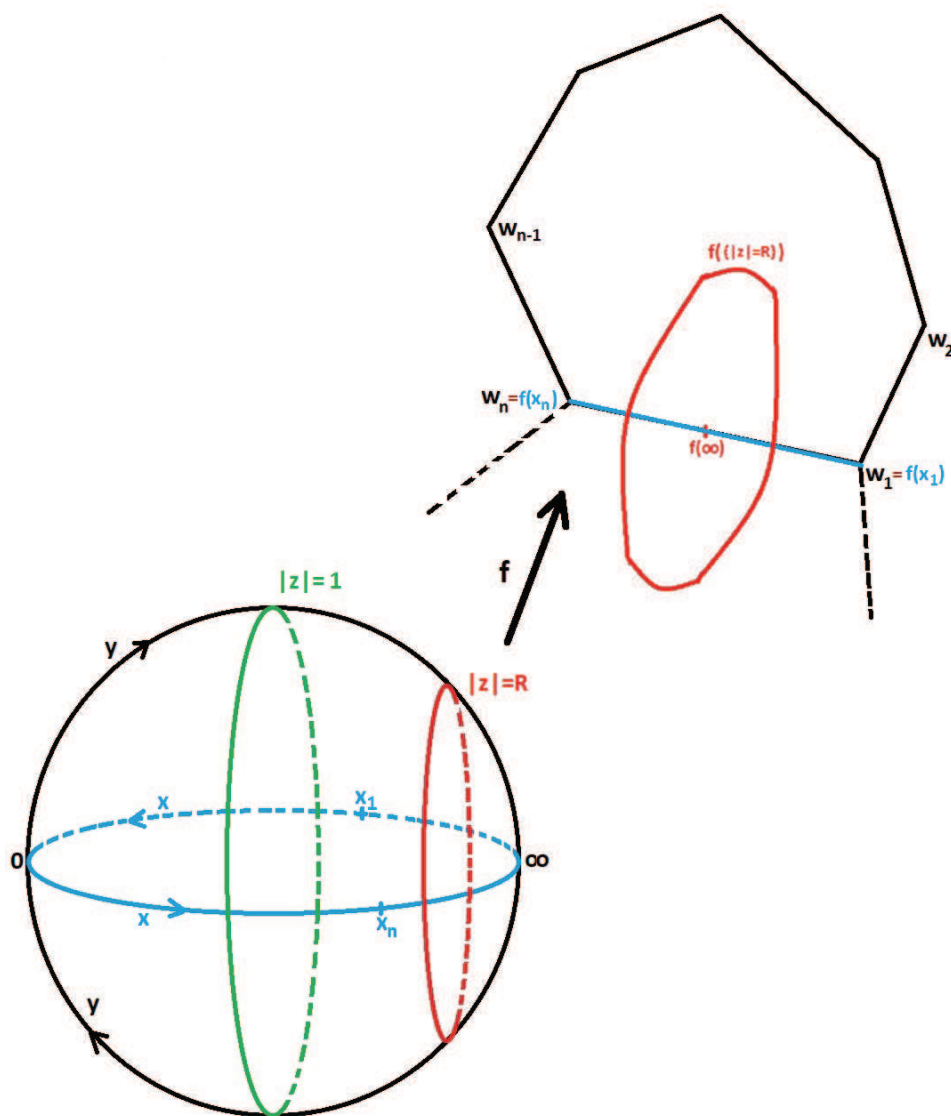
$$\frac{f''(z)}{f'(z)} - \frac{-k_\nu}{z - x_\nu}$$

is holomorphic on $\mathbb{C} - ((-\infty, x_{\nu-1}] \cup [x_{\nu+1}, \infty))$ and

$$\frac{f'(z)}{f''(z)} - \sum_{\nu=1}^n \frac{-k_\nu}{z - x_\nu}$$

is holomorphic on \mathbb{C} .

Since f maps the line-segments (x_n, ∞) and (∞, x_1) respectively to the line-segments $(w_n, f(\infty))$ and $(f(\infty), w_1)$, for $R > \max(|x_1|, \dots, |x_n|)$, we can reflect the bounded holomorphic function f on $\{|z| > R, \text{Im } z > 0\}$ with respect to the part $(x_n, \infty) \cup (\infty, x_1)$ of \mathbb{R} so that f can be continued to a deleted neighborhood $\{|z| > R\}$ of ∞ in the Riemann sphere and then by its boundedness on $\{|z| > R\}$ can be extended to be holomorphic in a neighborhood of ∞ in the Riemann sphere.



By using

$$f(z) = c_0 + \sum_{\nu=p}^{\infty} \frac{c_\nu}{z^\nu}$$

with $c_p \neq 0$ (actually $p = 1$ and $c_0 = f(\infty)$) and computing $f'(z)$ and $f''(z)$, we conclude that

$$\frac{f''(z)}{f'(z)} = -(p+1)(1+o(1))\frac{1}{z}$$

which approaches 0 as $z \rightarrow \infty$. This means that

$$\frac{f'(z)}{f''(z)} = \sum_{\nu=1}^n \frac{-k_\nu}{z - x_\nu}$$

on all of $\mathbb{C} \cup \{\infty\}$ and

$$(\log f'(z))' = \left(\log \frac{1}{\prod_{\nu=1}^n (z - x_\nu)^{k_\nu}} \right)'.$$

Integrating once and exponentiating, we obtain

$$f'(z) = \frac{A}{\prod_{\nu=1}^n (z - x_\nu)^{k_\nu}}$$

for some nonzero constant A . Another integration yields

$$f(z) = A \int \frac{dz}{\prod_{\nu=1}^n (z - x_\nu)^{k_\nu}} + B.$$

We now look at the case $x_n = \infty$. When $x_n = \infty$, we choose a biholomorphism $\zeta = b_n - \frac{1}{z}$ of \mathbb{H} to map $x_1, \dots, x_{n-1}, \infty$ in the z -plane to the n points b_1, \dots, b_n on \mathbb{R} in the ζ -plane. Let $g(\zeta) = f(z)$. Then

$$\frac{dg}{d\zeta} = \frac{\hat{A}}{(\zeta - b_1)^{k_1} \cdots (\zeta - b_n)^{k_n}}$$

so that by using $k_1 + \cdots + k_n = 2$ and $b_\nu = b_n - \frac{1}{x_\nu}$ for $1 \leq \nu \leq n-1$, we get

$$\begin{aligned} \frac{df}{dz} &= \frac{dg}{d\zeta} \frac{d\zeta}{dz} \\ &= \frac{\hat{A}}{\left(b_n - b_1 - \frac{1}{z}\right)^{k_1} \cdots \left(b_n - b_{n-1} - \frac{1}{z}\right)^{k_{n-1}} \left(-\frac{1}{z}\right)^{k_n}} \frac{1}{z^2} \\ &= \frac{\hat{A}}{\left((b_n - b_1)z - 1\right)^{k_1} \cdots \left((b_n - b_{n-1})z - 1\right)^{k_{n-1}} (-1)^{k_n}} \\ &= \frac{A}{(z - x_1)^{k_1} \cdots (z - x_{n-1})^{k_{n-1}}}, \end{aligned}$$

where

$$A = \frac{\hat{A}}{(b_n - b_1)^{k_1} \cdots (b_n - b_{n-1})^{k_{n-1}} (-1)^{k_n}}.$$

Example of a Conformal Mapping from the Upper Half Plane to an Equilateral Triangle. Given an equilateral triangle Ω with $w_1 = 0$ and $w_2 > 0$ and $\text{Im } w_3 > 0$. We seek a conformal map from the upper half-plane $\mathbb{H} = \{\text{Im } z > 0\}$ to Ω with $x_1 = -1$ and $x_2 = 1$. We do not specify $|w_2 - w_1| = |w_2|$ but instead normalize $A = 1$. The conformal map $w = f(z)$ will be given by the Schwarz-Christoffel transformation

$$f(z) = \int_{-1}^z \frac{d\zeta}{(\zeta + 1)^{\frac{2}{3}} (\zeta - 1)^{\frac{2}{3}}}.$$

We would like to determine the length of a side $|w_2|$ of the equilateral triangle Ω so that we know precisely what the equilateral triangle Ω with the normalization $|A| = 1$. In order for the Schwarz-Christoffel transformation to map 1 to $w_2 > 0$, according to the preceding discussion we must have $\arg A = \frac{2\pi}{3}$. We have the equation

$$\int_{-1}^1 \frac{dt}{(t+1)^{\frac{2}{3}} (t-1)^{\frac{2}{3}}} = w_2.$$

From the above discussion on the values of the chosen branches of the factors in the integrand, we have

$$e^{-i\frac{2\pi}{3}} \int_{-1}^1 \frac{dt}{|t+1|^{\frac{2}{3}} |t-1|^{\frac{2}{3}}} = w_2.$$

By taking the absolute values of both sides of the equation, we get

$$\int_{-1}^1 \frac{dt}{|1+t|^{\frac{2}{3}}|1-t|^{\frac{2}{3}}} = |w_2|.$$

We rewrite the equation as

$$\int_{-1}^1 \frac{dt}{(1-t^2)^{\frac{2}{3}}} = w_2,$$

because $w_2 > 0$. To evaluate the definite integral

$$\int_{-1}^1 \frac{dt}{(1-t^2)^{\frac{2}{3}}} = 2 \int_0^1 \frac{dt}{(1-t^2)^{\frac{2}{3}}},$$

we use the transformation $\tau = t^2$ and get $d\tau = 2tdt = 2\tau^{\frac{1}{2}}dt$ and

$$2 \int_0^1 \frac{dt}{(1-t^2)^{\frac{2}{3}}} = \int_0^1 \frac{d\tau}{\tau^{\frac{1}{2}}(1-\tau)^{\frac{2}{3}}}$$

which is equal to the value $B\left(\frac{1}{2}, \frac{1}{3}\right)$ of the beta function

$$B(x, y) = \int_0^1 \tau^{x-1} (1-\tau)^{y-1} d\tau.$$

Our final conclusion of this example is that the holomorphic map

$$z \mapsto e^{i\frac{2\pi}{3}} \int_{-1}^z \frac{d\zeta}{(\zeta+1)^{\frac{2}{3}}(\zeta-1)^{\frac{2}{3}}}$$

maps the upper half-plane to the equilateral triangle in the upper half-plane whose base is $[0, B\left(\frac{1}{2}, \frac{1}{3}\right)]$ and is the image of $[-1, 1]$ with the sense preserved.

Example of Using a Schwarz-Christoffel Transformation to Show that the Sine Function Maps a Vertical Upper Half-Strip to the Upper Half Plane. Let our 3-sided polygon Ω to be the vertical upper half-strip $\left\{-\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0\right\}$ with the two vertices $w_1 = -\frac{\pi}{2}$ and $w_2 = \frac{\pi}{2}$. We would like to write down the Schwarz-Christoffel transformation to map the upper half-plane $\mathbb{H} =$

$\{\operatorname{Im} z > 0\}$ to Ω with $x_1 = -1$ and $x_2 = 1$. The two angles at w_1 and w_2 are both $\frac{\pi}{2}$. The Schwarz-Christoffel transformation $w = f(z)$ is

$$f(z) = -\frac{\pi}{2} + A \int_{-1}^z \frac{d\zeta}{(\zeta + 1)^{\frac{1}{2}} (\zeta - 1)^{\frac{1}{2}}}.$$

for some $A > 0$, because $w_1 = -\frac{\pi}{2}$. To determine A , we put in the value $z = 1$ to get

$$\frac{\pi}{2} = -\frac{\pi}{2} + A \int_{-1}^1 \frac{d\zeta}{(\zeta + 1)^{\frac{1}{2}} (\zeta - 1)^{\frac{1}{2}}}.$$

because $w_2 = \frac{\pi}{2}$. From the above discussion on the values of the chosen branches of the factors in the integrand, after we move $-\frac{\pi}{2}$ from the left-hand side of the equation to the right-hand, we get

$$\pi = A \int_{-1}^1 \frac{i dt}{|1 + t|^{\frac{1}{2}} |1 - t|^{\frac{1}{2}}}.$$

We know that

$$\int_{-1}^1 \frac{dt}{|1 + t|^{\frac{1}{2}} |1 - t|^{\frac{1}{2}}} = \int_{-1}^1 \frac{dt}{\sqrt{1 - t^2}} = \sin^{-1} t \Big|_{t=-1}^{t=1} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Hence $A = -i$. Now our Schwarz-Christoffel transformation is

$$f(z) = -\frac{\pi}{2} + A \int_{-1}^z \frac{d\zeta}{(\zeta + 1)^{\frac{1}{2}} (\zeta - 1)^{\frac{1}{2}}}.$$

For $z = t$ in the interval $(-1, 1)$, according to the values of the chosen branches of the factors in the integrand we have

$$f(t) = -\frac{\pi}{2} + A \int_{-1}^t \frac{i ds}{\sqrt{1 - s^2}} = -\frac{\pi}{2} + \int_{-1}^t \frac{ds}{\sqrt{1 - s^2}} = \int_0^t \frac{ds}{\sqrt{1 - s^2}} = \sin^{-1} t,$$

because

$$\int_{-1}^0 \frac{ds}{\sqrt{1 - s^2}} = \frac{\pi}{2}.$$

The range of $\sin^{-1} t$ for $-1 < t < 1$ would have to be the range of $f(t)$ for $-1 < t < 1$ and hence must be $(-\frac{\pi}{2}, \frac{\pi}{2})$. Taking the inverse of $\sin^{-1} z$, we conclude that the holomorphic map $z = \sin w$ maps the vertical upper half-strip

$$\left\{ w = u + iv \mid -\frac{\pi}{2} < u < \frac{\pi}{2}, v > 0 \right\}$$

onto the upper half-plane $\{y > 0\}$ with $w = -\frac{\pi}{2}$ corresponding to $z = -1$ and $w = \frac{\pi}{2}$ corresponding to $z = 1$.