

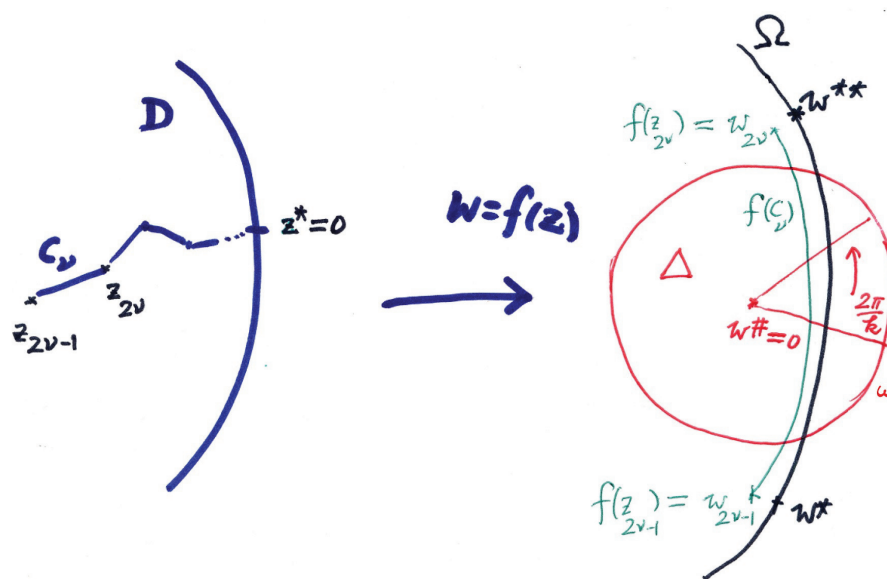
Boundary Behavior of Biholomorphism of Piecewise Smooth Domains

The continuous extendibility of biholomorphic maps up to piecewise smooth boundaries is needed to give a rigorous proof of the Schwarz-Christoffel which maps a given polygon to the open upper half-plane. It is used to locate the points on the real line which correspond to the vertices of the given polygon after the application of the Riemann mapping theorem to secure the abstract existence of a biholomorphic map between the given polygon and the open upper half-plane.

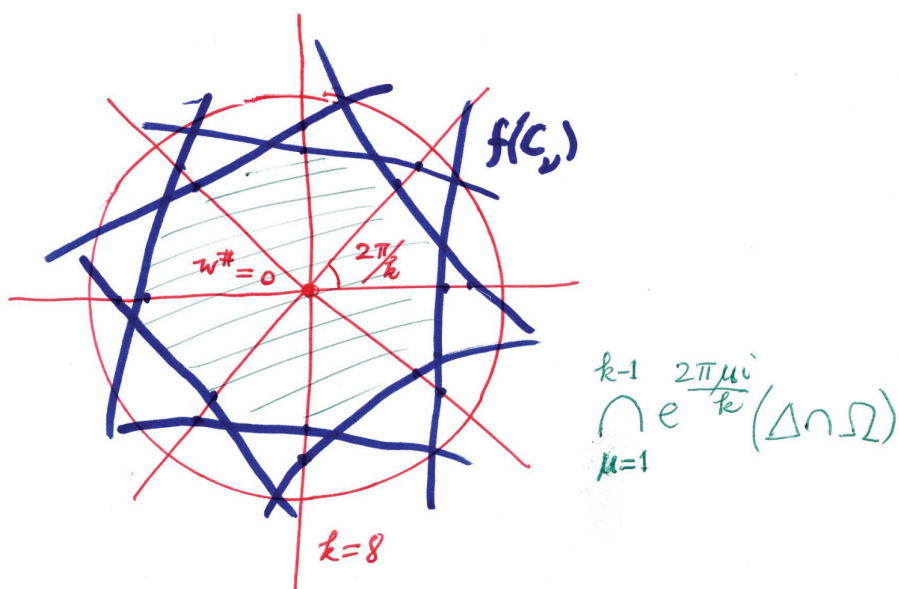
We now present the technique concerning the boundary behavior of biholomorphic mappings between domains of piecewise smooth boundaries. Suppose we have a univalent holomorphic map f between two domains D and Ω with piecewise smooth boundary. We want to show that f extends continuously to the boundary.

What is precisely needed for the continuous extension of f to $\bar{D} \rightarrow \bar{\Omega}$ is the following boundary condition of Ω . The boundary $\partial\Omega$ is parametrized continuously by the circle. For two distinct points w^* and w^{**} on $\partial\Omega$ such that to go from w^* to w^{**} along $\partial\Omega$ is in the counterclockwise sense (according to the continuous parametrization by the circle), there exists a closed disk $\bar{\Delta}$ of positive radius such that the center of $\bar{\Delta}$ is in Ω and the intersection of $\bar{\Delta}$ and $\partial\Omega$ is contained in the interior of the part of $\partial\Omega$ from w^* to w^{**} and there is a closed arc of $\bar{\Delta}$ of positive length (for example, subtended by an angle $\frac{2\pi}{k}$ at the center for some integer $k \geq 3$) which is outside of $\bar{\Omega}$. An example of a domain not satisfying this condition is the right-half of the open unit disk minus the line segment $(0, \frac{1}{2}]$ with the two points w^* and w^{**} on different sides of the open part $(0, \frac{1}{2})$ of the slit $(0, \frac{1}{2}]$.

We now prove that the biholomorphic map $f : D \rightarrow \Omega$ can be continuously extended to $\bar{D} \rightarrow \bar{\Omega}$. Suppose the contrary. Then we can find a sequence z_ν in D converging to some point z^* on the boundary of D such that $w_{2\nu-1} = f(z_{2\nu-1})$ converges to a boundary point w^* of Ω and $w_{2\nu} = f(z_{2\nu})$ converges to another boundary point w^{**} of Ω .



Since the boundary of D is piecewise smooth, we can join $z_{2\nu-1}$ to $z_{2\nu}$ by a curve C_ν with the property that for any open disk U_ϵ of positive radius ϵ centered at z^* all the curves C_ν are inside U_ϵ when ν is greater than some $\nu_0(\epsilon)$. Since w^* and w^{**} are distinct, we can find a disk Δ centered at some point $w^\#$ of Ω so that an arc ω of Δ with angle $\frac{2\pi}{k}$ for some k is outside Ω and for all ν sufficiently large the image of C_ν under f separates Δ into two parts with one part containing the $w^\#$ and one part containing the arc ω . Without loss of generality we can assume that z^* is the origin and $w^\#$ is also the origin.



Consider the function g on $\Delta \cap \Omega$ which is the inverse of the map f . Then $|g| \leq \epsilon$ on $f(C_\nu)$ for $\nu \geq \nu_0(\epsilon)$. Consider the function

$$G(w) = \prod_{\mu=0}^{k-1} g(e^{2\pi\mu\sqrt{-1}/k} w).$$

Then G is holomorphic on

$$\bigcap_{\mu=0}^{k-1} e^{2\pi\mu\sqrt{-1}/k} \cdot (\Delta \cap \Omega),$$

where

$$e^{2\pi\mu\sqrt{-1}/k} \cdot (\Delta \cap \Omega)$$

is the domain obtained by rotating $\Delta \cap \Omega$ by an angle of $\frac{2\pi\mu}{k}$. On the k curves

$$e^{2\pi\mu\sqrt{-1}/k} \cdot f(C_\nu)$$

obtained by rotating the curve $f(C_\nu)$ by an angle of $\frac{2\pi\mu}{k}$, the function G is bounded by $A^{k-1}\epsilon$, where A is the radius of some disk centered at the origin

which contains D . Since $w^\#$ is contained in the compact subset of

$$\bigcap_{\mu=0}^{k-1} e^{2\pi\mu\sqrt{-1}/k} \cdot (\Delta \cap \Omega)$$

bounded by

$$\bigcup_{\mu=0}^{k-1} e^{2\pi\mu\sqrt{-1}/k} \cdot f(C_\nu),$$

it follows from the maximum modulus principle that $G(w^\#) = g(w^\#)^k$ is bounded by $A^{k-1}\epsilon$. Thus $|g(w^\#)| \leq (A^{k-1}\epsilon)^{1/k}$. Since ϵ is just any positive number, we conclude that $g(w^\#) = 0$. The only condition on $w^\#$ is the existence of the disk Δ . Points sufficiently close to $w^\#$ would also satisfy this condition. So we get the vanishing of g in a neighborhood of $w^\#$ contradicting the univalence of f .