**Riemann Mapping Theorem**

In applications of conformal mappings, one seeks to construct harmonic functions on a given domain with given boundary value conditions and/or normal derivatives conditions on the boundary. If there is a way to map the given domain biholomorphically to a simpler domain, for example, the open unit disk $\mathbb{D}$, it would make the construction of the desired harmonic function easier. When the given domain in $\mathbb{C}$ is simply connected and is not equal to $\mathbb{C}$, the Riemann mapping theorem guarantees that there is a biholomorphic map from the given domain onto the open unit disk $\mathbb{D}$. The proof of the Riemann mapping theorem is not constructive and is done by *reductio ad absurdum.*

Later, we will discuss Schwarz-Christoffel transformations which explicitly give biholomorphic maps from polygons to the open upper half-plane (which is biholomorphic to $\mathbb{D}$). The derivation of the formula for a Schwarz-Christoffel transformation depends on the Riemann mapping theorem and on the continuous extendibility of biholomorphic maps up to piecewise smooth boundaries. Afterwards, we will also discuss the Poisson integral formula which yields a harmonic function on $\mathbb{D}$ with prescribed boundary values and which in a way can be regarded as the analogue of the Cauchy integral formula for harmonic functions. Now we precisely state and prove the Riemann mapping theorem

*Statement of Riemann Mapping Theorem.* Let $\Omega$ be a simply connected domain in $\mathbb{C}$ different from $\mathbb{C}$. Then there exits a biholomorphic map from the open unit disk $\mathbb{D}$ onto $\Omega$.

*Proof of Riemann Mapping Theorem.* Without loss of generality we assume that 0 is not in $\Omega$. By integrating $\frac{1}{z}$ from some chosen point of $\Omega$ to a variable point of $\Omega$ along a path composed of horizontal and vertical line-segments, on account of the simply connected property of $\Omega$ we obtain a branch of $\log z$ on $\Omega$, from which we can define a branch $f(z)$ of the $\sqrt{z}$ on $\Omega$. The other branch of $\sqrt{z}$ is $-f(z)$.

The map $f(z)$ is univalent on $\Omega$, because $f(z_1) = f(z_2)$ implies that

$$z_1 = (f(z_1))^2 = (f(z_2))^2 = z_2.$$

Moreover, the image of $\Omega$ under $f(z)$ is disjoint from the image of $\Omega$ under $-f(z)$. Otherwise, some $z_1$ and $z_2$ in $\Omega \subset \mathbb{C} - \{0\}$ satisfy $f(z_1) = -f(z_2)$. Squaring the identity gives $z_1 = (f(z_1))^2 = (f(z_2))^2 = z_2$ and $f(z_1) = -f(z_1)$ gives the contradiction that the two different values of the square root of $z_1$ are equal without $z_1$ being 0.

Since there is some closed disk of positive radius in the image of $\Omega$ under $-f(z)$ and since the image of $\Omega$ under $f(z)$ is disjoint from the image of $\Omega$ under $-f(z)$, it follows that the image of $\Omega$ under $f(z)$ is disjoint from some disk $|z - b| \leq r$. This means that

$$g(z) := \frac{r}{f(z) - b}$$

is univalent on $\Omega$ and maps $\Omega$ into the open unit disk $\Delta$.

Take $P_0 \in \Omega$. Let $\mathcal{F}$ be the set of all univalent functions from $\Omega$ to $\Delta$ which maps $P_0$ to 0. Let

$$\mu = \sup \left\{ |f'(P_0)| \,\middle|\, f \in \mathcal{F} \right\}$$

and let $f \in \mathcal{F}$ achieves $\mu$. This value $\mu$ is achieved by some element of $\mathcal{F}$ for the following reason. From $|f| \leq 1$ on $\Omega$ for $f \in \mathcal{F}$ it follows from Cauchy's formula for the first derivative of a holomorphic function that

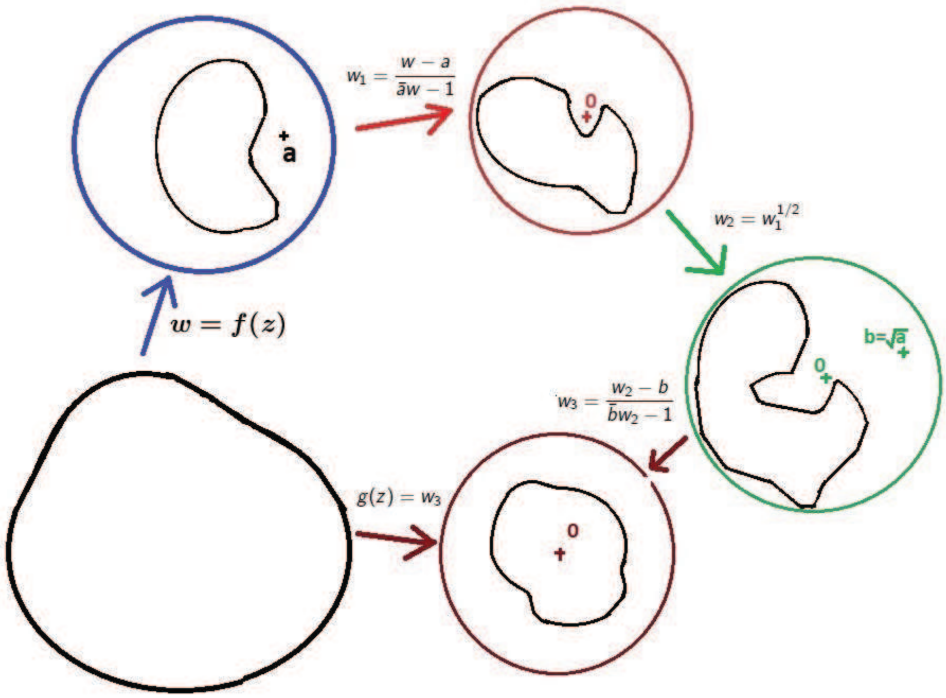
$$\sup_{z \in K} |f'(z)| = C_K < \infty$$

for any compact subset $K$ of $\Omega$. This means that the family of functions $\mathcal{F}$ is equicontinuous so that for any sequence in $\mathcal{F}$ there is a subsequence which converges uniformly on compact subsets to a limit function. Since the absolute value of the derivative of this limit function at 0 is $\mu > 0$, by Hurwitz's theorem on the limit function of univalent functions we know that this limit function is unvalent.

Suppose $a \in \mathbb{D}$ is not in the image of $\Omega$ under $f$. Introduce the following self-maps of $\mathbb{D}$:

$$w_1 = \frac{w - a}{1 - \bar{a}w}, \quad w_2 = \sqrt{w_1},$$
$$b = w_2\Big|_{w=0} = \sqrt{-a}, \quad w_3 = \frac{w_2 - b}{1 - \bar{b}w_2},$$

so that the value of $w_3$ at $w = 0$ is 0, which means that the value of $w_3$ at $z = P_0$ is 0 and $z \mapsto w_3$ belongs to $\mathcal{F}$.




Note that this construction of the function $z \mapsto w_3$ first moves the missed point $a$ to 0 to enable us to take the square root of the function and then moves $b$ back to 0 to get the normalization of having value 0 at 0 to make sure that the resulting function is in the family $\mathcal{F}$.

Calculation by the chain rule gives

$$\frac{dw_1}{dw} = \frac{(1-\bar{a}w) - (-\bar{a})(w-a)}{(1-\bar{a}w)^2} = \frac{1-\bar{a}w+\bar{a}w-a\bar{a}}{(1-\bar{a}w)^2},$$

$$\left.\frac{dw_1}{dw}\right|_{w=0} = 1-|a|^2,$$

$$\frac{dw_2}{dw_1} = \frac{1}{2\sqrt{w_1}},$$

$$\left.\frac{dw_2}{dw_1}\right|_{w_1=a} = \frac{1}{2\sqrt{-a}},$$

$$\frac{dw_3}{dw_2} = \frac{(1-\bar{b}w_2) - (-\bar{b})(w_2-b)}{(1-\bar{b}w_2)^2} = \frac{1-\bar{b}w_2+\bar{b}w_2-b\bar{b}}{(1-\bar{b}w_2)^2},$$

$$\left.\frac{dw_3}{dw_2}\right|_{w_2=b} = \frac{1}{1-b\bar{b}},$$

$$\left.\frac{dw_3}{dz}\right|_{z=0} = \frac{1-a\bar{a}}{2\sqrt{-a}\left(1-b\bar{b}\right)} = \frac{1-a\bar{a}}{2\sqrt{-a}\left(1-\sqrt{a\bar{a}}\right)}.$$

Thus the absolute value of

$$\left.\frac{dw_3}{dw_2}\right|_{w_2=b}$$

is

$$\frac{1-|a|^2}{2\sqrt{|a|}\,(1-|a|)} = \frac{1+|a|}{2\sqrt{|a|}}$$

and

$$\frac{1+|a|}{2\sqrt{|a|}} - 1 = \frac{1+|a|-2\sqrt{|a|}}{2\sqrt{|a|}} = \frac{\left(1-\sqrt{|a|}\right)^2}{2\sqrt{|a|}} > 0,$$

because $|a| < 1$. This means that the absolute value of the derivative of the element $z \mapsto w_3$ of $\mathcal{F}$ at $P_0$ is $> \mu$, which is a contradiction. Q.E.D.