Application to Fluid Flow, Temperature Distribution, Electrostatic Potential and Airfoil Lift

Velocity Potential of Irrotational Fluid Flow. Denote by $\vec{v} = (p, q)$ the velocity of a steady 2-dimensional fluid flow in a domain Ω. "Steady" means time-independent. The flow is said to be *irrotational* if

$$\oint_C \left(\vec{v} \cdot \vec{t}\right) ds = 0$$

for any curve C in Ω whose enclosure is completely in Ω, where $\vec{t}$ is the unit tangent vector of C. Since

$$\vec{t} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right),$$

it follows that irrotationality is equivalent to

$$\oint_C pdx + qdy = 0$$

for all C, which by Stokes's theorem is equivalent to

$$\int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy = 0$$

for any subdomain D in Ω. This means that

$$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$$

identically on Ω. When Ω is simply connected (*i.e.,* any loop in Ω can be continuously shrunk to a point inside Ω), we can find a function φ (called the *velocity potential*) such that $\vec{v} = (p, q) = \operatorname{grad} \varphi$.

Equation of Continuity. Let $\rho(x, y, t)$ be the fluid density at the point (x, y) and at time t. The law of conservation of mass yields the following *continuity equation*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\, (\rho \vec{v}) = 0,$$

which is derived from the conservation of mass and the the divergence theorem as follows. Divergence measures the rate of "moving away" so that for a small domain U, when the *flux* (which is the vector giving the direction and measuring the amount of material per unit volume leaving U across a unit area of ∂U per unit time) is $\vec{u}$, the total amount of material with density ρ which leaves U across the entire boundary ∂U per unit time is

$$\int_{\partial U} \rho \vec{u} \cdot \boldsymbol{n}$$

(where $\boldsymbol{n}$ is the outward pointing normal vector of ∂U) which according to the divergence theorem is equal to

$$\int_U \operatorname{div}(\rho \vec{u}) \,.$$

Because of the conservation of mass, the rate of decrease of all the material inside U is the integration of $-\frac{\partial \rho}{\partial t}$ over U, which is

$$-\int_U \frac{\partial \rho}{\partial t}$$

When U is allowed to shrink to a point P in Ω, the limit of the above equation divided by the volume of U yields the continuity equation at P

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\,(\rho \vec{v}) = 0,$$

Holomorphic Complex Vector Potential for Constant-Density Irrotational Fluid Flow. If the fluid is incompressible with constant density, then the continuity equation yields $\operatorname{div} \vec{v} = 0$, which means that $\Delta \varphi = 0$. Thus the velocity potential φ is harmonic. We will confine our discussion only to irrotational, incompressible, steady 2-dimensional flows with constant density. We are interested in determining the velocity $\vec{v}(x, y)$ at the point (x, y) and also the equations of the streamlines. We will seek the harmonic function φ as the real part of a holomorphic function $f(z)$ so that $f = \varphi + i\psi$ with real part φ and imaginary part ψ. We call the function f the *complex velocity potential.* We can express the velocity $\vec{v} = \operatorname{grad} \varphi$ in terms of f as follows.

$$\vec{v} = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \right)$$

which by the Cauchy-Riemann equations can be rewritten as

$$\vec{v} = \left(\frac{\partial \varphi}{\partial x}, -\frac{\partial \psi}{\partial x} \right).$$

When we use the notation of complex numbers to express a vector, we get

$$\vec{v} = \frac{\partial \varphi}{\partial x} - i \frac{\partial \psi}{\partial x} = \overline{\frac{\partial f}{\partial x}} = \overline{f'}.$$

Streamlines of Constant-Density Irrotational Fluid Flow. The streamlines are integral curves of the velocity vector so that along streamlines particles in the flow move. We are going to express the streamlines as the level curves of some function $g =$ constant. Then the gradient of g is normal to the streamlines and must be perpendicular to the velocity vector $\vec{v} = \operatorname{grad} \varphi$ According to the Cauchy-Riemann equations, the gradient of ψ is perpendicular to the gradient of φ. Hence we can choose g to be ψ and the streamlines are the level curves of the imaginary part of the complex velocity potential f.

Flow with Constant Speed at Infinity. For the flow with constant speed $A > 0$ from the left to the right on the upper half-plane the complex velocity potential is $f(z) = Az$, because $\overline{f'} = A$. For the flow with constant speed $A > 0$ and angle α measured from the real axis, the complex velocity potential is $f(z) = Aze^{-i\alpha}$, because $\overline{f'} = Ae^{i\alpha}$. We are interested in the determination of the complex velocity potential outside a given bounded obstacle when the flow at infinity is known to be of known constant speed at known angle.

Examples Involving the Exponential Function and the Function Relating the Exponential Function to the Sine Function. Consider the fluid flow on $\mathbb{C} - [-1, 1]$ which at ∞ is a flow with speed A and angle α measured from the real-axis. Find the complex velocity potential on $\mathbb{C} - [-1, 1]$.

Here is the solution.

Step One. First use a new complex variabale z_1 and the map $z = \frac{1}{2}\left(z_1 + \frac{1}{z_1}\right)$ which maps $z \in \mathbb{C} - [-1, 1]$ one-one onto the set of all z_1 in $\mathbb{C}$ minus the closed unit disk. The flow in the z_1-space at ∞ is a flow with speed $2A$ and angle α measured from the real-axis, because the behavior of $\frac{z_1}{2}$ is the same as the behavior of z at ∞.

Step Two. We now introduce another complex variable z_2 which is related to z_1 by $z_2 = z_1 e^{-i\alpha}$. Clearly the set of all z_1 in $\mathbb{C}$ minus the closed unit disk is in one-one correspondence with the set of all z_2 in $\mathbb{C}$ minus the closed unit disk. The flow in the z_1-space at ∞ is a flow with speed $2A$ and from left to right, because

$$\overline{\frac{df}{dz_2}} = \overline{\frac{df}{dz_1}\frac{dz_1}{dz_2}} = \overline{\frac{df}{dz_1}e^{i\alpha}} = \overline{\frac{df}{dz_1}}\, e^{-i\alpha}.$$

Step Three. We now introduce yet another complex variable w which is related to z_2 by $w = \frac{1}{2}\left(z_2 + \frac{1}{z_2}\right)$. The set of all z_2 in $\mathbb{C}$ minus the closed unit disk is in one-one correspondence with $w \in \mathbb{C} - [-1, 1]$. The flow in the w-space at ∞ is a flow with speed A and from left to right, because the behavior of $\frac{z_2}{2}$ is the same as the behavior of w at ∞. Thus the complex potential function f in terms of w is simply Aw. We now express f in terms of our original complex variable z. We have to invert $z = \frac{1}{2}\left(z_1 + \frac{1}{z_1}\right)$. To solve for z_1 in terms of z, we solve the quadratic equation

$$(z_1)^2 - 2zz_1 + 1 = 0.$$

According to the formula for the two roots of a quadratic equation, the roots are

$$\frac{2z \pm \sqrt{4z^2 - 4}}{2} = z \pm \sqrt{z^2 - 1}.$$

Since for our one-one correspondence between z and z_1, a large positive z corresponds to a large positive z_1, we should take the plus sign and get

$$z_1 = z + \sqrt{z^2 - 1}.$$

The other root $z - \sqrt{z^2 - 1}$ must be $\frac{1}{z}$, because the transformation $z \mapsto \frac{1}{z}$ leaves z_1 unchanged. Now we get as our final answer for the complex velocity potential $f(z)$ the expression

$$\begin{aligned} f = Aw = \frac{A}{2}\left(z_2 + \frac{1}{z_2}\right) = \frac{A}{2}\left(z_1 e^{-i\alpha} + \frac{e^{i\alpha}}{z_1}\right) \\ = \frac{A}{2}\left(e^{-i\alpha}\left(z + \sqrt{z^2 - 1}\right) + e^{i\alpha}\left(z - \sqrt{z^2 - 1}\right)\right). \end{aligned}$$

Temperature Distribution. We consider the situation of constant specific heat, which we normalize to be 1. We assume further that there is no heat source or heat sink inside the domain under consideration. For temperature distribution T, the heat flow vector is given by $-\operatorname{grad} T$. The total amount of heat leaving a domain U is

$$-\int_{\partial U} \operatorname{grad} T \cdot \boldsymbol{n}$$

(where $\boldsymbol{n}$ is the outward pointing normal vector of ∂U) which according to the divergence theorem is equal to

$$-\int_U \operatorname{div}(\operatorname{grad} T) = -\int_U \Delta T.$$

This is equal to the rate of heat loss in U which is

$$-\int_U \frac{\partial T}{\partial t}.$$

When U is allowed to shrink to a point P, the limit of the above equation divided by the volume of U yields the heat equation

$$\frac{\partial T}{\partial t} = \Delta T.$$

When the situation is steady (*i.e.,* independent of time), the left-hand side of the heat equation vanishes to yield the condition that $\Delta T = 0$, which means that the steady temperature function T satisfies the Laplace equation. The problem of finding the steady temperature distribution inside a domain with known temperature distribution along its boundary is the same as solving the Dirichlet problem of determining a harmonic function on a domain with given boundary values. If for part of the boundary, the boundary condition of known value is replaced by the condition of insulation of no heat going through, the boundary condition for T for that part of boundary is replaced by the condition that the normal derivative of T vanishes.

Example. Find the temperature distribution T on the upper half-plane so that $T = 0$ on $(-\infty, -1]$ and $T = 1$ on $[1, \infty)$ and the interval $[-1, 1]$ on the boundary is insulated.

Solution. We consider the map the function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ which relates the exponential function to the sine and cosine function. It maps the open upper unit half-disk to the lower half-plane so that

(i) the upper half-circle goes to $[-1,1]$,

(ii) the interval $(0,1]$ goes to $[1,\infty)$,

(ii) the interval $[-1,0)$ goes to $(-\infty,-1]$.

The exponent map $z \mapsto e^z$ maps the horizontal left half-strip $\{x<0, 0<y<\pi\}$ to the open upper unit half-disk so that

(i) the vertical line-segment $\{x=0, 0 \leq y \leq \pi\}$ goes to the upper half-circle, $[-1,1]$,

(ii) the horizontal line-segment $\{-\infty<x\leq 0, y=0\}$ goes to the interval $[0,1]$,

(iii) the horizontal line-segment $\{-\infty<x\leq 0, y=\pi\}$ goes to the interval $[-1,0]$.

Multiplication by i sends the vertical upper half-strip $\{0<x<\pi, y>0\}$ to the horizontal left half-strip $\{x<0, 0<y<\pi\}$. Thus the cosine function

$$z \mapsto \cos z = \frac{1}{2}\left(e^{iz}+e^{-iz}\right)$$

maps the vertical upper half-strip $\{0<x<\pi, y>0\}$ to the lower half-plane so that

(i) the horizontal line-segment $[0,\pi]$ goes to $[-1,1]$,

(ii) the vertical line-segment $\{x=0, 0\leq y<\infty\}$ goes to $[1,\infty)$,

(iii) the vertical line-segment $\{x=\pi, 0\leq y<\infty\}$ goes to $(-\infty,-1]$.

Let us now look at the sine function. We have $\sin z = -\cos\left(z+\frac{\pi}{2}\right)$ which maps the vertical upper half-strip $\left\{-\frac{\pi}{2}<x<\frac{\pi}{2}, y>0\right\}$ to the upper half-plane so that

(i) the horizontal line-segment $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ goes to $[-1,1]$,

(ii) the vertical line-segment $\left\{x=-\frac{\pi}{2}, 0\leq y<\infty\right\}$ goes to $(-\infty,-1]$,

(iii) the vertical line-segment $\left\{x=\frac{\pi}{2}, 0\leq y<\infty\right\}$ goes to $[1,\infty)$.

Thus the temperature distribution T is given by $T = \frac{1}{2} + \frac{1}{\pi}\text{Re}\,\sin^{-1} z$. We now use the formula for the real part of the inverse sine function to get

$$T = \frac{1}{2} + \frac{1}{\pi}\sin^{-1}\frac{1}{2}\left(\sqrt{(x+1)^2 + y^2} - \sqrt{(x+1)^2 + y^2}\right),$$

where the branch $\sin^{-1} t$ of the inverse sine function of a real variable t is chosen to have the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The elementary computational manipulations involving hyperbolas which are used to derive the formula for T from $T = \frac{1}{2} + \frac{1}{\pi}\text{Re}\,\sin^{-1} z$ are given in Appendix B at the end of these notes which depends on the discussion in Appendix A.

Electrostatic Potential. The problems of finding an electrostatic potential on a domain with given values on some part of the boundary and insulation on the rest of the boundary are the same as those for steady temperature distribution.

Electrostatic Potential Inside Non Coaxial Shielded Round Cable. Suppose we have a shielded round cable which is not coaxial, with the voltage of the shielding kept at 0 and the voltage of the inside non coaxial round connector kept at constant voltage V, we would like to determine the voltage of points between the inside non coaxial round connector and the outside shielding.

Suppose the outside shielding is represented by the unit circle $|z| = 1$ and the inside non coaxial round connector is represented by the circle $|z-c| = r$ insider $|z| < 1$ with $c \in (-1, 1)$ so that $|c| + r < 1$. The problem is to find a harmonic function u on the domain $\{|z| < 1, |z-c| > r\}$ with boundary value 0 on $|z| = 1$ and boundary value V at $|z-c| = r$.

We seek a biholomorphic map $w = f(z)$ (which will be a linear fractional transformation) between the given domain $\{|z| < 1, |z-c| > r\}$ and an annulus $\{1 < |w| < R\}$ for some $R > 1$ (to be determined) so that $a \log|w| + b$ for some $a, b \in \mathbb{R}$ will be the required harmonic function. Since the real line $\mathbb{R}$, which is orthogonal to the two circular boundaries $\{|z| = 1\}$ and $\{|z-c| = r\}$ of the given domain $\{|z| < 1, |z-c| > r\}$, is also orthogonal to the the two circular boundaries $\{|w| = 1\}$ and $\{|w| = R\}$ of the target annuls domain $\{1 < |w| < R\}$, by the conformality of the biholomorphic map $w = f(z)$ the real line $\mathbb{R}$ must be mapped to itself by $w = f(z)$. Thus the four points which are the intersection points of $\{|z| = 1\}$ and $\{|z-c| = r\}$ with $\mathbb{R}$ must be mapped by $w = f(z)$ to the four points which are the intersection points of $\{|w|=1\}$ and $\{|w|=R\}$ with $\mathbb{R}$ and the two cross-ratios of the two quadruples of points must be equal so that

$$\frac{\frac{-R-(-1)}{-R-1}}{\frac{R-(-1)}{R-1}}=\frac{\frac{-1-(c-r)}{-1-(c+r)}}{\frac{1-(c-r)}{1-(c+r)}}$$

or

$$\begin{aligned}\frac{(R-1)^2}{(R+1)^2}&=\frac{(1+(c-r))(1-(c+r))}{(1+(c+r))(1-(c-r))}\\&=\frac{(1-r+c)(1-r-c)}{(1+r+c)(1+r-c)}\\&=\frac{(1-r)^2-c^2}{(1+r)^2-c^2}\end{aligned}$$

or

$$\frac{R-1}{R+1}=\sqrt{\frac{(1-r)^2-c^2}{(1+r)^2-c^2}}<1,$$

because $1-r>|c|$ and $R>1$. Thus,

$$R=\frac{1+A}{1-A},$$

where

$$A=\sqrt{\frac{(1-r)^2-c^2}{(1+r)^2-c^2}}<1.$$

Again, by the preservation of cross-ratios, $w=f(z)$ is determined by

$$\frac{\frac{w-(-1)}{w-1}}{\frac{R-(-1)}{R-1}}=\frac{\frac{z-(c-r)}{z-(c+r)}}{\frac{1-(c-r)}{1-(c+r)}},$$

which yields

$$w=\frac{B+1}{B-1},$$

where

$$B=\frac{R+1}{R-1}\,\frac{\frac{z-(c-r)}{z-(c+r)}}{\frac{1-(c-r)}{1-(c+r)}}.$$

This linear fractional transformation $z \mapsto w$ indeed maps the given domain $\{|z| < 1, |z - c| > r\}$ to the annulus $\{1 < |w| < R\}$ with the two circular boundaries $\{|z| = 1\}$ and $\{|z-c| = r\}$ of the given domain $\{|z| < 1, |z-c| > r\}$ mapped respectively to two circular boundaries $\{|w| = 1\}$ and $\{|w| = R\}$ of the target annuls domain $\{1 < |w| < R\}$, because the linear fractional transformation $z \mapsto w$, which is conformal and maps the set of circles and lines to itself, maps $\mathbb{R}$ to $\mathbb{R}$. The real constants a and b in the harmonic function $u = a \log |w| + b$ are determined by

$$a \log R + b = 0 \quad \text{and} \quad a \log 1 + b = V,$$

which means that $b = V$ and $a = -\frac{V}{\log R}$ or

$$u = V \left(1 - \frac{\log |w|}{\log R}\right).$$

Example of Biholomorphic Map Between Open Upper Half-Plane and Open Horizontal Strip Minus Small Vertical Line Segment. We now give another example of putting together several basic biholomorphic maps (from the building blocks we have been discussing) to construct a desired biholomorphic map. Let $0 < h < \pi$. Consider the complement Ω of the line segment $[0, hi]$ in the open horizontal strip

$$\left\{ z = x + iy \in \mathbb{C} \,\middle|\, 0 < y < \pi \right\}.$$

We are going to construct a biholomorphic map $z \mapsto w$ from Ω onto the open upper half-plane $\mathbb{H}$ by using the following sequence of biholomorphic maps.

(i) The map

$$z \mapsto w_1 = e^z$$

maps Ω biholomorphically onto the complement Ω_1 of the arc

$$\left\{ e^{i\theta} \,\middle|\, 0 \leq \theta \leq h \right\}$$

in the open upper half-plane $\mathbb{H}$.

(ii) The linear fractional transformation

$$w_1 \mapsto w_2 = \frac{w_1 - 1}{w_1 + 1}$$

maps Ω_1 biholomorphically onto the complement Ω_2 of the line segment

$$\left[0,\ \frac{e^{ih}-1}{e^{ih}+1}\right]=\left[0, i\ \tan\frac{h}{2}\right]$$

in the open upper half-plane $\mathbb{H}$.

(iii) The square map

$$w_2 \mapsto w_3 = w_2^2$$

maps Ω_2 biholomorphically onto the complement Ω_3 of the half-line

$$\left[-\tan^2\frac{h}{2},\ \infty\right)$$

in the complex plane $\mathbb{C}$.

(iv) The translation map

$$w_3 \mapsto w_4 = w_3 + \tan^2\frac{h}{2}$$

maps Ω_3 biholomorphically onto the complement Ω_4 of the half-line $[0, \infty)$ in the complex plane $\mathbb{C}$.

(v) The square-root map

$$w_4 \mapsto w = \sqrt{w_4}$$

maps Ω_4 biholomorphically onto the open half-plane $\mathbb{H}$, where the value of the branch of the square root at -1 is i.

Putting all the building blocks together, we get

$$\begin{aligned}
w &= \sqrt{w_4} \\
&= \sqrt{w_3 + \tan^2\frac{h}{2}} \\
&= \sqrt{w_2^2 + \tan^2\frac{h}{2}} \\
&= \sqrt{\left(\frac{w_1-1}{w_1+1}\right)^2 + \tan^2\frac{h}{2}} \\
&= \sqrt{\left(\frac{e^z-1}{e^z+1}\right)^2 + \tan^2\frac{h}{2}} \\
&= \sqrt{\tanh^2\frac{z}{2} + \tan^2\frac{h}{2}}.
\end{aligned}$$

This biholomorphic map can be used to solve problems of electrostatic potential, heat temperature distribution, and fluid flow in the domain Ω.

Lift on Airfoil from Theorem of Kutta-Joukowski. We now discuss the computation of the lifting force on an airfoil due to the pressure of fluid flow (*i.e.,* air flow) from the movement of the airplane and the movement of air. An airfoil is the cross section of an airplane wing whose length is so large that it is considered infinite for the purpose of computation, *i.e.,* a cylinder.

Suppose the airfoil is the cylinder whose cross section has boundary C. Let p be the pressure of the fluid flow outside the airfoil. The force F on the airfoil due to the pressure of the fluid flow, as a complex number, is

$$-\oint_C p\,\boldsymbol{n}\,ds,$$

where ds is the arc-length element of C. Denote by φ the angle of the outward-pointing normal vector of C so that $\varphi+\frac{\pi}{2}$ is the tangent vector of C pointing in the counter-clockwise sense. Then, as complex numbers, we have $\boldsymbol{n}=e^{i\varphi}$ and

$$\bar{F}=-\oint_C p\,e^{-i\varphi}\,ds=-i\oint_C p\,d\bar{z},$$

because $(dx,dy)=(-\sin\varphi,\,\cos\varphi)\,ds$ and

$$d\bar{z}=dx-idy=(-\sin\varphi-i\cos\varphi)ds=-i(\cos\varphi-i\sin\varphi)ds=-ie^{-i\varphi}ds.$$

Since by Bernoulli's principle (see Appendix C below)

$$\frac{1}{2}\rho v^2+p=\text{constant}$$

along a streamline (when the density of the fluid flow is assumed to be the constant ρ), it follows that

$$\bar{F}=\frac{i\rho}{2}\oint_C v^2 d\bar{z},$$

because

$$\oint_C d\bar{z}=0.$$

Let f be the complex vector potential so that the velocity vector $\vec{v} = (v_1, v_2)$ as a complex number $v_1 + iv_2$ is equal to $\overline{f'}$. Then $v^2 d\bar{z} = f'(z)^2 dz$, because $d\bar{z} = -ie^{-i\varphi} ds$ and $dz = ie^{i\varphi} ds$ and

$$\overline{f'} = \vec{v} = ve^{i(\varphi + \frac{\pi}{2})} = ive^{i\varphi}$$

so that

$$f'(z)^2 dz = \left(\overline{ive^{i\varphi}}\right)^2 dz = -v^2 e^{-2i\varphi} dz = v^2 d\bar{z}$$

and

$$\bar{F} = \frac{i\rho}{2} \oint_C f'(z)^2 dz.$$

Let

$$f'(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots$$

be the power series expansion of $f'(z)$ centered at $z = \infty$. We obtain c_0 as the value of $f'(z)$ at $z = \infty$ so that $\bar{c}_0$ is the fluid velocity $\vec{v}_\infty$ at infinity. We compute

$$\begin{aligned} c_1 &= \frac{1}{2\pi i} \oint_C f'(z) dz \\ &= \frac{1}{2\pi i} \oint_C (v_1 - iv_2)(dx + idy) \\ &= \frac{1}{2\pi i} \oint_C (v_1 dx + v_2 dy) + \frac{1}{2\pi} \oint_C (v_1 dy - v_2 dx). \end{aligned}$$

The last term on the right-hand side vanishes, because its integrand $v_1 dy - v_2 dx$ is the inner product of the velocity vector and the normal direction $(dy, -dx)$ of its integral curve C.

We define the counterclockwise *circulation* Γ of the fluid flow around C as

$$\Gamma = \oint_C (v_1 dx + v_2 dy),$$

which is $2\pi i$ times c_1. Though the fluid flow is assumed to be irrotational so that the circulation vanishes around any closed loop whose enclosure is completely inside the domain of the fluid flow, yet the closed curve C is the boundary of the domain of the fluid flow and the circulation Γ is in general not zero. From

$$f'(z)^2 = c_0^2 + \frac{2c_0 c_1}{z} + \cdots$$

and (for R greater than the distance from the origin to C with coordinate change $\zeta = \frac{1}{z}$)

$$\begin{aligned}\oint_C f'(z)^2 dz = \int_{|z|=R} f'(z)^2 dz &= -\int_{|\zeta|=\frac{1}{R}} f'\left(\frac{1}{\zeta}\right)^2 \left(\frac{-d\zeta}{\zeta^2}\right) \\ &= 2\pi i \operatorname{Res}_{\zeta=0} f'\left(\frac{1}{\zeta}\right)^2 \frac{1}{\zeta^2} = (2\pi i) 2c_0 c_1,\end{aligned}$$

it follows that

$$\bar{F} = \frac{i\rho}{2}\oint_C f'(z)^2 dz = \frac{i\rho}{2} 2\pi i\, 2c_0 c_1 = \frac{i\rho}{2} 2\pi i\, 2c_0 \frac{1}{2\pi i}\Gamma = i\rho c_0 \Gamma.$$

Finally the vertical lift force $\operatorname{Im} F$ is equal to

$$\operatorname{Re}(i\bar{F}) = \operatorname{Re}\left(\rho(-\Gamma)\vec{v}_\infty\right) = \rho(-\Gamma)(v_1)_\infty.$$

In words, the lift force is equal to the product of the density ρ, the *clockwise* circulation $(-\Gamma)$, and the *right-pointing* horizontal component $(v_1)_\infty$ of the fluid velocity at infinity. This statement is known as *the theorem of Kutta-Joukowski* (for this special situation).

Lifting Force of Rotating Cylinder from Theorem of Kutta-Joukowski. As an example, we compute the lifting force of a rotating cylinder by using the theorem of Kutta-Joukowski. A cylinder of radius r spinning clockwise at ω revolutions per unit time so that the tangential velocity at the boundary is $2\pi r\omega$. The clockwise circulation is $2\pi r$ times $2\pi r\omega$. If the constant fluid density is ρ and the fluid velocity is v_∞ from left to right. Then the lifting force on the rotating cylinder is $\rho v_\infty(2\pi r)(2\pi r\omega)$ per unit length of the cylinder. The clockwise spinning provides faster left-to-right fluid motion above the cylinder than below the cylinder, resulting in uplifting force by Bernoulli's principle.

To relate this to actual applications of this conclusion in the real world, we would like to mention that "rotor ships" use clockwise rotating vertical metal cylinders, instead of sails, for forward propulsion with wind from the left.

Uplifting Force on Joukowski Airfoil. We would like to apply the theorem of Kutti-Joukowski to determine the uplifting force of a horizontal air flow on the wing.

First, we consider the case of the airfoil given simply by a line segment with angle $-\alpha$ (for some $\alpha > 0$), for example, the line segment $[-2e^{-i\alpha},\, 2e^{-i\alpha}]$ with the fluid flow which at a very far away point behaves like the flow with constant velocity vector $(1, 0)$.

For the explicit computation of the complex potential function for the flow, the trivial case with $\alpha = 0$ is clearly described with the complex potential function $f(z) = z$ whose imaginary part y has constant value 0 on the the line segment $[-2, 2]$ which is a streamline defined by the imaginary part of the complex potential function being constant.

For the explicit computation of the general case, we cannot just apply a rotation of $z \mapsto e^{-i\alpha} z$ to change $[-2, 2]$ to $[-2e^{-i\alpha},\, 2e^{-i\alpha}]$, because the rotation changes also the complex potential function to make the flow behave like the flow with constant velocity vector $e^{-i\alpha} = (\cos\alpha,\, -\sin\alpha)$, instead of with the constant velocity vector $(1, 0)$. In order to make the idea of applying a rotation work, we use a biholomorphic map to map the complement of $[-2,\, 2]$ to the exterior of the closed unit disk and then apply a rotation to the flow for the exterior of the closed unit disk and then use a rotated biholomorphic map to get the explicit flow outside the line segment $[-2e^{-i\alpha},\, 2e^{-i\alpha}]$. We now implement the details.

For the biholomorphic map between the complement of $[-2,\, 2]$ and the exterior of the closed unit disk, we use the *Joukowski map*

$$w = J(z) = z + \frac{1}{z},$$

whose inverse $z \mapsto J^{-1}(w)$ can be computed, by solving the quadratic equation $z^2 - wz + 1 = 0$ for z in terms of w and is given by

$$z = J^{-1}(w) = \frac{1}{2}\left(w + \sqrt{w^2 - 4}\right),$$

where the branch $\sqrt{w^2 - 4}$ is defined on $\mathbb{C} - [-2, 2]$ whose value is positive for $w \in \mathbb{R} - [-2, 2]$. The flow with complex potential function $f_1(w) = w$ (whose velocity vector is $(1, 0)$) for the complement of $[-2, 2]$ in the w-plane, via the Joukowski map $z \mapsto w = J(z)$, corresponds to the flow with complex potential function

$$f_2(z) = f_1(J(z)) = z + \frac{1}{z}$$

(which at a very far away point behaves like the constant flow with the velocity vector $(1,0)$) for the exterior of the closed unit disk $\{|z| \leq 1\}$ in the z-plane. The change of the rotation $z \to e^{i\alpha}z$ yields the flow with complex potential function

$$f_3(z) = f_2(ze^{-i\alpha}) = e^{-i\alpha}z + \frac{1}{e^{-i\alpha}z}$$

(which at a very far away point behaves like the constant flow with the velocity vector $e^{i\alpha} = (\cos\alpha, \sin\alpha)$) for the exterior of the closed unit disk $\{|z| \leq 1\}$ in the z-plane. Note that $e^{-i\alpha}$ is used instead of $e^{i\alpha}$ because the velocity is given by the complex conjugate of the derivative of the complex velocity potential. Again, via the Joukowski map $z \mapsto w = J(z)$, this flow corresponds to the flow with complex potential function

$$f_4(w) = f_3(J^{-1}(w)) = \frac{e^{-i\alpha}}{2}\left(w + \sqrt{w^2-4}\right) + \frac{2}{e^{-i\alpha}\left(w + \sqrt{w^2-4}\right)}$$

for the complement of $[-2,2]$ in the w-plane. At a very far away point in the w-plane the flow behaves like the flow with constant velocity vector $e^{i\alpha} = (\cos\alpha, \sin\alpha)$. We now apply the rotation $w \mapsto e^{i\alpha}w$ (in the other direction) to get the flow to behave like the flow with constant velocity vector $(1,0)$ at a very far away point in the w-plane. We end up the flow

$$\begin{aligned}
f(w) &= f_4\left(e^{i\alpha}w\right) \\
&= \frac{e^{i\alpha}}{2}\left(e^{i\alpha}w + \sqrt{e^{-2i\alpha}w^2-4}\right) + \frac{2}{e^{i\alpha}\left(e^{i\alpha}w + \sqrt{e^{2i\alpha}w^2-4}\right)} \\
&= \frac{1}{2}\left(w + \sqrt{w^2-4e^{-2i\alpha}}\right) + \frac{2}{w + \sqrt{w^2-4e^{-2i\alpha}}} \\
&= \frac{1}{2}\left(w + \sqrt{w^2-4e^{-2i\alpha}}\right) + \frac{2\left(w - \sqrt{w^2-4e^{-2i\alpha}}\right)}{\left(w - \sqrt{w^2-4e^{-2i\alpha}}\right)\left(w + \sqrt{w^2-4e^{-2i\alpha}}\right)} \\
&= \frac{1}{2}\left(w + \sqrt{w^2-4e^{-2i\alpha}}\right) + \frac{e^{4i\alpha}}{8}\left(w - \sqrt{w^2-4e^{-2i\alpha}}\right) \\
&= \left(\frac{1}{2} + \frac{e^{4i\alpha}}{8}\right)w + \left(\frac{1}{2} - \frac{e^{4i\alpha}}{8}\right)\sqrt{w^2-4e^{-2i\alpha}}
\end{aligned}$$

for the complement of the line segment $[-2e^{-i\alpha},\, 2e^{-i\alpha}]$ in the w-plane, which behaves like the constant flow with the velocity vector $(1,0)$ at a point very far away in the w-plane. We have

$$f'(w) = \left(\frac{1}{2} + \frac{e^{4i\alpha}}{8}\right) - \left(\frac{1}{2} - \frac{e^{4i\alpha}}{8}\right) \frac{w}{\left(w^2 - 4e^{-2i\alpha}\right)^{\frac{1}{2}}}$$

and

$$\begin{aligned} f'\left(\frac{1}{w}\right) &= \left(\frac{1}{2} + \frac{e^{4i\alpha}}{8}\right) - \left(\frac{1}{2} - \frac{e^{4i\alpha}}{8}\right) \frac{\frac{1}{w}}{\left(\frac{1}{w^2} - 4e^{-2i\alpha}\right)^{\frac{1}{2}}} \\ &= \left(\frac{1}{2} + \frac{e^{4i\alpha}}{8}\right) - \left(\frac{1}{2} - \frac{e^{4i\alpha}}{8}\right) \frac{1}{\left(1 - 4e^{-2i\alpha}w^2\right)^{\frac{1}{2}}} \\ &= 1 + \left(\frac{1}{2} - \frac{e^{4i\alpha}}{8}\right) 2e^{-2i\alpha}w^2 + g(w^2) \end{aligned}$$

is even in w, where $g(w)$ is holomorphic in an open neighborhood of $w = 0$, so that the residue of the even meromorphic function

$$f'\left(\frac{1}{w}\right)^2 \frac{1}{w^2}$$

with an isolated singularity at $w = 0$ must be zero. We conclude the uplifting force (or even the total force) on the airfoil represented by a line segment with angle of attack α must be 0 from the above argument.

From the above argument, the conclusion of zero force can also be obtained more easily by subjecting the flow for the exterior of the closed unit disk in the z-plane to a rotation around the origin to get the magnitude of the force independent of the rotation and observing that the force is zero for $\alpha = 0$.

We expect to get a lifting force from the angle of attack on the airfoil given by the line-segment $[-2, 2]$, but the above argument yields zero lifting force. We would like to understand in what way this mathematical model does not fit the actual physical situation.

Uniqueness Problem for Harmonic Function on Unbounded Domain. The reason is that the solution we write down for the unbounded domain (which is outside a slit or a disk) is not unique. We need some additional physical assumption (which is a special case of what is known as the *Kutta condition*) to make the solution unique from which we can compute the lifting force.

Consider the case of a flow outside $\overline{\mathbb{D}} = \{|z| \leq 1\}$ with velocity $e^{i\alpha}$ at infinity. A general solution for the complex velocity potential $f(z)$ (with $\operatorname{Im} f$ constant on $|z| = 1$ and the velocity $\overline{f'(z)}$ at $z = \infty$ equal to $e^{i\alpha}$) is locally of the form

$$f(z) = e^{-i\alpha}z + \frac{1}{e^{-i\alpha}z} + ai \log z$$

(with a being a nonzero real number), because the additional contribution $a \log |z|$ from its imaginary part has zero value on the unit circle and its derivative vanishes at infinity. To determine the real number a, the additional Kutta condition is that *the velocity at the sharp rear end of the airfoil is zero.*

The sharp rear end of the airfoil is at $w = 2$ which means $z = 1$ from the relation $w = z + \frac{1}{z}$. Since

$$f'(z) = e^{-i\alpha} - \frac{1}{e^{-i\alpha}z^2} + \frac{ai}{z},$$

it follows that $f'(1) = 0$ implies that

$$a = \frac{e^{i\alpha} - e^{-\alpha}}{i} = 2 \sin \alpha$$

to give $c_0 = e^{-i\alpha}$ and $c_1 = ai = 2i \sin \alpha$ in the expansion

$$f'(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots .$$

With $\rho = 1$, we now compute the force F as a complex number by

$$\bar{F} = -2\pi\rho c_0 c_1 = -4\pi i e^{-i\alpha} \sin \alpha$$

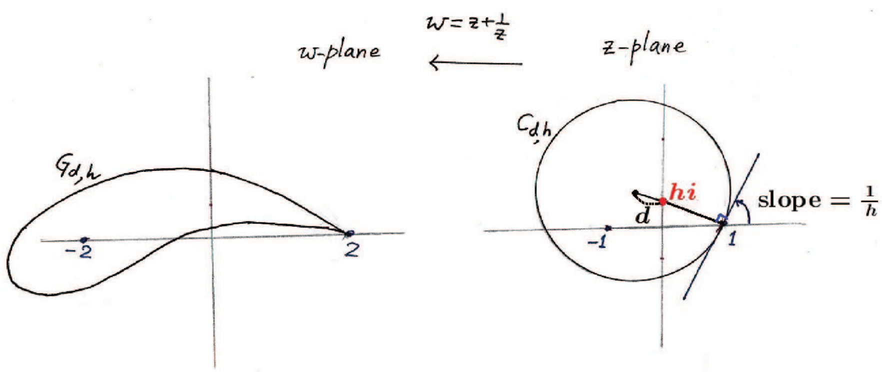
We have to use multiplication by $e^{-i\alpha}$ to rotate $[-2, 2]$ to $\mathbb{C} - [-2e^{-i\alpha}, 2e^{-i\alpha}]$. From this rotation and the expression for $\bar{F}$ we conclude that the vertical uplifting force is equal to $4\pi \sin \alpha$.

Airfoil of Joukowski. The above discussion on the uplifting force by a horizontal flow on an airfoil which is represented as a line segment $[-2e^{-i\alpha},\ -2e^{-i\alpha}]$. This representation may be good enough for the first airplane of the Wright brothers. For later airplanes, the representation of an airfoil is not just a slanted line segment, but has thickness which is not uniform, more rounded at the front and with a sharp rear end. Joukowski constructed such a representation by replacing the closed unit disk by some other closed disk in the

z-plane in the above argument. The *Joukowski airfoil* $G_{d,h}$ (with positive parameters d and h) is defined as the image, under the Joukowski transformation

$$z \mapsto z + \frac{1}{z},$$

of the closed disk $C_{d,h}$ which is tangential to the line through $z = 1$ with slope $\frac{1}{h}$ and whose center is of distance d from $z = hi$. The center of $C_{d,h}$ is on the line joining the two points $z = hi$ and $z = 1$. (see the figure below)



This definition is enough for the computation of the force on the Joukowski airfoil $G_{d,h}$ from the the irrotational incompressible fluid flow of constant density 1 outside the Joukowski airfoil $G_{d,h}$ with velocity $e^{i\alpha}$ at infinity, when the Kutta condition of the vanishing of the fluid velocity at its sharp rear end $z = 2$ of $G_{d,h}$ is assumed, because the Joukowski transformation

$$z \mapsto z + \frac{1}{z}$$

makes it possible to replace $G_{d,h}$ by $C_{d,h}$ and the case of $C_{d,h}$ can be obtained by modifying the case of a flow outside $\overline{\mathbb{D}}$. We are not carrying out the details here and let you do it in Problem 5 of Homework #8.

The following figure helps to better visualize the construction of Joukowski's airfoil $G_{d,h}$ by breaking down its construction into several steps. This visualization has nothing to do with the computation of the force on the Joukowski airfoil and is simply a geometric description of the behavior of maps.

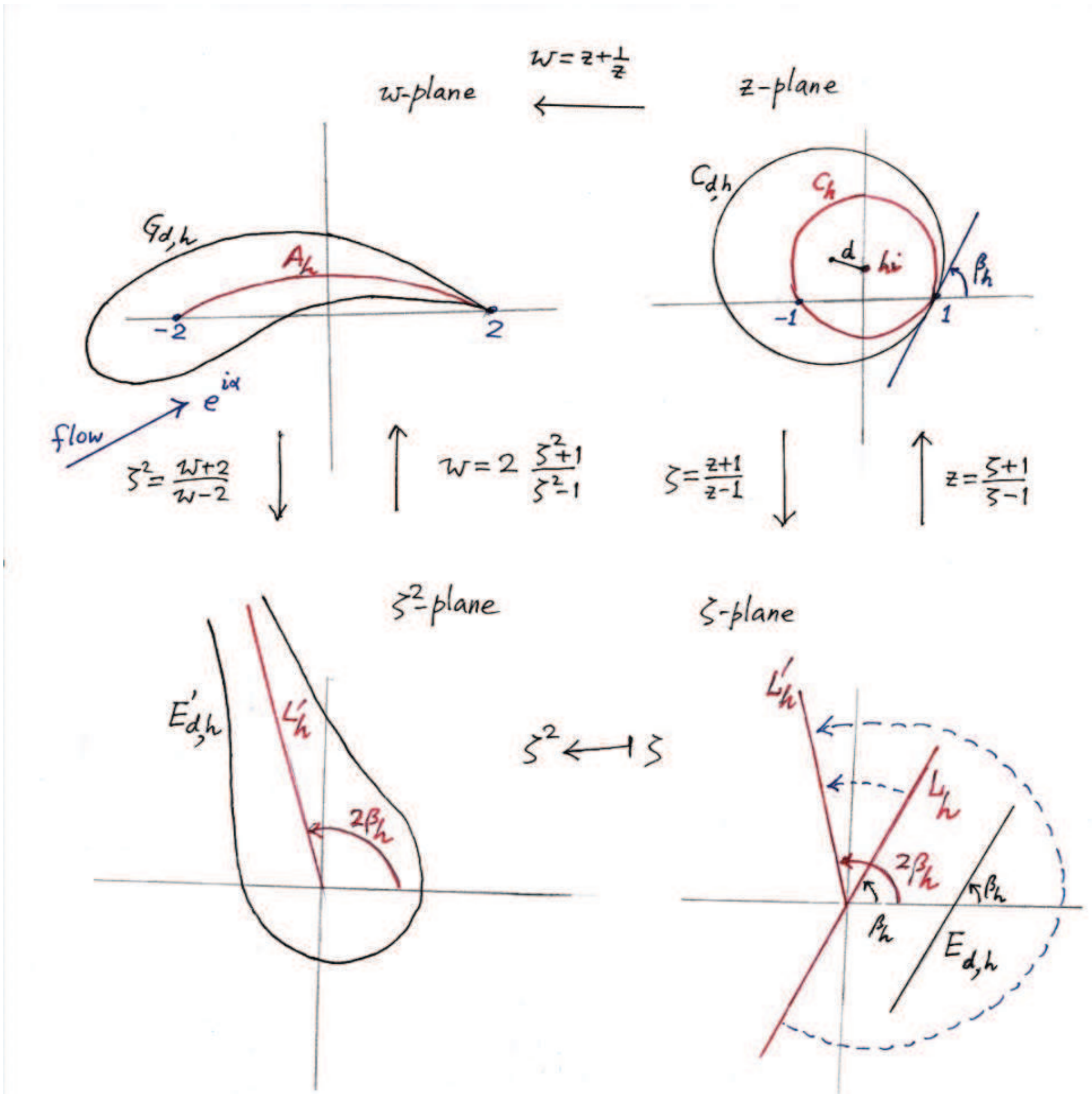


Figure 1: Maps used in the construction of Joukowski's airfoil

In the upper right-hand corner, C_h is the circle in the z-plane with center hi passing through -1 and 1 (with $h \geq 0$). The tangent of the angle at hi of the triangle with vertices $0, hi, 1$ is $\frac{1}{h}$, which is the same as the slope of the line tangent to the circle C_h at the point $z = 1$ (because the angle between the tangent line at $z = 1$ and the real axis is equal to the angle at hi of the triangle with vertices $0, hi, 1$ from the fact that both are complementary to the angle at 1 of the triangle with vertices $0, hi, 1$). We would like to determine the image of the exterior of C_h under the biholomorphic map $z \mapsto w = z + \frac{1}{z}$. The case of the unit circle is simply the special case of $h = 0$. The key to determine the image of C_h is to rewrite the relation $w = z + \frac{1}{z}$ as

$$\left(\frac{z+1}{z-1}\right)^2 = \frac{w+2}{w-2},$$

as we discussed earlier. We set

$$\zeta = \frac{z+1}{z-1}$$

to decompose the relation $w = z + \frac{1}{z}$ into the square map $\zeta \mapsto \zeta^2$ and another linear fractional transformation

$$w \mapsto \zeta^2 = \frac{w+2}{w-2}.$$

The circle C_h is mapped by

$$z \mapsto \zeta = \frac{z+1}{z-1}$$

to the line L_h in the ζ-plane (in the lower right-hand corner) which passes the origin with slope equal to $\frac{1}{h}$ for the following reason.

Since the coefficients of the map

$$z \mapsto \zeta = \frac{z+1}{z-1}$$

are real, the map maps the real line to the real line. The point $z = 1$ on the circle C_h is mapped to infinity, the circle C_h is mapped to a line. The point $z = -1$ which is on the circle is mapped to 0. The angle β_h made by the circle C_h and the real line is equal to $\tan^{-1}\left(\frac{1}{h}\right)$. Since the point ∞ is mapped to 1 by the map, the exterior of the circle C_h is mapped to the open half plane Ω_h in the ζ-plane which is on the side of L_h containing the point 1, *i.e.,* the open half plane

$$\Omega_h = \left\{ \zeta \in \mathbb{C} \,\middle|\, \beta_h - \pi < \arg \zeta < \beta_h \right\}.$$

The square map $\zeta \mapsto \zeta^2$ maps the closed half-line L_h to the closed half-line

$$L'_h = \{0\} \cup \left\{ \zeta \in \mathbb{C} - \{0\} \,\middle|\, \arg \zeta = 2\beta_h \right\}$$

and maps Ω_h to the complement of the closed half-line L'_h, with the image of the line L_h covering the half line L'_h twice. Here, we use the ζ^2-plane (in the lower left-hand corner), because we do not want to introduce another complex variable to denote ζ^2 to avoid the distraction of an additional complex variable. Finally, the transformation

$$\zeta^2 \to w = 2\frac{\zeta^2 + 1}{\zeta^2 - 1}$$

with real coefficients maps the closed half-line L'_h to the arc A_h in the w-plane (in the upper left-hand corner) which contains the points $w = -2$ (corresponding $\zeta^2 = 0$) and $w = -2$ (corresponding $\zeta^2 = \infty$) and makes an angle $2\beta_h$ with the real line in the w-plane.

Our earlier case of a line-segment airfoil is the case of $h = 0$. The case of a general positive h replaces the line-segment $[-2, 2]$ in the w-plane (which is covered twice by the image of the unit circle from the z-plane) by the arc A_h in the z-plane which is covered twice by the image of the circle C_h from the z-plane, once by the part of the circle C_h above the real axis of the z-plane and once by the part of the circle C_h below the real axis of the z-plane. As h increases, the part of the circle C_h above the real axis of the z-plane is increasingly greater than the part of the circle C_h below the real axis of the z-plane, but both cover the same arc A_h in the w-plane. The arc A_h can be used as an airfoil, but it is still of real dimension 1.

In order to thicken it to give a round front end and a sharp rear end, Joukowki replaced the circle C_h in the z-plane by another circle $C_{h,d}$ in the z-plane which touches C_h at the point $z = 1$ and encloses C_h inside and whose center is of distance $d > 0$ from hi. Since $C_{h,d}$ contains $z = 1$ which is being mapped by

$$z \mapsto \zeta = \frac{z + 1}{z - 1}$$

(with real coefficients) to $\zeta = \infty$, the image of $C_{h,d}$ in the ζ-plane is a straight line $E_{h,d}$ which is parallel to L_h and makes an angle β_h with the real axis of ζ-plane. The image $E'_{h,d}$ in the ζ^2-plane of the straight line $E_{h,d}$ under the square map $\zeta \mapsto \zeta^2$ is now a curve which, as a point on E_h comes up from the bottom to the top, is traced by a point which starts on the left of L'_h from the top down the left of L' and goes around the origin in the counterclockwise sense to the right of L'_h and then goes up on the right to disappear at the top.

The transformation
$$\zeta^2 \to w = 2\frac{\zeta^2 + 1}{\zeta^2 - 1}$$
with real coefficients maps the curve $E'_{h,d}$ from the ζ^2-plane to the boundary of the Joukowski airfoil $G_{h,d}$ in the w-plane. The front of the Joukowski airfoil $G_{h,d}$ is its round left end going around $w = -2$ (which is the image of $z = -1$ and $\zeta = 0$) in the counterclockwise sense while its sharp rear end on its right is tangential to the arc A_h at the point $w = 2$ (which is the image of $z = 1$ and $\zeta = \infty$).

APPENDIX A: Hyperbola as Locus of a Point With Difference of Distances to Two Fixed Points Kept Constant

Let $c > a > 0$. Consider the locus of a point $P = (x, y)$ constrained by the condition that the distance between $P = (x, y)$ and $(c, 0)$ and the distance between $P = (x, y)$ and $(-c, 0)$ is always kept constant to be $2a$ or $-2a$. The equation of the locus of the point $P = (x, y)$ is given by

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

Moving the second term on the left-hand side to the right-hand side and squaring both sides, we get

$$\left((x+c)^2 + y^2\right) = 4a^2 \pm 4a\sqrt{(x-c)^2 + y^2} + \left((x-c)^2 + y^2\right).$$

Moving the last term and the first term on the right-hand side to the right-hand side yields

$$4cx - 4a^2 = \pm 4a\sqrt{(x-c)^2 + y^2}.$$

Dividing the equation by $4ac$ and squaring it yields

$$\left(\frac{x}{a} - \frac{a}{c}\right)^2 = \frac{(x-c)^2 + y^2}{c^2}.$$

After expanding the squares, we get

$$\frac{x^2}{a^2} - \frac{2x}{c} + \frac{a^2}{c^2} = \frac{x^2 - 2cx + c^2 + y^2}{c^2}$$

or

$$\frac{x^2}{a^2} + \frac{a^2}{c^2} = \frac{x^2}{c^2} + 1 + \frac{y^2}{c^2}.$$

Rearranging the terms, we get

$$\left(\frac{1}{a^2} - \frac{1}{c^2}\right)x^2 - \frac{y^2}{c^2} = 1 - \frac{a^2}{c^2}.$$

We let $b = \sqrt{c^2 - a^2}$ so that $b^2 = c^2 - a^2$ and we can rewrite the equation as

$$\frac{b^2}{a^2c^2}x^2 - \frac{y^2}{c^2} = \frac{b^2}{c^2}$$

which is equivalent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

APPENDIX B: Real Part of Arc Sine Computed by Characterization of Hyperbola as Locus of a Point With Difference of Distances to Two Fixed Points Kept Constant

Consider the function $z = \sin w$. We would like to compute the imaginary part of its inverse $w = \sin^{-1} z$ with a choice of a branch which we will specify later. First of all we observe that

$$\sin iy = \frac{1}{2i}\left(e^{i(iy)} - e^{-i(iy)}\right) = i\,\frac{1}{2}\left(e^{y} - e^{-y}\right) = i\,\sinh y$$

and

$$\cos iy = \frac{1}{2}\left(e^{i(iy)} + e^{-i(iy)}\right) = \frac{1}{2}\left(e^{y} - e^{-y}\right) = \cosh y.$$

As a consequence,

$$\begin{aligned} x + iy = \sin w &= \sin(u + iv) \\ &= \sin u \cos iv + \cos u \sin iv \\ &= \sin u \cosh v + i \cos u \sinh v. \end{aligned}$$

Separating the real parts and the imaginary parts, we get

$$\begin{cases} x = \sin u \cosh v \\ \\ y = \cos u \sinh v \end{cases}$$

Eliminating v by using the identity

$$\cosh^2 v - \sinh^2 v = 1,$$

we get

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1.$$

Let $a = \sin u$, $b = \cos u$ and $c = \sqrt{a^2 + b^2} = 1$. An equivalent description of this hyperbola is

$$\sqrt{(x+1)^2 + y^2} - \sqrt{(x+1)^2 + y^2} = 2 \sin u$$

which yields

$$u = \sin^{-1} \frac{1}{2}\left(\sqrt{(x+1)^2 + y^2} - \sqrt{(x+1)^2 + y^2}\right).$$

APPENDIX C: Bernoulli's Principle

We introduce and prove Bernoulli's Principle for our very special situation. Let $\vec{v}(x,y,t)$ be the velocity vector at the location (x,y,t) as observed by somebody stationed at (x,y) (*i.e.,* Eulerian description of the flow). For an observer who travels alongside a fluid particle (*i.e,.* Lagrangian description of the flow), the rate of change of velocity $\frac{d\vec{v}}{dt}$ is given by the chain rule

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + \frac{\partial\vec{v}}{\partial x}\frac{dx}{dt} + \frac{\partial\vec{v}}{\partial y}\frac{dy}{dt} = \frac{\partial\vec{v}}{\partial t} + (\nabla\vec{v})\cdot\vec{v},$$

because the two components of the velocity vector $\vec{v}$ are precisely $\frac{dx}{dt}$ and $\frac{dy}{dt}$ when $t \mapsto (x(t), y(t))$ is the fluid flow in its Lagrangian description. When the fluid flow is steady, the velocity vector $\vec{v}(x,y,t)$ in the Eulerian description is independent of t so that

$$\frac{d\vec{v}}{dt} = (\nabla\vec{v})\cdot\vec{v}.$$

Suppose the internal pressure density of the fluid flow is $p(x,y,t)$ which is assumed to be steady and is therefore independent of t. It means that the force from the pressure on a body in the flow equals the integral of $-p\vec{n}\,ds$ over the boundary of the body, where $\vec{n}$ is the outward-pointing unit normal vector and ds is the arc-length element of the boundary. Let the density of incompressible fluid be the constant 1. When we consider a small rectangle of fluid particles with vertices at

$$(x,y),\ (x+\Delta x,\ y),\ (x,\ y+\Delta y),\ (x+\Delta x,\ y+\Delta y),$$

the force on the vertical side with abscissa $x+\Delta x$ is approximately

$$-p(x+\Delta x, y)\Delta y$$

(up to an error of higher order) while the force on the vertical side with abscissa x is approximately

$$p(x+\Delta x, y)\Delta y$$

(up to an error of higher order), which means that the net horizontal force is approximately $-\frac{\partial p}{\partial x}(\Delta x)(\Delta y)$. Since the mass of the small rectangle of fluid particles is approximately $(\Delta x)(\Delta y)$, by Newton's second law,

$$-\nabla p = \frac{d\vec{v}}{dt},$$

which means that
$$(\nabla \vec{v}) \cdot \vec{v} = -\nabla p.$$
We can rewrite it as
$$\nabla \left(\frac{\|\vec{v}\|^2}{2} \right) = -\nabla p,$$
because
$$\frac{\partial}{\partial x} \|\vec{v}\|^2 = \frac{\partial}{\partial x} \left(u^2 + v^2 \right) = 2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 2 \left(\frac{\partial}{\partial x} \vec{v} \right) \cdot \vec{v}$$
and
$$\frac{\partial}{\partial y} \|\vec{v}\|^2 = \frac{\partial}{\partial y} \left(u^2 + v^2 \right) = 2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 2 \left(\frac{\partial}{\partial y} \vec{v} \right) \cdot \vec{v}$$
so that
$$\nabla \left(\frac{\|\vec{v}\|^2}{2} \right) = (\nabla \vec{v}) \cdot \vec{v}.$$
The vanishing of the gradient for $\frac{\|\vec{v}\|^2}{2} + p$ means that
$$\frac{\|\vec{v}\|^2}{2} + p = c$$
for some constant c. This relation between pressure and velocity for fluid flow is known as *Bernoulli's principle* (for our very special situation).