Holomorphicity Characterized by Conformality and Preservation of Orientation

This course covers a number of application of complex analysis. The first application is the computation of certain definite integrals and infinite sums by residue theory. From the computation of infinite sums, we get also partial fraction decomposition and infinite produce decomposition of certain meromorphic functions on $\mathbb{C}$. We now go into the second application which is the application of conformal mappings to fluid flow, temperature distribution, and electrostatic potential. Fluid flow includes the theory of lifting force for airplane wings with certain cross sections constructed by using conformal mappings. Before we do the applications of conformal mappings, we first discuss the definition of conformal mappings and their characterization by the conformal factor in the pullback of a metric. Along that line, a holomorphic function is characterized by its conformal property and the preservation of orientation. We give the two historical examples of the making of navigation maps by stereographic projection and Mecarto projection. In discussing stereographic projections, we look at the complex structure of the Riemann sphere (which is the simplest case of a Riemann surface) and the extended Gauss plane $\mathbb{C} \cup \{\infty\}$. Later in the course we will look at other applications of complex analysis to number theory and special functions like the elliptic functions and theta functions. In this set of lectures, we focus on the characterization of conformal mappings in terms of pullbacks of metrics and the characterization of holomorphic functions by conformality and preservation of orientation and the discussion of stereographic and Mercato projections.

Definition (Conformal Property of Smooth Diffeomorphism between Plane Domains). Let $\Phi = (\varphi, \psi) : U \to W$ with $u = \varphi(x, y)$ and $v = \psi(x, y)$ be a smooth diffeomorphism between two open neighborhoods U and W of the origin in $\mathbb{R}^2$, which maps the origin to itself. The map Φ is said to be *conformal* at 0 is the angle at 0 between two smooth nonsingular curves in U is equal to the angle at 0 of their two image curves in W.

Euclidean Metric of the Plane and its Pullback by Diffeomorphism. The Euclidean metric of $\mathbb{R}^2$ with coordinates u, v is $du^2 + dv^2$. It measures the length a curve C in $\mathbb{R}^2$ with paramterization $u = u(t), v = v(t)$ for $a \leq t \leq b$

by the formula

$$\text{length of } \ C = \int_C \sqrt{du^2 + dv^2} = \int_{t=a}^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

When the coordinates (u, v) are replaced by local coordinates (x, y) related to (u, v) by Φ, the metric $du^2 + dv^2$ becomes

$$\begin{aligned} &(u_x dx + u_y dy)^2 + (v_x dx + v_y dy)^2 \\ &= E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2, \end{aligned}$$

where $E(x, y) = u_x^2 + v_x^2$, $F(x, y) = u_x v_x + u_y v_y$, and $G(x, y) = v_x^2 + v_y^2$. We can consider $Edx^2 + 2Fdxdy + Gdy^2$ as the pullback of the metric $du^2 + dv^2$ by the map Φ. In notations,

$$\Phi^*(du^2 + dv^2) = Edx^2 + 2Fdxdy + Gdy^2.$$

Geometrically, E is the length of the vector (u_x, v_x) which is the Φ-image of the vector $(1, 0)$, G is the length of the vector (u_y, v_y) which is the Φ-image of the vector $(0, 1)$, and F is the inner product of the two vectors (u_x, v_x) and (u_y, v_y).

Theorem (Conformality Characterized by Pullback Metric of Euclidean Metric). Φ is conformal at 0 if and only if $E = G$ and $F = 0$ in $\Phi^*(du^2 + dv^2) = Edx^2 + 2Fdxdy + Gdy^2$ at 0. In other words, $(x, y) \mapsto (u, v)$ is conformal at 0 if and only if $u_x^2 + v_x^2 = u_y^2 + v_y^2$ and $u_x v_x + u_y v_y = 0$, in which case E is known as the *conformal factor*.

Representation of Vector by Directional Derivative. Before we give the proof, we introduce a way of identifying a vector $\vec{\xi} = (\alpha, \beta)$ in $\mathbb{R}^2$ with the directional derivative

$$\vec{\xi} = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}.$$

This identification is a coordinate-free way of describing a vector. It offers an easier, less error-prone way of keeping track of the various kinds of partial differentiation when the chain rule is applied to changing a set of variables to another set of variables. Given another local coordinate system (u, v), a component of the vector $\vec{\xi}$ is equal to the result of applying it to the corresponding coordinate so that

$$\vec{\xi} = (\vec{\xi}(u)) \frac{\partial}{\partial x} + (\vec{\xi}(v)) \frac{\partial}{\partial v}.$$

Proof of Theorem on Conformality Characterized by Pullback Metric of Euclidean Metric. First, we prove the "only if" part. Suppose Φ is conformal at 0. Consider the pair of orthogonal vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Their images under Φ are the same as their respectively representations in the coordinate system (u, v), which means that

$$u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v}$$

and

$$u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v}$$

respectively. The orthogonality of the two image vectors means

$$u_x u_y + v_x v_y = 0.$$

That is $F = 0$. We can consider another pair of orthogonal vectors

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

whose images under Φ are

$$(u_x + u_y)\frac{\partial}{\partial u} + (v_x + v_y)\frac{\partial}{\partial v}$$

and

$$(u_x - u_y)\frac{\partial}{\partial u} + (v_x - v_y)\frac{\partial}{\partial v}$$

respectively. The orthogonality of the two image vectors means

$$(u_x^2 - u_y^2) + (v_x^2 - v_y^2) = 0,$$

which is the same as

$$u_x^2 + v_x^2 = u_y^2 + v_y^2.$$

That is $E = G$. This finishes the proof of the "only if" part.

For the proof of the "if" part, take two tangent vectors (α, β) and (γ, δ) in $\mathbb{R}^2$ at 0. The cosine of the angle made by these two vectors is equal to their inner product divided by the product of their lengths, which means

$$\frac{\alpha\gamma + \beta\delta}{\sqrt{\alpha^2 + \beta^2}\sqrt{\gamma^2 + \delta^2}}.$$

The images of (α, β) and (γ, δ) are

$$\left(\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\right)u\right)\frac{\partial}{\partial u}+\left(\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\right)v\right)\frac{\partial}{\partial v}$$

and

$$\left(\left(\gamma\frac{\partial}{\partial x}+\delta\frac{\partial}{\partial y}\right)u\right)\frac{\partial}{\partial u}+\left(\left(\gamma\frac{\partial}{\partial x}+\delta\frac{\partial}{\partial y}\right)v\right)\frac{\partial}{\partial v},$$

or, when expressed in the form of of a pair of components in terms of (u, v)-coordinates,

$$(\alpha u_x+\beta u_y,\, \alpha v_x+\beta v_y) \quad \text{and} \quad (\gamma u_x+\delta u_y,\, \gamma v_x+\delta v_y)$$

whose inner product is

$$\begin{aligned}&(\alpha u_x+\beta u_y)(\gamma u_x+\delta u_y)+(\alpha v_x+\beta v_y)(\gamma v_x+\delta v_y)\\ &=\alpha\gamma(u_x^2+v_x^2)+(\alpha\gamma+\beta\delta)(u_xu_y+v_xv_y)+\beta\delta(u_y^2+v_y^2)\\ &=E(\alpha\gamma+\beta\delta).\end{aligned}$$

The length of $(\alpha u_x+\beta u_y,\, \alpha v_x+\beta v_y)$ is the square root of the inner product with itself and therefore is equal to $\sqrt{E(\alpha^2+\beta^2)}$. Likewise, the length of $(\gamma u_x+\delta u_y,\, \gamma v_x+\delta v_y)$ is the square root of the inner product with itself and therefore is equal to $\sqrt{E(\gamma^2+\delta^2)}$. Hence the cosine of the angle made by

$$(\alpha u_x+\beta u_y,\, \alpha v_x+\beta v_y) \quad \text{and} \quad (\gamma u_x+\delta u_y,\, \gamma v_x+\delta v_y)$$

is equal to

$$\frac{E(\alpha\gamma+\beta\delta)}{\sqrt{E(\alpha^2+\beta^2)}\sqrt{E(\gamma^2+\delta^2)}}=\frac{\alpha\gamma+\beta\delta}{\sqrt{\alpha^2+\beta^2}\sqrt{\gamma^2+\delta^2}}.$$

Thus, the angle between (α, β) and (γ, δ) is equal to the angle between their images under Φ. This finishes the proof of the "if" part. Q.E.D.

Theorem (Holomorphicity Characterized by Conformality and Presevation of Orientation). The map $\Phi : (x, y) \mapsto (u, v)$ is conformal and orientation-preserving at 0 if and only the function $w = f(z)$ with $z = x + iy$ and $w = u + iv$ is holomorphic at $z = 0$.

Proof. First, we do this "only if" part. Suppose the map $\Phi : (x, y) \mapsto (u, v)$ is conformal and orientation-preserving at 0. At 0, the vector $\frac{\partial}{\partial x}$ is mapped to $u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v}$ by Φ whose length is $\sqrt{u_x^2 + v_x^2} = E$. Likewise, the vector $\frac{\partial}{\partial y}$ is mapped to $u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v}$ by Φ whose length is $\sqrt{u_y^2 + v_y^2} = E$. The two lengths are the same. Since $\frac{\partial}{\partial y}$ is obtained by rotating $\frac{\partial}{\partial x}$ by 90 degrees and since their images under Φ have the same length, by conformality and preservation of orientation of Φ, the image of $\frac{\partial}{\partial y}$ under Φ is obtained by rotating by 90 degrees the image of $\frac{\partial}{\partial x}$ under Φ. It means that $u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v}$ is equal to $v_x \frac{\partial}{\partial u} - u_x \frac{\partial}{\partial v}$, which means $u_y = v_x$ and $v_y = -u_x$. Thus the Cauchy-Riemann equations are satisfied at 0 and $w = f(z)$ is holomorphic at $z = 0$, from the assumption that both u and v are smooth functions of x and y.

For the proof of the "if" part, we assume that $w = f(z)$ is holomorphic at $z = 0$. Then the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ hold. This implies that $u_x^2 + v_x^2 = u_y^2 + v_y^2$ and $u_x u_y + v_x v_y = 0$. Moreover, the Jacobian determinant

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

is equal to $u_x^2 + v_x^2 > 0$ from the Cauchy equations $v_y = u_x$ and $u_y = -v_x$. The positivity of the Jacobian determinant means the preservation of the orientation. One can also see this from

$$du \wedge dv = \frac{\partial(u, v)}{\partial(x, y)} (dx \wedge dy) .$$

This finishes the proof of the "if" part. Q.E.D.

Remark. The map of complex reflection $z \mapsto \bar{z}$ in $\mathbb{C}$ is conformal but not orientation preserving. This shows that the two properties, conformality and preservation of orientation, are independent.

Stereographic Projection of Map-Making with Minimal Distortion Near South Pole. The unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ with coordinates (x, y, z) represents the earth. The map is to be made on the (u, v)-plane $\Pi := \{z = -1\}$ which touches the unit sphere at the south pole S defined by $(x, y, z) = (0, 0, -1)$. The stereographic projection π_N uses the north pole $N = (0, 0, 1)$ as the light-source and the (u, v)-plane Π as the target screen. Any point $P = (x, y, z)$ on the unit sphere other than the north pole N is mapped to the point $Q=(u,v)$ on Π such that the three points N, P, Q are collinear. The map-making from the stereographic projection π_N minimizes the distortion near the south pole S.

Let $P'=(x,y,0)$. From the similar triangles $\triangle NPP'$ and $\triangle NQS$, we get the formula

$$u=\frac{2x}{1-z} \quad \text{and} \quad v=\frac{2y}{1-z}.$$

Straightforward computation yields

$$du^2+dv^2=\frac{4}{(1-z)^2}\left(dx^2+dy^2+dz^2\right),$$

which shows that the steregraphic projection π_N is conformal, with conformal factor $\frac{4}{(1-z)^2}$. The details of the straightforward computation are as follows. From

$$du=\frac{2dx}{1-z}+\frac{2xdz}{(1-z)^2} \quad \text{and} \quad dv=\frac{2dy}{1-z}+\frac{2ydz}{(1-z)^2}$$

and

$$\begin{cases} du^2=\frac{4}{(1-z)^4}\left((1-z)^2dx^2+2x(1-z)dxdz+x^2dz^2\right) \\[2ex] dv^2=\frac{4}{(1-z)^4}\left((1-z)^2dy^2+2y(1-z)dydz+y^2dz^2\right), \end{cases}$$

by using $xdx+ydy+zdz=0$ and $x^2+y^2+z^2=1$ we conclude that

$$du^2+dv^2=\frac{4}{(1-z)^4}\left((1-z)^2(dx^2+dy^2)+2z(1-z)dz^2+(1-z^2)dz^2\right).$$

Finally, from

$$-2z(1-z)+(1-z^2)=-2z(1-z)+(1-z)(1+z)=(1-z)^2$$

it follows that

$$du^2+dv^2=\frac{4}{(1-z)^2}\left(dx^2+dy^2+dz^2\right).$$

We now look at how the orientation behaves under the stereographic projection. The orientation on a surface in $\mathbb{R}^3$ is determined by the choice of the direction of the normal according to the right-hand rule. The counter-clockwise orientation on the (u,v)-plane is given by the normal vector pointing upward.

By considering the situation near the south pole, we know that for the stereographic projection π_N to preserve the orientation, the orientation chosen for the unit sphere must be given by its normal vector pointing inward to the origin of the unit sphere. If there is a coordinate chart on the unit sphere near the south pole such that with respect to such a coordinate chart the stereographic projection π_N is both conformal and orientation-preserving, then π_N is holomorphic with respect to such a coordinate chart on the unit sphere and the coordinate $u + iv$. Of course, no such coordinate chart is defined on the unit sphere near its south pole. We can turn around and use the stereographic projection π_N and the coordinate $u+iv$ on the (u, v)-plane to define a coordinate chart on the unit sphere minus its north pole.

We can exchange the roles of the north pole N and the south pole S to define a stereographic projection π_S which uses the south pole as the light source and the plane $\Pi' := \{z = 1\}$ as the target screen. Use the coordinates ξ, η to denote x, y on the target screen $\Pi' := \{z = 1\}$. For any point P on the unit sphere other than the south pole, the image $Q' := \pi_S(P)$ on Π' is defined so that the three points S, P, Q' are collinear. The formula for the coordinate (ξ, η) of $Q' := \pi_S(P)$ on Π' is

$$\xi = \frac{2x}{z+1} \quad \text{and} \quad \eta = \frac{2y}{z+1}.$$

The stereographic projection π_S is conformal and orientation-preserving if the orientation for the target screen $\Pi' := \{z = -1\}$ is defined by its normal vector pointing downward, to be consistent with the inward-pointing normal vector of the unit sphere at the north pole. We can define a coordinate chart on the unit sphere minus its south pole by using the stereographic projection π_S and the coordinate $\xi + i\eta$ on the (ξ, η)-plane. On the unit sphere minus both the north and south poles we have two coordinate charts $u+iv$ and $\xi+i\eta$. Using both stereographic projections π_N and π_S, we have a holomorphic map between $w = u + iv$ and $\zeta = \xi - i\eta$ when the complex variable $\zeta = \xi - i\eta$ is used instead of $\xi + i\eta$ because the orientation for the (ξ, η)-plane should be from a counterclockwise rotation of 90 degrees from the vector $\frac{\partial}{\partial\xi}$ to the vector $\frac{\partial}{\partial\eta}$.

This holomorphic map between the two coordinate charts $w = u+iv$ and $\zeta = \xi - i\eta$ is given by $\zeta = \frac{1}{w}$, as one can see from the following argument in plane Euclidean geometry of using two similar right-angled triangles. The plane to which Euclidean geometry of similar triangles is applied is the vertical plane containing the vertical axis NS of the unit sphere. Since NS is a diameter of the unit sphere, $\angle SNP$ is a right angle. Both $\angle SNQ'$ and $\angle QNS$ are right angles. The two angles $\angle SNQ$ and $\angle NQ'S$ are equal, because they are both complementary to the angle $\angle NSP$. Thus, the triangle $\triangle SNQ'$ is similar to $\triangle QSN$. Since the corresponding ratios of lengths of sides of the two triangles are equal, we have

$$\frac{\overline{SN}}{\overline{NQ'}} = \frac{\overline{QS}}{\overline{SN}}$$

and

$$\overline{NQ'} \cdot \overline{QS} = \overline{SN} \cdot \overline{SN} = 1.$$

Thus, $\sqrt{\xi^2 + \eta^2}\,\sqrt{u^2 + v^2} = 1$, or $|\zeta|\,|w| = 1$. Since both $Q' = \bar{\zeta}$ and $Q = w$ are on the vertical plane containing the vertical axis NS of the unit sphere, it follows that $\zeta = \frac{1}{w}$.

The unit sphere with the two holomorphically related coordinate charts w, ζ is the *Riemann sphere*. In general, any real 2-dimensional surface which is covered by holomorphically related coordinate charts is called a *Riemann surface*. Using such holomorphically related coordinate charts one can do complex analysis on any Riemann surface. Instead of going through the geometric process of stereographic projections from the north pole and the south pole, one can also introduce the Riemann sphere in the very simple way of defining it as the *extended Gauss plane* $\mathbb{C} \cup \{\infty\}$ with ∞ given by the point of the vanishing of the coordinate $w = \frac{1}{z}$ when z is the coordinate of the Gauss plane $\mathbb{C}$.

Mercato Projection of Map-Making with Minimal Distortion Near Equator. The unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ with coordinates represents the earth. The map is to be made on the vertical cylinder $x^2 + y^2 = 1$ which is tangential to the unit sphere at points of the equator of the unit sphere. We use the longitude λ and the latitude φ of a point on the unit sphere to denote its position. We use the longitude λ and the height z of a point on the cylinder to denote its position. The Mercato projection introduces a smooth bijective function $\varphi \mapsto z$ so that the diffeomorphism $(\lambda, \varphi) \mapsto (\lambda, z)$ between the unit sphere minus its two poles and the cylinder is conformal. The map-making from the Mercato projection minimizes the distortion near the equator.

The metric on the unit sphere (which is induced by the Euclidean metric $dx^2 + dy^2 + dz^2$ on $\mathbb{R}^3$) is given by

$$\cos^2 \varphi d\lambda^2 + d\varphi^2.$$

It can be verified straightforwardly as follows, by using the spherical coordinates of the unit sphere in terms of the Euler angles λ, φ. From

$$\begin{cases} x = \cos \varphi \cos \lambda \\ y = \cos \varphi \sin \lambda \\ z = \sin \varphi \end{cases}$$

and

$$\begin{cases} dx = -\sin \varphi \cos \lambda d\varphi - \cos \varphi \sin \lambda d\lambda \\ dy = -\sin \varphi \sin \lambda d\varphi + \cos \varphi \cos \lambda d\lambda \\ dz = \cos \varphi d\varphi \end{cases}$$

it follows (with the cancellation of the coefficients of $d\varphi\, d\lambda$) that the metric $dx^2 + dy^2 + dz^2$ on the unit sphere is equal to

$$\left(\sin^2 \varphi \cos^2 \lambda + \sin^2 \varphi \sin^2 \lambda\right) d\varphi^2 + \left(\cos^2 \varphi \sin^2 \lambda + \cos^2 \varphi \cos^2 \lambda\right) d\lambda^2 + \cos^2 \lambda d\lambda^2$$

which is simplified to be

$$\cos^2 \varphi d\lambda^2 + d\varphi^2.$$

An intuitive, yet unrigorous, way of seeing this is to argue that the radius of the horizontal circle on the unit sphere which contains the point (λ, φ) is $\cos \varphi$. The contribution to this metric from the change $d\lambda$ of the longitude comes form this horizontal circle of radius $\cos \varphi$ and is therefore $\cos^2 \varphi d\lambda^2$. This, together with the contribution $d\varphi^2$ from the change $d\varphi$ of the latitude, provides the metric

$$\cos^2 \varphi d\lambda^2 + d\varphi^2$$

on the unit sphere.

Our task is to construct a a smooth bijective function $\varphi \mapsto z$ so that the diffeomorphism $(\lambda, \varphi) \mapsto (\lambda, z)$ between the unit sphere minus its two poles and the cylinder is conformal. It means that

$$d\lambda^2 + dz^2 = d\lambda^2 + \left(\frac{dz}{d\varphi}\right)^2 d\varphi^2$$

should be equal to

$$E\left(\cos^2\varphi d\lambda^2 + d\varphi^2\right)$$

for some conformal factor E. Comparing the coefficients of $d\lambda^2$, we conclude that E should be set to be $\sec^2\varphi$. It means that $\frac{dz}{d\varphi}$ should be set to be $\sec\varphi$ so that $z = \log(\tan\varphi + \sec\varphi)$. The Mercato projection is given by $(\lambda, \varphi) \mapsto (\lambda, z)$ with $z = \log(\tan\varphi + \sec\varphi)$.

This was done by Mercato in the year 1569 AD in the making of his famous *World Map* for navigation. Mercato did it almost one hundred years before the discovery of calculus by Isaac Newton in the year 1665-1666 during the bubonic plague of Europe when all classes of Trinity College of Cambridge University where he was a student was cancelled and he was forced to work at home by himself to produce his important work of laws of motion, gravity, optics, and calculus.

Remark on Including Orientation Preservation in Definition of Conformal Mapping. In the textbook by Stein-Shakarchi used in this course, the property of the preservation of orientation is included in the definition of conformal mapping so that automatically all conformal mappings are orientation preserving. In the lecture notes we separate the property of preservation of angles and the property of preservation of orientation.

The reason is that we would like to introduce the Riemann sphere geometrically as a sphere with coordinate charts from stereographical projections, and not just formally as the extended Gauss plane obtained by adding ∞ to $\mathbb{C}$. The question of different orientations arises when the two stereographical projections from the north pole and the south pole are considered.

Moreover, in Riemannian geometry the characterization of a conformal map is that the pullback of the metric of the target space is equal to a positive function times the metric of the domain space. In particular, an isometry in Riemannian geometry is conformal. The complex conjugation map $z \to \bar{z}$ which sends (x, y) to $(x, -y)$ preserves the metric $dx^2 + dy^2$ and is an isometry and from the viewpoint of Riemannian geometry is considered conformal though the orientation is reversed.

In complex analysis, some books allow conformal mappings which are not orientation preserving. For example, Section 6.12 on p.190 of the Titchmarsh's *The Theory of Functions (2nd ed.)* states, "There are also conformal representations in which the magnitude of angles is conserved, but their sign is changed. Consider, for example, the transformation $w = \bar{z}$, where $\bar{z}$ is the complex number conjugate to z."

When the convention of including orientation preservation in the definition of conformal map is used, an angle-preserving map which reverses orientation is usually called an *anticonformal map*.