

**Homework #11 Assigned on April 18, 2024
due April 25, 2024**

**Please submit the PDF file of your homework
to the CANVAS website for Math 113**

Problem 1 (*Symmetric Form of Addition Formula for Weierstrass \wp Function*). Prove

$$\begin{vmatrix} \wp(w_1) & \wp'(w_1) & 1 \\ \wp(w_2) & \wp'(w_2) & 1 \\ \wp(w_3) & \wp'(w_3) & 1 \end{vmatrix} = 0$$

for $w_1 + w_2 + w_3 = 0$ by arguing as follows. Choose complex numbers A and B such that $\wp'(w) + A\wp(w) + B$ vanishes at $w = w_j$ for $j = 1, 2$ (whenever the given pair (w_1, w_2) makes it possible to so choose). Verify that $\wp'(w) + A\wp(w) + B$ vanishes also at $w = w_3$ from one of the three fundamental properties of elliptic functions concerning their zeroes and poles.

Problem 2 (*Elliptic Sine Function Expressed in Terms of Weierstrass \wp Function*). Let ω_1, ω_2 be complex numbers which are \mathbb{R} -linearly independent. Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Let $\wp(w)$ be the Weierstrass \wp function for the lattice L . Let $e_1 = \wp\left(\frac{\omega_1}{2}\right)$, $e_2 = \wp\left(\frac{\omega_2}{2}\right)$, and $e_3 = \wp\left(\frac{\omega_1 + \omega_2}{2}\right)$.

(a) Prove that the following branches

$$S(w) = \left(\frac{e_1 - e_2}{\wp(w) - e_2}\right)^{1/2}, \quad C(w) = \left(\frac{\wp(w) - e_1}{\wp(w) - e_2}\right)^{1/2}, \quad D(w) = \left(\frac{\wp(w) - e_3}{\wp(w) - e_2}\right)^{1/2}$$

on \mathbb{C} can be defined as elliptic functions (*i.e.*, doubly periodic meromorphic functions on \mathbb{C}).

Hint: The zero-set of the odd function \wp' consists of the three points $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1 + \omega_2}{2}$, which are all simple zeroes, so that the zero-set of $\wp(w) - e_j$ consists only of a single point, which is a zero of order 2.

(b) Let $k^2 = \frac{e_3 - e_2}{e_1 - e_2}$. Prove that

$$\operatorname{sn} w = S((e_1 - e_2)^{-1/2}w),$$

in which the modulus of the elliptic sine function $\operatorname{sn} w$ is k .

Hint: Use

$$S(w)^2 + C(w)^2 = 1, \quad k^2 S(w)^2 + D(w)^2 = 1$$

and

$$S'(w)^2 = (e_1 - e_2)C(w)^2 D(w)^2.$$

Problem 3 (*Weierstrass \wp function and σ function*). **(a)** Prove the formula

$$\wp(z) - \wp(u) = - \frac{\sigma(z-u)\sigma(z+u)}{\sigma^2(u)\sigma^2(v)}$$

by comparing the zeroes and the poles in z of both sides for fixed u and considering the principal parts of both sides at $z = 0$.

(b) Prove the formula

$$\begin{vmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(w) & \wp'(w) \end{vmatrix} = 2 \frac{\sigma(v-w)\sigma(w-u)\sigma(u-v)\sigma(u+v+w)}{\sigma^3(u)\sigma^3(v)\sigma^3(w)}$$

Hint: If $\wp(v) \neq \wp(w)$, then both sides of the formula are elliptic functions of the variable u , having the same triple pole at the point $u = 0$ and the same roots at the points $u = v$, $u = w$, and $u = -(v+w)$. This makes both sides equal up to a constant. Use Part **(b)** to determine the constant.

Problem 4 (*Location of Poles of Elliptic Sine Function*). Recall that the elliptic sine function $z = \operatorname{sn} w$ with modulus k is defined as the inverse of the indefinite integral

$$\operatorname{sn}^{-1}(z) = \int_{\zeta=0}^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}$$

when the value of the square root in the denominator is 1 at the initial point of integration $\zeta = 0$.

(a) Verify that the poles of $\operatorname{sn} w$, $\operatorname{cn} w$, $\operatorname{dn} w$ are at $2mK + (2n+1)iK'$ for $m, n \in \mathbb{Z}$, which are all simple.

Hint: One pole of $\operatorname{sn} w$ is at the image $w = iK'$ of the point $y = \infty$ under the Schwarz-Christoffel transformation

$$y \mapsto \int_{x=0}^y \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Another is at the reflection of the point $w = iK'$ on the rectangle with vertices $\pm K$, $\pm K + iK'$ with respect to the side $[K, K + iK']$.

(b) Verify that the zeroes of $\operatorname{sn} w$ are at $2mK + 2niK'$ for $m, n \in \mathbb{Z}$, the zeroes of $\operatorname{cn} w$ are at $(2m + 1)K + 2niK'$ for $m, n \in \mathbb{Z}$, and the zeroes of $\operatorname{dn} w$ are at $(2m + 1)K + (2n + 1)iK'$ for $m, n \in \mathbb{Z}$.

Problem 5 (*Arc Length of Ellipse as Integral Involving Jacobian Elliptic Function*). For positive numbers a and b , consider the ellipse defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in \mathbb{R}^2 (with coordinates (x, y)). The arc-length function of the ellipse measured from the point with abscissa a to the point with abscissa \hat{x} and positive ordinate (where $\frac{a}{2} < \hat{x} < a$) is given by

$$s = \int_{x=\hat{x}}^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Let k denote the eccentricity

$$\sqrt{\frac{a^2 - b^2}{a^2}}$$

of the ellipse. Suppose

$$\hat{\varphi} \in \left[0, \frac{\pi}{3}\right] \quad \text{and} \quad \hat{w} \in \mathbb{C}$$

(with $|\hat{w}|$ less than the distance between 0 and any pole of the elliptic sine function $\operatorname{sn} w$) satisfy

$$\hat{x} = a \cos \hat{\varphi} \quad \text{and} \quad \sin \hat{\varphi} = \operatorname{sn} \hat{w}.$$

Verify that the arc-length s is given by

$$s = \int_{w=0}^{\hat{w}} \operatorname{dn}^2 w \, dw$$

along some path from $w = 0$ to $w = \hat{w}$, where $\operatorname{dn} w$ is the third Jacobian elliptic function with modulus k .