Argument Principle and Rouché's Theorem from Residue Computation of Logarithmic Derivative

If a holomorphic function $f(z)$ defined on an open neighborhood of a in $\mathbb{C}$ has a zero of order $k \geq 1$ at a, we can write $f(z) = (z-a)^k g(z)$ with $g(z)$ being a holomorphic function on U which is nonzero at a. The point a is an isolated singularity for the logarithmic derivative

$$\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)}$$

with the principal part given by $\frac{k}{z-a}$, because

$$\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)} = \frac{k}{z-a} + \frac{g'(z)}{g(z)}$$

and $\frac{g'(z)}{g(z)}$ is holomorphic on an open neighborhood of a and can therefore be expanded in a convergent power series centered at a.

Let Ω be a bounded open subset of $\mathbb{C}$ with piecewise smooth boundary. If $f(z)$ is holomorphic on an open neighborhood U of the topological closure $\bar{U}$ of U which is nowhere zero on $\partial\Omega$, then the total number of zeroes of $f(z)$ inside Ω (with multiplicities counted) is equal to

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'}{f} \, dz = \frac{1}{2\pi i} \int_{\partial\Omega} d \log f,$$

because of the residue theorem applied to the meromorphic function $\frac{f'}{f}$ on U.

The argument can be applied to the more general setting when $f(z)$ is a meromorphic function on U which has no zero and no pole on $\partial\Omega$ to yield the statement that the total number of zeroes of $f(z)$ (with multiplicities counted) minus the total number of poles (with orders counted) inside Ω is equal to

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'}{f} \, dz = \frac{1}{2\pi i} \int_{\partial\Omega} d \log f.$$

Here the total number of poles with orders counted means the sum of all the pole orders.

When the boundary $\partial\Omega$ of Ω is a single connected simple closed curve C, from

$$\frac{1}{2\pi i}\int_C d\log f = \frac{1}{2\pi i}\int_C (d\log|f| + i\,d\arg f) = \frac{1}{2\pi}\int_C d\arg f$$

it follows that the total number of zeroes of $f(z)$ (with multiplicities counted) minus the total number of poles (with orders counted) inside Ω is equal to the total change $\Delta_C \arg f$ of (the numerical value of) the argument of f. Here $\Delta_C \arg f$ means that one starts out with any chosen numerical value of the argument $\arg f$ of f at any one point P of C and let this numerical value continuously change as the point P moves along C in a counter-clockwise sense and then determines by how much this numerical value has changed when the moving point gets back to the original position on C and this change of the numerical value of the argument of f is $\Delta_C \arg f$.

The statement that the sum of the orders of the zeroes of $f(z)$ minus the sum of orders of the poles of $f(z)$ inside Ω is equal to the total change $\Delta_C \arg f$ of (the numerical value of) the argument of f along the connected boundary C of Ω is known as the *argument principle*.

When the boundary $\partial\Omega$ is not connected, the argument principle still holds but the sum of the orders of the zeroes of $f(z)$ minus the sum of orders of the poles of $f(z)$ inside Ω is equal to the total change $\Delta_{\partial\Omega} \arg f$ of (the numerical value of) the argument of f along the entire boundary of Ω with the corrected orientation used for each component of the boundary.

When the meromorphic function f on the open neighborhood U of $\bar{\Omega}$ is perturbed by another meromorphic function g to yield the resulting meromorphic function $f+g$, if $|g| < |f|$ on $\partial\Omega$, then the sum of the orders of the zeroes of $f(z)$ minus the sum of orders of the poles of $f(z)$ inside Ω is equal to the sum of the orders of the zeroes of $f(z)+g(z)$ minus the sum of orders of the poles of $f(z)+g(z)$ inside Ω. This is known as *Rouché's theorem*. Its proof is as follows. For $0 \le t \le 1$, $f(z)+tg(z)$ has no zeros and no poles on $\partial\Omega$, because $|g| < |f|$ on $\partial\Omega$ and $f(z)$ has no poles on $\partial\Omega$. Since for any fixed $0 \le t \le 1$, the integral

$$\frac{1}{2\pi i}\int_{\partial\Omega} \frac{f'(z)+tg'(z)}{f(z)+tg(z)}\,dz,$$

being equal to the sum of the orders of the zeroes of $f(z) + tg(z)$ minus the sum of the orders of the poles of $f(z) + tg(z)$ inside Ω, is always an integer, its continuity in $0 \leq t \leq 1$ implies that it is constant. Hence the sum of the orders of the zeroes of $f(z)$ minus the sum of orders of the poles of $f(z)$ inside Ω (which is its value at $t = 0$) is equal to the sum of the orders of the zeroes of $f(z) + g(z)$ minus the sum of orders of the poles of $f(z) + g(z)$ inside Ω (which is its value at $t = 1$).

Remark. The argument principle for a holomorphic function can be roughly regarded as being related to the following statement concerning the zeroes of a continuous real-valued function of a real variable. A continuous real-valued function $g(x)$ of a real variable $x \in [a, b]$ has at least one zero in (a, b) if $g(a)$ and $g(b)$ are nonzero and have different signs.

Proof of Fundamental Theorem of Algebra by Rouche's Theorem. The fundamental theorem of algebra is an immediate consequence of Rouché's theorem applied to $f(z) = a_n z^n$ (with $a_n \in \mathbb{C} - \{0\}$ and $n \geq 1$) and $g(z) = \sum_{k=1}^{n-1} a_k z^k$ (withy $a_0, \cdots, a_{n-1} \in \mathbb{C}$) on $\Omega = \{|z| < R\}$ for R sufficiently large.

Hurwitz's Theorem on Limits of Univalent Functions as Application of Rouché's Theorem. Let Ω be a connected open subset of $\mathbb{C}$ and $f_n(z)$ be a sequence of holomorphic functions on Ω which approaches a nonconstant (holomorphic) function $f(z)$ uniformly on compact subsets of Ω. If each $f_n(z)$ is univalent (in the sense that the map from Ω to $\mathbb{C}$ defined by $f_n(z)$ is injective), then $f(z)$ is also univalent. This is known as *Hurwitz's theorem* and it proved as follows as an application of Rouché's theorem.

Since $f(z)$ is assumed to be nonconstant, the zeros of its derivative $f'(z)$ must be a discrete subset of the connected open subset Ω of $\mathbb{C}$. Assume that at two distinct points a_1 and a_2 of Ω the function $f(z)$ assumes that same value b. Since we have the power series expansion

$$f'(z) = \sum_{m=k_j}^{\infty} c_{m,j}(z - a_j)^m$$

with $c_{k_j,j} \neq 0$ for some nonnegative integer k_j on some open disk centered at a_j for $j = 1, 2$, it follows that we have the power series expansion

$$f(z) = b + \sum_{m=k_j}^{\infty} \frac{c_{m,j}}{m+1}(z - a_j)^{m+1}.$$

on some open disk centered at a_j for $j = 1, 2$. With $c_{k_j,j} \neq 0$ for $j = 1, 2$, this implies that there exists some $r > 0$ and some $\varepsilon > 0$ such that

$$|f(z) - b| \geq \varepsilon$$

for $|z - a_j| = r$ and $j = 1, 2$ and the two closed disks $\{|z - a_1| \leq r\}$ and $\{|z - a_2| \leq r\}$ are inside Ω and are disjoint. There exists some $n \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \frac{\varepsilon}{2}$ on the compact subset $\{|z - a_1| \leq r\} \cup \{|z - a_2| \leq r\}$ of Ω. By applying Rouché's theorem to the perturbation

$$f_n(z) - b = f(z) - b + (f_n(z) - f(z))$$

of $f(z) - b$ on $\{|z - a_j| \leq r\}$ (for $j = 1, 2$), from

$$|f_n(z) - f(z)| < \frac{\varepsilon}{2} < \varepsilon \leq |f(z) - b|$$

on $\{|z - a_j| \leq r\}$ we conclude that $f_n(z) - b$ and $f(z) - b$ have the same number of zeroes in $\{|z - a_j| < r\}$ for $j = 1, 2$, which means that $f_n(z) - b$ vanishes at some point of $\{|z - a_1| < r\}$ and also at some point of $\{|z - a_2| < r\}$, contradicting the assumption that $f_n(z)$ is univalent on Ω.