Computation of Infinite Sums by Residues and Partial Fraction Expansion of Meromorphic Functions

One kind of definite integrals which we can compute by methods of residues is

$$\int_{x=-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx,$$

where $P(x)$ and $Q(x)$ are polynomials with the degree of $Q(x)$ at least 2 more than than of $P(x)$ under the additional assumption that $Q(x)$ is nowhere zero on $\mathbb{R}$.

We now discuss the application of the methods of residues to discrete infinite sums instead of definite integrals, for example, to compute explicitly the infinite sum

$$\sum_{n=-\infty}^{\infty}\frac{P(n)}{Q(n)},$$

where $P(x)$ and $Q(x)$ are polynomials with the degree of $Q(x)$ at least 2 more than that of $P(x)$ under the additional assumption that $Q(n)$ is nonzero for any $n\in\mathbb{Z}$.

The idea is to use a meromorphic function $f(z)$ on $\mathbb{C}$ whose poles include $z=n$ for $n\in\mathbb{Z}$ and to use a sequence of contours C_n with the property that the domain enclosed by C_n is increasing as n increases and approach $\mathbb{C}$ as $n\to\infty$ and

$$\lim_{n\to\infty}\int_{C_n}f(z)dz=0$$

and the residue of $f(z)$ at $z=n$ is $\frac{P(n)}{Q(n)}$, so that

$$\sum_{n=-\infty}^{\infty}\frac{P(n)}{Q(n)}$$

is equal to the negative of the sum of the residues of $f(z)$ at poles other than the points of $\mathbb{Z}$.

We have to choose a meromorphic function $f(z)$ on $\mathbb{C}$ whose residue at $z=n$ is $\frac{P(n)}{Q(n)}$. One choice is the function

$$\frac{P(z)}{Q(z)}\,\pi\cot\pi z$$

because

$$\pi\cot\pi z=\cos\pi z\,\frac{\pi}{\sin\pi z},$$

has a simple pole at $z=n$ with residue 1, from $\cos\pi n=(-1)^n$ and

$$\lim_{z\to n}\frac{\pi(z-n)}{\sin\pi z}=\lim_{z\to n}\frac{\pi(z-n)}{(-1)^n\sin\pi(z-n)}=(-1)^n.$$

If $Q(n)$ is nonzero for $n\in\mathbb{Z}$, we conclude that $z=n$ is a simple pole for

$$\frac{P(z)}{Q(z)}\,\pi\cot\pi z$$

with residue precisely equal to $\frac{P(n)}{Q(n)}$.

For the contour C_n we use the square with vertices at $(n+\frac{1}{2})(\pm 1\pm i)$. Observe that when $|y|>\frac{1}{2\pi}$ we have

$$|\cot\pi z|\le\left|\frac{e^{2i\pi z}+1}{e^{2i\pi z}-1}\right|\le\frac{e^{2\pi y}+1}{e^{2\pi y}-1}=1+\frac{2}{e^{2\pi y}-1}\le 1+\frac{2}{e-1}$$

and hence uniformly bounded. Also observe that $\cot\pi z$ is bounded on the line segment joining $\frac{1}{2}(1-i)$ to $\frac{1}{2}(1+i)$ and we can use the periodicity

$$\cot\pi(z+1)=\cot\pi z$$

of $\cot\pi z$ with period 1 to conclude that $\cot\pi z$ is uniformly bounded on C_n. From the assumption that the degree of $Q(z)$ is at least 2 more than the degree of $P(z)$ it now follows that

$$\lim_{n\to\infty}\int_{C_n}\frac{P(z)}{Q(z)}\,\pi\cot\pi z\,dz=0.$$

Finally from the residue theorem (which now simply says that the sum of all the residues is zero) we have the formula

$$\sum_{n=-\infty}^{\infty}\frac{P(n)}{Q(n)}=-\sum_{j=1}^{k}\mathrm{Res}_{z=a_j}\left(\frac{P(z)}{Q(z)}\,\pi\cot\pi z\right),$$

where $a_1,\cdots,a_k$ are the distinct zeroes of the polynomial $Q(z)$ (*i.e.,* each zero being counted only once by ignoring its multiplicity).

As a simple example, we compute

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2}$$

for $a > 0$. The two points whose residue for

$$\frac{1}{z^2+a^2}\,\pi\cot\pi z$$

we have to compute are the two zeroes ai and $-ai$ of z^2+a^2. Since both poles are simple, we have

$$\begin{aligned}
\operatorname{Res}_{z=ai}\left(\frac{1}{z^2+a^2}\,\pi\cot\pi z\right) &= \lim_{z\to ai}\left(\frac{z-ai}{z^2+a^2}\,\pi\cot\pi z\right)\\
&= \left(\frac{1}{z+ai}\,\pi\cot\pi z\right)_{z=ai}\\
&= \frac{\pi\cot\pi ai}{2ai}\\
&= \frac{\pi}{2ai}\,\frac{i(e^{i\pi ai}+e^{-i\pi ai})}{e^{i\pi ai}-e^{-i\pi ai}}\\
&= \frac{\pi}{2a}\,\frac{e^{-\pi a}+e^{\pi a}}{e^{-\pi a}-e^{\pi a}}\\
&= -\frac{\pi}{2a}\coth\pi a
\end{aligned}$$

and, with a replaced by $-a$ and the odd property of the function $z\mapsto\coth\pi z$,

$$\operatorname{Res}_{z=-ai}\left(\frac{1}{z^2+a^2}\,\pi\cot\pi z\right) = -\frac{\pi}{2a}\coth\pi a.$$

Hence

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} &= -\left(\operatorname{Res}_{z=ai}\left(\frac{1}{z^2+a^2}\,\pi\cot\pi z\right)+\operatorname{Res}_{z=-ai}\left(\frac{1}{z^2+a^2}\,\pi\cot\pi z\right)\right)\\
&= -\left(-\frac{\pi}{2a}\coth\pi a-\frac{\pi}{2a}\coth\pi a\right) = \frac{\pi\coth\pi a}{a}.
\end{aligned}$$

Another kind of infinite sum which can be similarly computed in an explicit way is

$$\sum_{n=-\infty}^{\infty}(-1)^n\frac{P(n)}{Q(n)},$$

where $P(x)$ and $Q(x)$ are polynomials with the degree of $Q(x)$ at least 2 more than than of $P(x)$ under the additional assumption that $Q(n)$ is nonzero for any $n \in \mathbb{Z}$. For this kind of infinite sum the meromorphic function $f(z)$ to be used is modified to be

$$\frac{P(z)}{Q(z)}\,\pi \operatorname{cosec} \pi z,$$

because $\cos \pi z$ is $(-1)^n$ at $z = n$. The same contour C_n of the square with vertices at $(n+\frac{1}{2})(\pm 1 \pm i)$ is used.

$$\begin{aligned}
\operatorname{Res}_{z=n}\left(\frac{P(z)}{Q(z)}\,\pi \operatorname{cosec} \pi z\right) &= \lim_{z\to n} \frac{P(z)}{Q(z)}\,\frac{\pi(z-n)}{\sin \pi z} \\
&= \frac{P(n)}{Q(n)} \lim_{z\to n} \frac{\pi(z-n)}{(-1)^n \sin \pi(z-n)} \\
&= (-1)^n \frac{P(n)}{Q(n)}.
\end{aligned}$$

When $y > \frac{1}{\pi}$ we have

$$\begin{aligned}
|\operatorname{cosec} \pi z| &= \left|\frac{2i}{e^{i\pi z} - e^{-i\pi z}}\right| \\
&= \left|\frac{2}{e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{\pi y}}\right| \\
&\leq \frac{2}{e^{\pi y} - e^{-\pi y}} \leq \frac{2}{e - e^{-1}}.
\end{aligned}$$

When $y < \frac{-1}{\pi}$ we have

$$\begin{aligned}
|\operatorname{cosec} \pi z| &= \left|\frac{2i}{e^{i\pi z} - e^{-i\pi z}}\right| \\
&= \left|\frac{2}{e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{\pi y}}\right| \\
&\leq \frac{2}{e^{-\pi y} - e^{\pi y}} \leq \frac{2}{e - e^{-1}}.
\end{aligned}$$

Hence $\operatorname{cosec} \pi z$ is uniformly bounded on $|y| > \frac{1}{\pi}$. Moreover, $\operatorname{cosec} \pi z$ is bounded on the line segment joining $\frac{1}{2}(1-i)$ to $\frac{1}{2}(1+i)$ and we can use the periodicity

$$|\operatorname{cosec} \pi(z+1)| = |\operatorname{cosec} \pi z|$$

of $|\operatorname{cosec}\pi z|$ with period 1 to conclude that $\operatorname{cosec}\pi z$ is uniformly bounded on C_n. From the assumption that the degree of $Q(z)$ is at least 2 more than the degree of $P(z)$ it now follows that

$$\lim_{n\to\infty}\int_{C_n}\frac{P(z)}{Q(z)}\,\pi\operatorname{cosec}\pi z\,dz=0.$$

Finally from the residue theorem (which now simply says that the sum of all the residues is zero) we have the formula

$$\sum_{n=-\infty}^{\infty}(-1)^n\frac{P(n)}{Q(n)}=-\sum_{j=1}^{k}\operatorname{Res}_{z=a_j}\left(\frac{P(z)}{Q(z)}\,\pi\operatorname{cosec}\pi z\right),$$

where $a_1,\cdots,a_k$ are the distinct zeroes of the polynomial $Q(z)$ (*i.e.,* each zero being counted only once by ignoring its multiplicity).

Infinite Product Expansion of Sine Function. The identity

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2}=\frac{\pi\coth\pi a}{a},$$

which was derived for $a>0$, holds when a is replaced by any complex number z, because the left-hand side of

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+z^2}=\frac{\pi\coth\pi z}{z}$$

defines a meromorphic function on $\mathbb{C}$ and the identity simply follows from applying the identity theorem the meromorphic function which is the difference of the two sides. Note that the coefficients of a Laurent series with an isolated singularity are computed along a circle centered at the isolated singularity so that the identity theorem applied to the complement of its poles determine a meromorphic function completely. To make it more convenient to factor the denominator of each term on the left-hand side, we replace z by iz to get

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{1}{n^2-z^2}&=\frac{\pi\coth\pi iz}{iz}\\&=\frac{\pi(e^{\pi iz}+e^{-\pi iz})}{iz(e^{\pi iz}-e^{-\pi iz})}\\&=-\frac{\pi\cot\pi z}{z}\end{aligned}$$

or

$$\pi\cot\pi z = \sum_{n=-\infty}^{\infty} \frac{z}{z^2-n^2},$$

which can be rewritten as

$$\begin{aligned}\pi\cot\pi z &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} \\ &= \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n}+\frac{1}{z-n}\right) \\ &= \frac{1}{z} + \sum_{n\in\mathbb{Z}-\{0\}}\left(\frac{1}{z-n}+\frac{1}{n}\right).\end{aligned}$$

The identity

$$\pi\ \cot\pi z = \frac{1}{z} + \sum_{n\in\mathbb{Z}-\{0\}}\left(\frac{1}{z-n}+\frac{1}{n}\right)$$

is the *partial fraction expansion* of the cotangent function or the expansion of the cotangent function into a sum of its principal parts. Note that we are forced to use the convergent series

$$\sum_{n\in\mathbb{Z}-\{0\}}\left(\frac{1}{z-n}+\frac{1}{n}\right)$$

instead of the meaningless non-convergent series

$$\sum_{n\in\mathbb{Z}-\{0\}}\left(\frac{1}{z-n}\right).$$

We would like to remark that the convergence

$$\sum_{n\in\mathbb{Z}-\{0\}}\left(\frac{1}{z-n}+\frac{1}{n}\right)$$

is uniform and absolute on any compact subset (*i.e.,* bounded closed subset) of $\mathbb{C}$ after removing a finite number of terms (which depend on the compact subset). We are going to use this partial fraction expansion of the cotangent function to get an infinite product expansion of the trigonometric sine function. Let us first say something about an infinite product.

Convergence of Infinite Product. The statement $\prod_{j=1}^{\infty} a_n = L$ for $a_n \in \mathbb{C}$ and $L \in \mathbb{C} - \{0\}$ means that the sequence of partial products $p_n = \prod_{j=1}^{n} a_n$ approaches the limit L. An important point here is that the limit L, by definition, must be nonzero. Automatically this means that every a_n is nonzero, otherwise the sequence of the partial products will be zero from some point on and the limit must be 0.

The condition that the sequence of partial products $p_n = \prod_{j=1}^{n} a_n$ approaches the limit L is equivalent to the following pair of conditions.

(i) The partial sum $\sum_{j=1}^{n} \log |a_n|$ approaches $\log |L|$.

(ii) The point in the unit circle defined by the angle $\sum_{j=1}^{n} \arg a_n$ converges to the point on the unit circle defined by the angle $\arg L$.

The equivalence is obtained by using polar coordinates to representing a point of $\mathbb{C} - \{0\}$ by the pair consisting of the logarithm of its distance to the origin and the point on the unit circle determined by its angle in polar coordinates.

In practice, the convergence of an infinite product is verified by using the equivalent pair of conditions.

Partial Product Expansion of Trigonometric Sine Function. The equation

$$\frac{d}{dz}\left(\log \sin \pi z\right) = \pi \cot \pi z$$

holds as an identity between two meromorphic functions, with the left-hand side interpreted as the logarithmic derivative

$$\frac{\frac{d}{dz} \sin \pi z}{\sin \pi z}$$

of the entire function $\sin \pi z$. Note that the logarithmic derivative

$$\frac{d}{dz}\left(\log F(z)\right)$$

of a meromorphic function $F(z)$, by definition, means the meromorphic function

$$\frac{\frac{d}{dz} F(z)}{F(z)}$$

and it does not mean the derivative of $\log F(z)$ unless a branch of $\log F(z)$ can be defined.

For any chosen $m \in \mathbb{Z}$, since $\sin \pi z$ is holomorphic and nowhere zero on the simply connected subdomain

$$\mathbb{C} - ((-\infty, m] \cup [m+1, \infty))$$

of $\mathbb{C}$, we can define a branch of $\log \sin \pi z$ on

$$\mathbb{C} - ((-\infty, m] \cup [m+1, \infty)).$$

Interpreted in terms of that particular branch, the left-hand side of

$$\frac{d}{dz}(\log \sin \pi z) = \pi \cot \pi z = \frac{1}{z} + \sum_{n \in \mathbb{Z} - \{0\}} \left(\frac{1}{z-n} + \frac{1}{n} \right)$$

is the derivative of the holomorphic function $\log \sin \pi z$ on the simply connected subdomain

$$\mathbb{C} - ((-\infty, m] \cup [m+1, \infty))$$

of $\mathbb{C}$. Pick any point P_0 in the subdomain

$$\mathbb{C} - ((-\infty, m] \cup [m+1, \infty))$$

and integrate from P_0 to an arbitrary point z on both sides of

$$\frac{d}{dz}(\log \sin \pi z) = \frac{1}{z} + \sum_{n \in \mathbb{Z} - \{0\}} \left(\frac{1}{z-n} + \frac{1}{n} \right)$$

and then exponentiate to get

$$\frac{\sin \pi z}{\pi z} = C \prod_{n \in \mathbb{Z} - \{0\}} \left(1 - \frac{z}{n} \right) e^{\frac{z}{n}}$$

on

$$\mathbb{C} - ((-\infty, m] \cup [m+1, \infty)),$$

where C is a constant from putting together, in the exponent, all the constants of integration of each term when we integrate from P_0 to z. By setting $z = 0$, we determine C to be 1 and get the following factorization of $\sin \pi z$ as an infinite product

$$\sin \pi z = \pi z \prod_{n \in \mathbb{Z} - \{0\}} \left(1 - \frac{z}{n} \right) e^{\frac{z}{n}}$$

on

$$\mathbb{C} - \left((-\infty, m] \cup [m+1, \infty)\right).$$

Since $m \in \mathbb{Z}$ is arbitrary chosen, we conclude that the infinite product expansion of

$$\sin \pi z = \pi z \prod_{n \in \mathbb{Z} - \{0\}} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}$$

holds on all of $\mathbb{C}$.

Partial Fraction Expansion of Trigonometric Cosecant Function. By using the cosecant function to sum up

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2},$$

for $a > 0$ in a completely analogous manner, we obtain the following partial fraction expansion

$$\operatorname{cosec} z = \frac{1}{z} + \sum_{n \in \mathbb{Z} - \{0\}} (-1)^n \left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right)$$

of the cosecant function. In general, for a certain kind of meromorphic function on $\mathbb{C}$ we can use the reside theorem to get its partial fraction expansion as follows.

Partial Fraction Expansion by Modified Cauchy Kernel for Meromorphic Functions on $\mathbb{C}$ with Growth of Polynomial Order on a Sequence of Contours. Suppose $f(z)$ is a meromorphic function on $\mathbb{C}$ whose poles $\{a_n\}_{1 \leq n < \infty}$ are simple with $0 < |a_1| \leq |a_2| \leq \cdots$ so that the residue of $f(z)$ at a_n is b_n. Suppose that there is a sequence of closed contours C_n such that the enclosure of C_n includes $a_1, \cdots, a_n$ but no other poles. Assume that the distance R_n from C_n to the origin goes to infinity as $n \to \infty$ and the length L_n of C_n is of the order $O(R_n)$. Assume that on C_n we have $f(z) = o(R_n^{p+1})$. We are going to apply the theorem of residue to the integral

$$\frac{1}{2\pi i} \int_{C_n} \frac{f(w)}{w^{p+1}(w - z)} dw.$$

The reason for choosing this integrand for the contour integration is that we want to reproduce $f(z)$, which forces us to consider the use of the Cauchy kernel

$$\frac{1}{w-z}$$

as a factor and we also want the contour to approach 0 as $n \to \infty$, which motivates the introduction of the factor w^{p+1} into the modified Cauchy kernel

$$\frac{1}{w^{p+1}(w-z)}$$

in order to make use of the condition that $f(z) = o(R_n^{p+1})$ on C_n whose length is of the order $O(R_n)$. The residue at $w = 0$ is obtained by expanding

$$\frac{f(w)}{w-z}$$

in Laurent series in w around $w = 0$. The function

$$\frac{f(w)}{w^{p+1}(w-z)}$$

is now equal to

$$\frac{-1}{w^{p+1}}\left(\frac{1}{z}+\frac{w}{z^2}+\frac{w^2}{z^3}+\cdots\right)\left(f(0)+f'(0)w+\frac{1}{2}f''(0)w^2+\cdots\right),$$

where the coefficient of $\frac{1}{w}$ is

$$-\frac{1}{z}\left(\frac{f(0)}{z^p}+\frac{f'(0)}{z^{p-1}}+\cdots+\frac{f^{(p)}(0)}{p!}\right).$$

The residue at $w = z$ is given by

$$\frac{f(z)}{z^{p+1}}.$$

The residue at a_n is

$$\frac{b_n}{a_n^{p+1}(a_n-z)}.$$

Since as $n \to \infty$ the integral becomes zero, we get

$$-\frac{1}{z}\left(\frac{f(0)}{z^p}+\frac{f'(0)}{z^{p-1}}+\cdots+\frac{f^{(p)}(0)}{p!}\right)+\frac{f(z)}{z^{p+1}}+\sum_{n=1}^{\infty}\frac{b_n}{a_n^{p+1}(a_n-z)}=0$$

which means that

$$f(z) = f(0) + z\,f'(0) + \cdots + \frac{z^p}{p!}f^{(p)}(0) - \sum_{n=1}^{\infty} \frac{b_n z^{p+1}}{a_n^{p+1}(a_n - z)}$$
$$= \sum_{\nu=0}^{p} \frac{z^\nu}{\nu!} f^{(\nu)}(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} + \frac{z}{a_n^2} + \cdots + \frac{z^p}{a_n^{p+1}} \right).$$

The last expression comes from writing z^{p+1} as $(z^{p+1} - a_n^{p+1}) + a_n^{p+1}$ and then factoring

$$z^{p+1} - a_n^{p+1} = (z - a_n) \sum_{\nu=0}^{p} z^\nu a_n^{p-\nu}$$

to get

$$-\frac{z^{p+1}}{a_n^{p+1}(a_n - z)} = \frac{a_n^{p+1} + (z - a_n)\sum_{\nu=0}^{p} z^\nu a_n^{p-\nu}}{a_n^{p+1}(z - a_n)}$$
$$= \frac{1}{z - a_n} + \frac{1}{a_n} + \frac{z}{a_n^2} + \cdots + \frac{z^p}{a_n^{p+1}}.$$

Infinite Product Expansion of Gamma Function. The Gamma function $\Gamma(z)$ which is originally defined for $\operatorname{Re} z > 0$ by

$$\Gamma(z) = \int_{t=0}^{\infty} t^{z-1} e^{-t} dt$$

can be extended to a meromorphic function to all of $\mathbb{C}$ from the functional equation

$$\Gamma(z+1) = z\Gamma(z)$$

by defining

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{(z+1)z} = \cdots = \frac{\Gamma(z+n)}{(z+n-1)(z+n-2)\cdots(z+1)z}$$

for $\operatorname{Re} z > -n$. Inductively on n, starting from $n = 0$, the residue of $\Gamma(z)$ at $z = 0$ is $1 = \Gamma(1)$ from

$$\operatorname{Res}_{z=0}\Gamma(z) = \lim_{z\to 0} (z\Gamma(z)) = \lim_{z\to 0} \Gamma(z+1) = G(1) = 1.$$

and the residue of $\Gamma(z)$ at $z = -n$ is computed by

$$\operatorname{Res}_{z=-n}\Gamma(z) = \lim_{z\to -n} (z+n)\Gamma(z)) = \lim_{z\to -n} \frac{\Gamma(z+n+1)}{(z+n-1)\cdots(z+1)z} = \frac{(-1)^n}{n!}.$$

Recall that when we discussed the computation of definite integrals with the use of branches of holomorphic functions, an example is the computation of the integral

$$\int_{x=0}^{1} \frac{dx}{x^{\alpha}(1-x)^{1-\alpha}} = \frac{\pi}{\sin \alpha\pi}.$$

The Beta function defined by

$$B\left(x,y\right) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma\left(x+y\right)}$$

is the generalization of the binomial coefficient

$$\binom{m+n}{m} = \frac{(m+n)!}{m!\,n!}$$

to the case of variable arguments, in the same way as the Gamma function $\Gamma(x)$ is the generalization of the factorial function to the case of a continuous argument. The rewriting of a double integral as an iterated integral and the changing of variables for the integrands $u = tv$ and then $w = t\left(1+v\right)$ yield

$$\begin{aligned}
\Gamma(x)\Gamma(y) &= \left(\int_{t=0}^{\infty} t^{x-1}e^{-t}dt\right)\left(\int_{u=0}^{\infty} u^{y-1}e^{-u}du\right) \\
&= \left(\int_{t=0}^{\infty} t^{x-1}e^{-t}dt\right)\left(\int_{t=0}^{\infty} t^{y}v^{y-1}e^{-tv}dv\right) \\
&= \int_{t=0}^{\infty}\int_{v=0}^{\infty} t^{x+y-1}e^{-t(1+v)}v^{y-1}dvdt \\
&= \int_{w=0}^{\infty}\int_{v=0}^{\infty} \frac{w^{x+y-1}}{(1+v)^{x+y-1}}\, e^{-w}\, v^{y-1}\frac{1}{1+v}\, dv\, dw \\
&= \left(\int_{w=0}^{\infty} w^{x+y-1}e^{-w}dw\right)\left(\int_{v=0}^{\infty} \frac{v^{y-1}dv}{(1+v)^{x+y}}\right) \\
&= \Gamma\left(x+y\right)\int_{v=0}^{\infty} \frac{v^{y-1}dv}{(1+v)^{x+y}}
\end{aligned}$$

and

$$B\left(x,y\right) = \int_{v=0}^{\infty} \frac{v^{y-1}dv}{(1+v)^{x+y}},$$

which takes on the following symmetric form in x, y under the coordinate change $v = \frac{\lambda}{1-\lambda}$

$$B\left(x,y\right) = \int_{t=0}^{1} t^{x-1}\left(1-t\right)^{y-1}dt.$$

The setting of $x+y=1$ yields the *Euler's reflection law for the Gamma function*

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin \pi z}$$

which holds on $\mathbb{C}$ as an equation for meromorphic functions, because

$$\begin{aligned}\Gamma(x)\Gamma(1-x) &= \left(B(x,y)\right)_{x+y=1}\\ &= \left(\int_{t=0}^{1} t^{y-1}\left(1-t\right)^{x-1}dt\right)_{x+y=1}\\ &= \int_{t=0}^{1} t^{-x}\left(1-t\right)^{x-1}dt=\frac{\pi}{\sin \pi x}\end{aligned}$$

for $0<x<1$ and because of the identity theorem for meromorphic functions.

Euler's reflection law for the Gamma function suggests an infinite product expansion for the Gamma function which consists of half of the factors in the infinite product expansion for the sine function. One way to get an infinite product expansion of the Gamma function is to get an infinite sum expansion for the logarithmic derivative $\frac{\Gamma'(z)}{\Gamma(z)}$ of the Gamma function and then, by integrating and then exponentiating, get an infinite product expansion for the Gamma function. One way to get an infinite sum expansion for $\frac{\Gamma'(z)}{\Gamma(z)}$ is to replace the factor $\Gamma(z-h)$ in the Beta function formula

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}=\int_{0}^{1}(1-t)^{z-h-1}t^{h-1}dt$$

by its Taylor expansion

$$\Gamma(z-h)=\Gamma(z)-h\,\Gamma'(z)+\frac{(-h)^2}{2}\Gamma''(z)+\cdots$$

and then to equate the coefficients of like powers of h on both sides of the equation. More precisely, for $\operatorname{Re} z>h>0$ with z fixed and h variable, we use

$$\frac{1}{h}=\int_{0}^{1}t^{h-1}dt$$

to get

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}=\int_{0}^{1}(1-t)^{z-h-1}t^{h-1}dt=\frac{1}{h}+\int_{0}^{1}\left((1-t)^{z-h-1}-1\right)t^{h-1}dt.$$

From the binomial expansion of

$$(1-t)^{z-h-1}=e^{(z-h-1)\log(1-t)}=e^{(z-h-1)\left(-\sum_{n=1}^{\infty}\frac{t^n}{n}\right)}$$

in t for $|t|<1$ and $\operatorname{Re} z > h > 0$, we conclude that the integral on the right-hand side now is holomorphic for h in a small open neighborhood of 0 in $\mathbb{C}$ and we can write it as its value at $h=0$ plus a term of the order $o(h)$ as $h \to 0$. Thus

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}=\frac{1}{h}+\int_0^1\left((1-t)^{z-1}-1\right)t^{-1}dt+o(h).$$

This is the Laurent expansion of the Beta function $B\left(z-h,z\right)$ in the variable h at $h=0$.

We compare this to the Laurent series expansion of

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}$$

in h and get

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}=\frac{1}{\Gamma(z)}\left(\Gamma(z)-h\,\Gamma'(z)+\cdots\right)\left(\frac{1}{h}+A+\cdots\right),$$

where A is a constant. Equating the constant terms of

$$\begin{aligned}&\frac{1}{\Gamma(z)}\left(\Gamma(z)-h\,\Gamma'(z)+\cdots\right)\left(\frac{1}{h}+A+\cdots\right)\\&=\frac{1}{h}+\int_0^1\left((1-t)^{z-1}-1\right)t^{-1}dt+o(h),\end{aligned}$$

we get

$$\frac{\Gamma'(z)}{\Gamma(z)}=\int_0^1\left(1-(1-t)^{z-1}\right)t^{-1}dt-A$$

for $\operatorname{Re} z>0$. Using

$$\frac{1}{t}=\frac{1}{1-(1-t)}=\sum_{n=0}^{\infty}(1-t)^n,$$

we get

$$\begin{aligned}\frac{\Gamma'(z)}{\Gamma(z)} &= -A + \int_0^1 \left(1-(1-t)^{z-1}\right)\left(\sum_{n=0}^{\infty}(1-t)^n\right)dt \\ &= -A + \int_0^1 \left(\sum_{n=0}^{\infty}\left((1-t)^n-(1-t)^{n+z-1}\right)\right)dt \\ &= -A + \sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right).\end{aligned}$$

We can rewrite it as

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z} = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right) - C$$

for some constant C. To determine the constant C, we integrate and take exponents of both sides and get

$$\frac{1}{\Gamma(z)} = e^{Cz} z \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}.$$

Setting $z=1$, we get

$$1 = e^{C}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}}.$$

Hence

$$C = -\log\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}} = \lim_{N\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}-\log N\right)$$

which is equal to the Euler constant γ. We have finally the following infinite product decomposition for $\Gamma(z)$.

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}.$$

We can put together the infinite product decomposition of $\frac{1}{\Gamma(z)}$ and $\frac{1}{\Gamma(1-z)}$ and use Euler's reflection formula for the Gamma function to get recover the infinite product expansion of the sine function as follows. From

$$
\begin{aligned}
\frac{\sin\,\pi z}{\pi} &= \frac{1}{\Gamma(z)\Gamma(1-z)} \\
&= \frac{1}{\Gamma(z)(-z)\Gamma(-z)} \\
&= \frac{1}{(-z)}\left(e^{\gamma z} z \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right)\left(e^{\gamma(-z)}(-z)\prod_{n=1}^{\infty}\left(1+\frac{z}{-n}\right)e^{-\frac{z}{-n}}\right) \\
&= z \prod_{n\in\mathbb{Z}-\{0\}}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}
\end{aligned}
$$

it follows that

$$\sin\,\pi z = \pi z \prod_{n\in\mathbb{Z}-\{0\}}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}.$$