Homework #10 Assigned on April 11, 2024
due April 18, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 113

Problem 1 *(Nontangential Limit of Poisson Integral for Unit Disk).* Given a continuous function f on the unit circle, the solution F of the Dirichlet problem for a harmonic function on the open unit disk $\mathbb{D}$ with boundary value f is given by the Poisson integral

$$F(z) = \frac{1}{2\pi} \int_{|\zeta|=1} f(\zeta) \operatorname{Re}\left(\frac{\zeta + z}{\zeta - z}\right) \frac{d\zeta}{i\zeta}$$

for $z \in \mathbb{D}$. At the end of the lecture notes for Lecture 18 on "Poisson Kernel", the function $F(re^{i\theta})$ is shown to approach $f(e^{i\theta})$ uniformly in $\theta \in \mathbb{R}$ as $r \to 1^-$. Suppose $0 < \alpha < \frac{\pi}{2}$. For $\zeta \in \partial\mathbb{D}$, prove that $F(z)$ approaches $f(\zeta)$ uniformly (with respect to ζ) as $|z| \to 1^-$ if the angle between the directed line segment $[0, \zeta]$ and the directed line segment $[z, \zeta]$ (from $[0, \zeta]$ to $[z, \zeta]$) is in the closed interval $[-\alpha, \alpha]$. In other words, on the unit circle the uniform nontangential limit of F is the given boundary-value function f.

Problem 2 *(Nontangential Limit of Poisson Integral for Upper Half-Plane).* Given a continuous function f on $\mathbb{R}$ such that

$$\lim_{x\to\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x)$$

exist and are equal to the same finite real number. Define the function $F(x, y)$ on the open upper half-plane $\mathbb{H}$ by

$$F(x, y) = \frac{1}{\pi} \int_{t=-\infty}^{\infty} f(t) \, \frac{y}{(x - t)^2 + y^2} \, dt$$

for $x \in \mathbb{R}$ and $y > 0$. Suppose $0 < \alpha < \frac{\pi}{2}$. For $t \in \mathbb{R}$, prove that $F(x, y)$ approaches $f(t)$ uniformly (with respect to t) as $(x, y) \to (t, 0)$ if the angle

$$\tan^{-1} \frac{x - t}{y}$$

is in the closed interval $[-\alpha, \alpha]$.

Problem 3 *(Addition Formula for Elliptic Sine Function as Generalization of Addition Formula for Trigonometric Sine Function).* For this problem we use the notation f' to mean the first derivative of f and use the notation f'' to mean the second derivative of f.

(a) Verify that the addition formula

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

for the trigonometric sine function is equivalent to the statement that for any constant c, the two functions $\varphi(u) = \sin u$ and $\psi(u) = \sin(c-u)$ of u satisfy the equation

$$\left(\psi\varphi' - \varphi\psi'\right)' \equiv 0.$$

Hint: To get the addition formula from the equation, integrate and then evaluate at $u = 0$ to determine the constant of integration.

(b) Verify the addition formula

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$$

for the elliptic sine function by using the following steps.

(i) Verify from

$$(\operatorname{sn} u)' = \sqrt{(1-\operatorname{sn}^2 u)(1-k^2\operatorname{sn}^2 u)}$$

that

$$(\operatorname{sn} u)'' = -(1+k^2)\operatorname{sn} u + 2k^2 \operatorname{sn}^3 u.$$

(ii) Let c be a constant. Let $\xi(u) = \operatorname{sn} u$ and $\eta(u) = \operatorname{sn}(c-u)$. Use **(i)** to verify

$$\left(\eta\xi' - \xi\eta'\right)' = 2k^2\xi\eta(\xi^2 - \eta^2).$$

Remark. When $k = 0$, this identity is reduced to the identity for trigonometric functions in Part**(a)**.

(iii) Use **(ii)** to verify

$$\left(\log\left(\frac{\eta\xi' - \xi\eta'}{1 - k^2\xi^2\eta^2}\right)\right)' \equiv 0.$$

(iv) By integrating and exponentiating the equation in **(iii)** and evaluating at $u = c$, verify that

$$\frac{\eta\xi' - \xi\eta'}{1 - k^2\xi^2\eta^2} \equiv \operatorname{sn} c.$$

Finally, use $(\operatorname{sn} u)' = \operatorname{cn} u \operatorname{dn} u$ to obtain the addition formula

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

for the elliptic sine function.

Problem 4 *(Double Periodicity of Weierstrass $\wp$ Function Proved Without Differentiating Defining Series Term-by-Term – from Stein & Sharkachi, p.279, #4).* Let ω_1, ω_2 be two complex numbers which are $\mathbb{R}$-linearly independent. Let Λ be the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\Lambda^* = \Lambda - \{0\}$. Recall that the Weierstrass $\wp$ function $\wp(z)$ (for the lattice Λ) is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right].$$

By rearranging the series

$$\frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right],$$

show directly, without differentiation, that $\wp(z+\omega) = \wp(z)$ whenever $\omega \in \Lambda$.

Hint: For R sufficiently large, note that

$$\wp(x) = \wp^R(z) + O\left(\frac{1}{R}\right),$$

where

$$\wp^R(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda, \\ 0 < |\omega| < R}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right].$$

Next, observe that both $\wp^R(z+\omega_1) - \wp^R(z)$ and $\wp^R(z+\omega_2) - \wp^R(z)$ are

$$O\left(\sum_{\substack{\omega \in \Lambda, \\ R-c < |\omega| < R+e}} \frac{1}{|\omega|^2} \right) = O\left(\frac{1}{R}\right).$$

Problem 5 *(Weierstrass $\wp$ Function as Series of Squares of Translated Cosecant Function – from Stein & Sharkachi, p.281, #2).* Let τ be a complex number with positive imaginary part. Let Λ be the lattice $\mathbb{Z} + \mathbb{Z}\tau$ and $\Lambda^* = \Lambda - \{0\}$. Recall that the Weierstrass $\wp$ function $\wp(z)$ (for the lattice Λ) is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right].$$

Prove that for some complex constant c,

$$\wp(z) = c + \pi^2 \sum_{m=-\infty}^{\infty} \frac{1}{\sin^2((z+m\tau)\pi)}$$

holds.

Hint: Verify that the series

$$\sum_{m=-\infty}^{\infty} \frac{1}{\sin^2((z+m\tau)\pi)}$$

defines a meromorphic function on $\mathbb{C}$ with double periods 1 and τ, whose principal part at any point z is the same as that of $\wp(z)$.