Problem Set 5.
The Peter-Weyl theorem.

Due to the length of this problem set, I am giving it two weeks.

October 7, 2014 Due Oct. 21

Typos: Ignore the commas in the subscripts on pages 9 and 10. Please tell me of additional typos.

The purpose of this problem set is to work through some of the fundamentals of the representation theory of compact groups. We will find that a key tool is our old friend, the spectral theorem for compact self-adjoint operators. When we are done, we will find that we have a vast generalization of the theorem which asserts that the e^{inx} form an orthonormal basis of $L_2(\mathbb{T})$. The Peter-Weyl theorem generalizes this result from $\mathbb{T}$ to an arbitrary compact group. This great result was published by Weyl and his student Peter in 1927

A more basic tool than the spectral theorem for compact self-adjoint operators is "averaging over the group". This requires the existence of an "invariant measure" on the group. The existence of such a measure is due to Haar, and the corresponding measure is known as "Haar measure". We will give a poor man's version of the existence theorem for Haar measures on compact groups. The tool we will use is a variant of the "mean ergodic theorem" which I hope we will use later in the course for other purposes. For the full rich version of Haar's theorem see Loomis, where he follows the treatment by A. Weil.

Throughout this problem set, all vector spaces are over the complex numbers.

Contents

1 The mean ergodic theorem.

Let T be a linear transformation on a normed vector space V which is such that

1. There is some constant c such that

$$\|T^n v\| \leq c\|v\|$$

for all $v \in V$ and all positive integers n.

2. For some fixed $w \in V$ the sequence of vectors

$$S_n w := \frac{1}{n}\left(w + Tw + \cdots + T^{(n-1)}w\right)$$

possesses a subsequence which converges to some $\hat{w} \in V$.

Theorem 1.1. The mean ergodic theorem. *$T\hat{w} = \hat{w}$ and the (full) sequence $S_n w$ converges to $\hat{w}$.*

The purpose of the next few problems is to give a proof of this theorem.

1. Let $(I - T)V$ denote the space of all vectors of the form $v - Tv$. Show that if $z = (I - T)v$ then $S_n z \to 0$. Conclude that $T\hat{w} = \hat{w}$.

2. Show that the space of all z such that $S_n z \to 0$ is a closed subspace. Consequently every $z \in \overline{(I-T)V}$ (the closure of $(I-T)V$) satisfies $S_n z \to 0$.

3. Show that $z - S_n z \in (I-T)V$ for any $z \in V$ and any positive integer n. Conclude that $\overline{(I-T)V}$ consists precisely of those $z \in V$ for which $S_n z \to 0$.

Now $T^n w = T^n \hat{w} + T^n(w - \hat{w})$ so

$$S_n w = \hat{w} + S_n(w - \hat{w}).$$

By assumption the subsequence $S_{n_j} w \to \hat{w}$ so $w - S_{n_j} w \to w - \hat{w}$. By problem **3**, $w - S_{n_j} w \in (I-T)V$ and hence $(w - \hat{w}) \in \overline{(I-T)V}$ and therefore $S_n(w-\hat{w}) = S_n w - \hat{w} \to 0$ which is the conclusion of the mean ergodic theorem. □

We can weaken hypothesis 2 to

"2weak: For some fixed $w \in V$ the sequence of vectors

$$S_n w := \frac{1}{n}\left(w + Tw + \cdots + T^{(n-1)} w\right)$$

possesses a subsequence which converges weakly to some $\hat{w} \in V$,"

if we assume that V is a Banach space and draw the same conclusion. Recall: a sequence $y_n \in V$ "converges weakly to y" means that for any continuous function ℓ on V we have $\ell(y_n) \to \ell(y)$. Indeed, we can argue as before to conclude that $\ell(T\hat{w} - \hat{w}) = 0$ for any continuous function ℓ. Hence, by the Hahn-Banach theorem, $T\hat{w} = \hat{w}$. We can also prove that $w - \hat{w} \in \overline{(I-T)V}$ as follows: If $(w - \hat{w}) \notin \overline{(I-T)V}$ then by Hahn-Banach there would exist some continuous linear function ℓ vanishing on $\overline{(I-T)V}$ with $\ell(w - \hat{w}) \neq 0$. But $w - S_{n_j} w \in (I-T)V$ and $\ell(w - S_{n_j} w) \to \ell(w - \hat{w})$ by hypothesis, a contradiction. □

The unit ball in a Hilbert space $\mathfrak{H}$ is weakly compact. Indeed, if ℓ is continuous linear function on $\mathfrak{H}$, then by the Riesz representation theorem, ℓ is given by scalar product with some element ϕ of $\mathfrak{H}$ and by the Cauchy-Schwarz inequality,

$$|(w, \phi)| \leq \|w\|\|\phi\| \leq \|\phi\|$$

so ℓ maps the unit ball in $\mathfrak{H}$ onto the closed disk of radius $\|\phi\|$ in $\mathbb{C}$.

So if T is an operator on $\mathfrak{H}$ which satisfies $\|Tw\| \leq \|w\|$ for all w in $\mathfrak{H}$, then 2weak is satisfied for any $w \in \mathfrak{H}$. In particular, we have

Theorem 1.2. [Mean ergodic theorem, usual form.] *Let U be a unitary operator on a Hilbert space $\mathfrak{H}$. The for any $w \in \mathfrak{H}$ the sequence*

$$S_n w = \frac{1}{n}\left(w + Uw + \cdots + U^{(n-1)} w\right)$$

converges to an element $\hat{w}$ which satisfies $U\hat{w} = \hat{w}$.

We will use the first statement given above (using hypotheses 1. and 2.) for getting Haar measure on a compact group.

2 Haar measure on totally bounded groups.

2.1 Compact groups, totally bounded groups.

A **topological group** is a topological space G which is also a group such that the multiplication map $G \times G \to G$ (sending $(a,b) \mapsto ab$) and the inverse map $G \to G$ (sending $a \mapsto a^{-1}$) are continuous. In a topological group, if U_α form a fundamental system of neighborhoods of the identity element e, then the translates aU_α form a fundamental system of neighborhoods of a.

If the group is compact (as a topological space) and separable (so that there is a countable family of fundamental neighborhoods U_i of the identity), then for each i we can find finitely many elements $g_{i,1} \ldots, g_{i,j(i)}$ such that $g_{i,1}U_i, \ldots, g_{ij(i)}U_i$ cover G. A group with this property will be called **totally bounded**. From now on we assume that G is totally bounded.

Let V be the space of uniformly continuous functions on G, and put the sup norm on V, i.e. define

$$\|f\| := \sup_{g \in G} |f(g)|.$$

(If G is compact then any continuous function is uniformly continuous.) Take the elements $g_{i,k}$ that enter into the definition of total boundedness and renumber them into a single sequence $\{g_i\}$ which is clearly dense in G. Define the operator T on V by

$$(Tf)(g) := \sum_{k=1}^{\infty} \frac{1}{2^k} f(g_k^{-1}g).$$

If $|f(a) - f(b)| < \epsilon$ whenever $a^{-1}b \in U$ where U is some neighborhood of e, then $(g_k^{-1}a)^{-1}(g_k^{-1}b) = a^{-1}b$ so

$$|Tf(a) - Tf(b)| < \epsilon$$

and hence T maps V into itself and clearly $\|Tf\| \leq \|f\|$ and so $\|T^n f\| \leq \|f\|$. Recalling the definition of S_n from the preceding section, we see that $S_n f(g) = \sum a_i f(h_i g)$ for some sequence of numbers $a_i > 0$ with $\sum_i a_i = 1$ and elements $h_i \in G$. So the same argument we just gave shows that $|S_n f(a) - S_n f(b)| < \epsilon$ if $a^{-1}b \in U$. Since G possesses a dense sequence of elements, we can choose a subsequence n_j so that $S_{n_j} f(h)$ converges at each element of of the subsequence, and hence everywhere by what we have just proved. So we conclude from the mean ergodic theorem that for each $f \in V$ the sequence $S_n f$ converges to an element $\hat{f}$ which satisfies

$$T\hat{f} = \hat{f}.$$

4. Show that $\hat{f}$ is a constant. [Hint: Let $M = \sup_{g \in G} f(g)$. If $\hat{f}$ is not a constant, there is an $\epsilon > 0$, an $h \in G$ and a neighborhood U of the origin such that $\hat{f}(g) < M - \epsilon$ for all g with $gh^{-1} \in U$. A finite number of g_iU will cover G. Conclude that $\sup_{g\in G} T\hat{f}(g) < M - \frac{\epsilon}{2^N}$ for some sufficiently large N.

It follows immediately from the definition that

1. The map $f \mapsto \hat{f}$ is linear.
2. If $\mathbf{1}$ denotes the function which is identically 1 then $\hat{\mathbf{1}} = \mathbf{1}$.
3. If $f(g) \geq 0 \forall g \in G$ then $\hat{f}(g) \geq 0$.
4. If $r_h : V \to V$ is the map given by $(r_h(f))(g) = f(gh)$ then $T \circ r_h = r_h \circ T$ and hence $\widehat{r_h f} = r_h \hat{f} = \hat{f}$ since $\hat{f}$ is constant.

Items 1 and 3 say that the the map $f \mapsto \hat{f}$ has the properties of an integral, so we will write

$$\int f(g)dg$$

instead of $\hat{f}$. Item 2 says that the total volume of G is one. Item 4 says that

$$\int (r_h f)(g)dg = \int f(g)dg$$

It follows from our construction that for any fixed $f \in V$ and and fixed $\epsilon > 0$ we can find a sequence of real numbers $a_i \geq 0$ with $\sum a_k = 1$ and group elements h_i such that

$$\sup_{g\in G} \left| \sum a_k f(h_k^{-1} g) - \int f(g)dg \right| < \epsilon. \tag{1}$$

Now we could have started our whole procedure with a map $\tilde{T}V \to V$ given by

$$(\tilde{T}f)(g) = \sum \frac{1}{2^k} f(gp_k)$$

for a suitable dense sequence p_k of elements of G. The arguments would have gone just as before, and we would have ended up with an integral $\int^*$ which would satisfy 1,2 and 3 with 4 replaced by

$$\int^* \ell_h f(g)dg = \int^* f(g)dg \quad \text{where} \quad (\ell_h f)(g) = f(h^{-1}g)$$

and with (1) replaced by

$$\sup_{g\in G} \left| \sum b_k f(gq_k) - \int^* f(g)dg \right| < \epsilon. \tag{2}$$

5. Show that (1) and (2) imply that $\int = \int^*$. In other words, $\int$ satisfies

$$\int \ell_h f(g)dg = \int f(g)dg.$$

6. Show that any integral $\int'$ which satisfies 1-4 must equal $\int$. In other words, properties 1-4 uniquely characterize $\int$. [Hint: use (2).]

It follows from this characterization that

$$\int_G f(g^{-1})dg = \int_G f(g)dg.$$

3 Representations.

Let G be a topological group and V a topological (complex) vector space. By a **representation** of G on V we will mean a continuous homomorphism of G into $\mathrm{End}(V)$, the space of continuous linear maps of V into itself, which sends the identity element e of G into the identity transformamtion I of V. Here $\mathrm{End}(V)$ is endowed with one of its suitable topologies, and the choice of such a topology will affect the definition of what we mean by a representation. If V is a finite dimensional, there is only one reasonable definition, the topology given by identifying $\mathrm{End}(V)$ with the set of all $n \times n$ matrices (using a choice of basis) and then identifying the set of $n \times n$ matrices with $\mathbb{C}^{2n}$, if V is a complex vector space (as we will be assuming) or with $\mathbb{R}^{2n}$ if V is a real vector space. The topology is clearly independent of the basis.

If necessary, we will denote a representation by a letter, such as R, so the image of an element $g \in G$ is $R(g) \in \mathrm{End}(V)$ and the action of $R(g)$ on $v \in V$ is $R(g)v$. Frequently, when R is fixed or understood, we will write $g \cdot v$ or even more simply gv instead of $R(g)v$. So we have

$$(gh)v = g(hv) \quad \text{and} \quad ev = v.$$

If we fix v, we obtain a map $f_v : G \to V$ given by

$$f_v(g) := gv.$$

Then the preceding equations translate into

$$f_v(e) = v$$

and

$$(\ell_h f_v)(g) = f_v(h^{-1}g) = (h^{-1}g)v = h^{-1}f_v(g),$$

i.e.

$$\ell_h f_v = h^{-1} \circ f_v.$$

3.1 Averaging over the group.

f_v is a continuous function of g and so we can integrate it over G to obtain

$$\int_G f_v(g)dg.$$

Then

$$h^{-1}\left(\int_G f_v(g)dg\right) = \int_G (\ell_h f_v)(g)dg = \int_G f_v(g)dg.$$

In other words (replacing h by h^{-1}) the vector $w \in V$ given by

$$w = \int_G f_v(g)dg = \int_G gvdg$$

is invariant under the action of G, i.e. satisfies

$$hw = w \quad \forall\ h \in G.$$

In words, averaging over the group produces an invariant element.

3.1.1 Maschke's theorem.

Let V be a (finite dimensional complex) vector space, and let $Q = Q(V)$ be the complex vector space of sesquilinear forms on V. That is, an element $q \in Q$ is a rule which assigns to every pair of vectors $v, w \in V$ a complex number $q(v, w)$ subject to the rules

- $q(v, w)$ is linear in v for fixed w and
- $q(w, v) = \overline{q(v, w)}$.

Suppose we are given a representation of G on V. Define $R_Q(g)$ acting on Q by

$$[R_Q(g)q](v, w) = q(g^{-1}v, g^{-1}w).$$

7.

- Show that R_Q is a representation of G on Q.
- Show that if q is positive definite, i.e. satisfies $q(v, v) > 0$ for all $v \neq 0$ then so is $\int_G R_Q(g)qdg$.
- Conclude that for any finite dimensional representation of G there exists an invariant Hilbert space structure on V, i.e. a Hilbert space scalar product (,) such that

$$(gv, gw) = (v, w) \ \ \forall g \in G,\ v, w \in V.$$

- Conclude that every finite dimensional representation (of a compact group) is completely reducible in the sense that if $W \subset V$ is an invariant subspace (meaning that if $w \in W$ then $gw \in W$ for all $g \in G$) then W possesses an invariant complement.

3.1.2 Averaging a linear map between two representation spaces.

Let R_V be a representation of G on a vector space V, and let R_W be a representation of G on a vector space W. Consider the space $\mathrm{Hom}(V,W)$ of all linear maps from V to W. Define a representation of G on $\mathrm{Hom}(V,W)$ by

$$[R_{\mathrm{Hom}(V,W)}(g)T](v) = \left(R_W(g) \circ T \circ (R_V(g^{-1})\right)(v).$$

(You should check for yourselves that this is indeed a representation.) You can see that the notation is getting cumbersome. It is neater (if less precise) to write

$$g: \quad T \mapsto gTg^{-1}.$$

To say that T is invariant under this action is the same as saying that T **intertwines** the two action is the sense that

$$g \circ T = T \circ g$$

or more formally as

$$R_w(g) \circ T = T \circ R_V(g)$$

for all $g \in G$. The space of T which satisfy this equation is denoted by $\mathrm{Hom}_G(V,W)$.

If we start with any $S \in \mathrm{Hom}(V,W)$ and average, i.e. set

$$T = \int_G g \circ S \circ g^{-1} dg$$

then $T \in \mathrm{Hom}_G(V,W)$.

3.1.3 Schur's lemma.

We say that a representation R_V is **irreducible** (or using more sloppy language that V is irreducible) if there is no non-trivial subspace V_1 of V invariant under G, i.e. if V_1 an invariant subspace then either $V_1 = \{0\}$ or $V_1 = V$.

We say that to representations R_V and R_W are **equivalent** and write $R_V \sim R_W$ if there is an *isomorphism* $T \in \mathrm{Hom}_G(V,W)$.

Theorem 3.1. [Schur's lemma.] *If R_V and R_W are irreducible finite dimensional representations then*

- *If $R_V \not\sim R_W$ then* $\mathrm{Hom}_G(V,W) = \{0\}$.
- $\mathrm{Hom}_G(V,V)$ *consists of all scalar multiples of the identity I.*

Proof.

- Suppose that $0 \neq T \in \mathrm{Hom}_G(V,W)$. Then the kernel $N(T) = \{v \in V | Tv = 0\}$ is an invariant subspace of V and so it is either all of V, in which case $T = 0$ or it is $\{0\}$ in which case T is injective. The image of T is an invariant subspace of W which is therefore either $\{0\}$ in which case $T = 0$ or is all of W so that T is an isomorphism, contrary to the hypothesis.

- Since V is a finite dimensional, any $T \int \mathrm{Hom}(V, V)$ has an eigenvalue, λ which means that $T - \lambda I$ has a non-zero kernel. By the first part this implies that $T - \lambda I = 0$.

□

3.1.4 Orthogonality of the matrix elements of inequivalent finite dimensional representations.

Let us combine the results of the previous three sections: Let R_V and R_W be inequivalent finite dimensional representations.

8. Show that for any $S \in \mathrm{Hom}(V, W)$ we have

$$\int_G R_W(g) \circ S \circ R_V(g^{-1}) dg = 0.$$

Choose bases of V and W so that relative to these bases we can write R_V, R_W, and S as matrices:

$$R_V(g) = (r_{Vij}(g)), \quad R_W(g) = (r_{Wk\ell}(g)), \quad S = (s_{\ell i}).$$

Then Problem **8** says that

$$\sum_{\ell i} \int_G r_{Wk,\ell}(g) s_{\ell i} r_{V,ij}(g^{-1}) dg = 0.$$

Since this is true for all $(s_{\ell i})$ we conclude that for all i, j, k, ℓ

$$\int_G r_{Wk,\ell}(g) r_{Vij}(g^{-1}) dg = 0.$$

Suppose that we choose our bases of V and W to be orthonormal bases relative to scalar products which are invariant under G. Then the matrices $(r_{Vij}(g)$ and $r_{Wk\ell}(g)$ are unitary. In particular, $r_{Vij}(g^{-1}) = (r_{Vij}(g))^* = (\overline{r_{Vji}(g)})$. So the previous equation reads

$$\int_G (r_{Wk\ell}(g) \overline{r_{Vji}(g)} dg = 0.$$

Let us put the scalar product

$$(f_1, f_2) := \int_G f(g) \overline{f_2(g)} dg$$

on the space of continuous functions on G. Then we can write

$$(r_{W,k\ell}, r_{V,ij}) = 0.$$

9. Show that when we take $V = W$ and use an orthonormal basis relative to an invariant scalar product then $(r_{Vij}, r_{Vk\ell}) = \int_G r_{V,ij}(g)\overline{r_{V,k\ell}(g)}dg = 0$ unless $i = k$ and $j = \ell$.

10. What is (r_{Vij}, r_{Vij})? [Hint: Use the fact that if $T = \int_G R_V(g)SR_V(g^{-1})dg$ then $\operatorname{tr} T = \operatorname{tr} S$.]

3.1.5 Characters and their orthogonality relations.

Let $r = R_V$ be a representation of G on a finite dimensional space. The **character** of this representation is the function χ^r on G given by

$$\chi^r(a) = \operatorname{tr} r(a).$$

So in terms of matrices,

$$\chi^r(a) = \sum_i r_{ii}(a).$$

If we take $a = e$, so that $r(e)$ is the identity operator we see that

$$r(e) = \dim V.$$

For any two linear transformations on a finite dimensional vector space we have $\operatorname{tr} AB = \operatorname{tr} BA$ so if B is invertible, $\operatorname{tr} BAB^{-1} = \operatorname{tr} A$. So it follows that

$$\chi(bab^{-1}) = \chi(a)$$

if χ is the character of any finite dimensional representation. Also, if we put an invariant scalar product on V so that $r(g^{-1}) = r(g)^*$ we see that

$$\chi(a^{-1}) = \overline{\chi(a)}.$$

If $V = V_1 \oplus V_2$ is a decomposition of V into invariant subspaces, then

$$\chi^r = \chi^{r_1} + \chi^{r_2}$$

in the obvious notation.

If r and s are inequivalent finite dimensional representations the it follows from Problem **8** and its consequences that

$$(\chi^r, \chi^s) = 0.$$

On the other hand, if r and s are equivalent representations, the $\chi^r = \chi^s$. In words: two equivalent (finite dimensional) representations have the same character.

If r is an irreducible representation then it follows from Problem **10** that

$$(\chi^r, \chi^r) = 1.$$

Let r be a representation of G on some finite dimensional vector space V which is not necessarily irreducible. Decompose V into a direct sum of irreducible representations. Let ϕ be the character of r, and χ_i the character of r_i, the i-th irreducible component. So

$$\phi = \chi_1 + \cdots + \chi_k.$$

Let s be some particular finite dimensional irreducible representation of G and χ its character. Then

$$(\phi, \chi) = (\chi_1, \chi) + \cdots + (\chi_k, \chi).$$

Each term on the right is zero or one according to whether $r_i \not\sim s$ or $r_i \sim s$. Thus (ϕ, χ) is the number of terms in the decomposition of s which are equivalent to s. In particular, this number does not depend on the particular decomposition of s into irreducibles, and it follows also that two representations with the same character are equivalent. The character *charactrizes* the representation - hence its name.

Suppose we do the decomposition of s as above, but now collect together all the terms with the same character so that, with mildly different notation,

$$\phi = m_1\chi_1 + \cdots + m_p\chi_p$$

where the χ_i are inequivalent (hence orthogonal) irreducible characters. Then

$$(\phi, \phi) = m_1^2 + \cdots + m_p^2.$$

It follows that the character

$$\phi \quad \text{is the character of an irreducible representation if and only if} \quad (\phi, \phi) = 1.$$

3.1.6 The irreducible representations of a direct product.

Let G and H be compact groups and $G \times H$ their direct product as groups and as topological spaces. So $G \times H$ is again a compact group. If R_V is a representation of G and S_W a representation of H then we get a representation $R_V \otimes S_W$ on $V \otimes W$ where

$$(R_V \otimes S_W)(g, h) = R_V(g) \otimes S_W(h).$$

Assume that V and W are finite dimensional. If χ is the character of R_V and ϕ is the character of S_W then $\chi\phi$ is the character of $R_V \otimes S_W$, where, of course,

$$(\chi\phi)(g, h) = \chi(g)\phi(h).$$

The direct product of the Haar measures of G and H is an (and hence the) invariant measure on $G \times H$ with total volume one. By Fubini's theorem,

$$(\chi\phi, \chi\phi)_{G\times H} = (\chi, \chi)_G \cdot (\phi, \phi)_H$$

in the obvious notation. So if R_V is an irreducible representation of G and S_W is an irreducible representation of H then $R_V \otimes S_W$ is an irreducible representation of $G \times H$.

For the proof of the Peter-Weyl theorem we will need the converse of this result. Namely

Proposition 3.1. *Any finite dimensional irreducible representation r of $G \times H$ on a vector space Z is of the form $R_V \otimes S_W$ where R_V is an irreducible representation of G and S_W is an irreducible representation of H.*

Proof. Consider the restriction of r to the group G, considered as the subgroup $G \times \{e_2\} \subset G \times H$ where e_2 is the identity element of H. The space Z decomposes into a finite direct sum of irreducible subspaces

$$Z = V_1 \oplus \cdots \oplus V_k$$

under G. I claim that all the V_i are equivalent to one another as representation spaces of G. Indeed, let Z_1 be the sum of all G invariant spaces U for which $\mathrm{Hom}_G(V_1, U) = \{0\}$. Thus Z_1 is the maximal such subspace., and is clearly G invariant. I claim that it is also H invariant. Indeed,

$$r_{(e_1,h)} r(_{g,e_2)} = r_{(g,h)} = r_{(g,e_2)} r_{(e_1,h)}.$$

Let us write this more succinctly (but more sloppily) as

$$r_g r_h = r_h r_g.$$

So

$$r_h \circ T \in \mathrm{Hom}_G(V_1, r_h U) \iff T \in \mathrm{Hom}_G(V_1, U).$$

So Z_1 is $G \times H$ invariant, and since $Z_1 \cap V_1 = \{0\}$ and $V_1 \neq \{0\}$, the irreducibility of r implies that $Z_1 = \{0\}$. Thus all the V_i are equivalent as representations of G. Let $V := V_1$.

Let $W := \mathrm{Hom}_G(V_1, Z)$. Schur's lemma tells us that $\dim W = k$. Also, we have an action of H on W given by

$$hT := r_h \circ T.$$

We have the "evaluation map" $\mathrm{ev} : V \times W \to Z$ determined by

$$v \otimes T \mapsto Tv.$$

Under this map

$$(gv) \otimes (hT) \mapsto r_h(T r_g v) = r_h(r_g T v) = r_g r_h(Tv) = r_{(g,h)} \mathrm{ev}(v \otimes w).$$

In other words, ev is a $G \times H$ equivariant map. It is clearly not zero, so its image must be all of Z. If the representation of H on W had a non-trivial invariant subspace W_1, then the image of $V \otimes W_1$ under ev would be a non-trivial invariant subspace of Z. So the representation of $G \times H$ on $V \otimes W$ is irreducible, hence ev must be an isomorphism.

□

3.2 The regular representation.

Let us complete the space of (uniformly) continuous functions on G with respect to the scalar product $(\ ,\ )$ and call this space $L_2(G)$. The group G acts on the space of (uniformly) continuous function of G by

$$G:\ f \mapsto \ell_g f \quad \text{where } (\ell_g f)(h) = f(g^{-1}h).$$

As usual we will sometimes write gf instead of $\ell_g(f)$. We have

$$(\ell_g f_1, \ell_g f_2) = \int_G \ell_g f_1(h)\overline{l_g f_2(h)}dh = \int_G f_1(h)\overline{f_2(h)}dh = (f_1, f_2).$$

In short

$$(gf_1, gf_2) = (f_1, f_2).$$

So the representation of G on the space of (uniformly) continuous functions extends to a unitary representation of G on $L_2(G)$. This representation (which is infinite dimensional unless G if finite) is known as the **regular representation**.

3.2.1 Representative functions and matrix elements of finite dimensional representations.

A (uniformly) continuous function f is called a **representative function** if the space spanned by all the $\ell_g f,\ g \in G$ is finite dimensional. For example, let R_V be a finite dimensional representation and $(r_{ij}(h)) = r_{V,ij}(h))$ the matrix of R_V relative a basis of V. Then the equation

$$r_{ij}(g^{-1}h) = \sum_{jk} r_{ij}(g^{-1})r_{jk}(h)$$

tells us that the set of functions $\ell_g r_{ij}$ lies in the finite dimensional space of functions spanned by all the $r_{k\ell}$. In other words, any linear combination of the matrix elements of a finite dimensional representation (considered as a function on G) is a representative function. We will now establish the converse:

Proposition 3.2. *Any representative function is a linear combination of the matrix elements of a finite dimensional representation (considered as functions on G).*

Proof. Let f be a representative function, and let

$$f_1 := g_1 f, \ldots, f_n := g_n f$$

be a maximal set of linearly independent elements among the $gf,\ g \in G$. Then

$$gf = \sum_{i=1}^{n} h_i(g) f_i$$

where the h_i are (uniformly) continuous functions on G. Replacing g by g^{-1} in the above equation gives

$$f(gu) = (g^{-1}f)(u) = \sum h_i(g^{-1})f_i(u), \quad \forall\ u \in G.$$

This shows that the set of functions $g \mapsto f(gu)$ also spans a finite dimensional space.

Consider the function $\check{f}$ given by $\check{f}(a) = f(a^{-1})$. Then $\check{f}$ is again a representative function. Let W be the finite dimensional space spanned by the gf, so that the f_i form a basis of W. then

$$gf_j = (gg_j)f = \sum_i h_i(gg_j)f_i$$

so that the values $h_i(gg_j)$ are the matrix elements of the representation of G on W relative to the basis $\{f_i\}$. But

$$\check{f}(g) = f(g^{-1}) = f(g^{-1}e) = \sum h_i(gg_j)f_i(e).$$

The $f_i(e)$ are constants, and the functions $g \mapsto h_i(gg_j)$ are matrix elements of a finite dimensional representation. This shows that $\check{f}$ is a linear combination of matrix elements. But $(\check{f})\check{} = f$. So if we started with $\check{f}$ we conclude that f is linear combination of matrix elements of a finite dimensional representation.

□

Since every finite dimensional representation is a (finite) direct sum of finite dimensional irreducible representations, we conclude that every representative function is a finite linear combination of matrix elements of finite dimensional irreducible representations.

3.2.2 An irreducible closed invariant subspace of $L_2(G)$ is finite dimensional.

We say that a closed invariant subspace W of $L_2(G)$ is irreducible if it possesses no closed proper invariant subspace; that is no closed invariant subspace other that itself of $\{0\}$. We want to prove the assertion in the title of this subsection. We will use our spectral theorem for compact self-adjoint operators.

Proof. Let f be a unit vector in $L_2(G)$. Consider the function k_f defined on $G \times G$ by

$$k_f(a,b) := \int_H \overline{f(ga)}f(gb)dg.$$

Since $f \in L_2(G)$ it is easy to check that k_f is a continuous function on $G \times G$.

Let P_f denote orthogonal projection onto the line spanned by f, so

$$P_f v = (v,f)f, \quad \forall v \in L_2(G).$$

Let K_f denote the operator

$$K_f := \int_G g^{-1} P_f g dg.$$

Then for any $v, w \in L_2(G)$ we have

$$\begin{aligned}
(K_f v, w) &= \int_G (g^{-1} P_f g v, w) dg \\
&= \int_G (P_f g v, g w) dg \\
&= \int_G (g v, f)(f, g w) dg.
\end{aligned}$$

Suppose that v and w are (uniformly) continuous functions. Then

$$\begin{aligned}
\int_G (gv, f)(f, gw) dg &= \int_G \left(v(g^{-1}a)\overline{f(a)} da \int_G f(b)\overline{w(g^{-1}b)} db \right) dg \\
&= \int_G \left(v(g^{-1}a)\overline{f(a)} da \int_G f(gb)\overline{w(b)} db \right) dg \\
&= \int_G \int_G k_f(a,b) v(a) \overline{w(b)} da db \\
&= \left(\int_G k_f(a, \cdot) v(a) da, w \right).
\end{aligned}$$

In other words, K_f is given by the integral operator

$$K_f v = \int_G k_f(a, \cdot) v(a) da$$

and this is valid therefore for all $v \in L_2(G)$. Since K_f has the continuous integral kernel k_f, it carries any bounded set in $L_2(G)$ into an equicontinuous set:

$$|(K_f v)(b1) - (K_f v)(b_2)| \leq \|v\|_2 \sup_{a \in G} |k_f(a, b_1) - k_f(a, b_2)|.$$

Hence K_f is compact. (In fact, any integral operator whose integral kernel is in $L_2(G \times G)$ is compact as we shall see in a later lecture.)

Since P_f is self-adjoint, so is K_f. If $f \in W$ where W is an invariant subspace then $P_f : L_2(G) \to W$ and hence so does $g^{-1} P_f g$ and hence so does K_f. Now

$$(p_f(gf), (gf) \geq 0$$

for all g, is continuous in g, and is $= 1$ when $g = e$ so

$$(K_f f, f) > 0.$$

Thus $K_f \neq 0$.

So K_f is a non-zero compact self-adjoint operator. It has a finite dimensional eigenspace corresponding to a non-zero eigenvalue. By construction, if $K_f w = \lambda w$, then $K_f g w = g g^{-1} K_f g w = g(K_f w) = \lambda g w$ so this eigenspace in invariant under the action of G. Hence, by the assumed irreducibility of W, it must coincide with W. □

3.3 Statement of the Peter-Weyl theorem.

Theorem 3.2.

1. *The representative functions are dense in* $L_2(G)$.
2. $L_2(G)$ *decomposes into a Hilbert space direct sum of irreducible representations of* G *each of which is finite dimensional.*
3. *Every irreducible representation of* G *is finite dimensional.*
4. *Each irreducible representation of* G *occurs in* $L_2(G)$ *with a multiplicity equal to its dimension.*
5. *Any unitary representation of* G *on any separable Hilbert space decomposes into a Hilbert space direct sum of (finite dimensional) irreducible representations.*

We have proved a special case of 3.

3.4 Proofs.

3.4.1 Proof of 2.

Let U_i be a fundamental system of neighborhoods of e and let $f_i \geq 0$ be a (uniformly) continuous function with $f_i(e) > 0$ and supp $(f_i) \subset U_i$. Multiplying f_i by a suitable positive constant, we may assume that

$$\int f_i(g)dg = 1.$$

Thus the f_i form a "sequence of approximations to the delta function". For any (uniformly) continuous function f, and any representation R of G, let us define the operator

$$R(f) := \int_G f(g)R(g)dg.$$

In particular, let us take R to be the regular representation and set

$$R_i = \ell((f_i)) = \int_G f_i(a)\ell_g dg.$$

Then

$$(R_i v)(b) = \int_G f_i(a)v(a^{-1}b)da = \int_G f_i(ab^{-1})v(a^{-1})da = \int_G f_i(ba^{-1})v(a)da..$$

Thus R_i is an integral operator with integral kernel

$$h_i(a,b) = f_i(ba^{-1})$$

and so is compact.

On the other hand, $R_i v \to v$ as $i \to \infty$. Indeed,

$$\begin{aligned} \|R_i v - v\| &= \left\| \int_G f_i(a)(av - v)da \right\| \\ &\leq \int_G f_i(a)\|av - v\|dg \end{aligned}$$

which approaches zero since $\int_G f_i(a)da = 1$ and $av \to v$ as $a \to e$. The operator R_i is compact and

$$R_i^* = \int_G f_i(a)(a^*)da = \int_G f_i(a)a^{-1}da.$$

So if we also choose f_i so that $f_i(a^{-1}) = f_i(a)$ then R_i is a self-adjoint compact operator.

Decompose

$$L_2(G) = H_{0,i} \oplus \bigoplus_j H_{j,i}$$

(Hilbert space direct sum) where $H_{0,i}$ is the zero eigenspace of R_i and $H_{j,i}$ are the non-zero eigenspaces, so that the $H_{j,i}$ are finite dimensional for $j \geq 1$.

Lemma 3.1. *$L_2(G)$ is the closure of the sum $\sum_i \sum_{j\geq 1} H_{j,i}$.*

Proof. If v is orthogonal to this subspace, then $v \in \bigcap H_{0,i}$ so $R_i v = 0$ for all i. But $R_i v \to v$ as $i \to \infty$ so $v = 0$ □

For each j and i, let $W_{j,i}$ denote the intersection of all the closed invariant subspaces of $L_2(G)$ which contain $H_{j,i}$. It is a closed invariant subspace containing $H_{j,i}$ and hence is the minimal such subspace.

Lemma 3.2. *Any closed non-zero invariant subspace of $W_{j,i}$ has a non-zero intersection with $H_{j,i}$.*

Proof. Suppose U is a closed invariant subspace of $W_{j,i}$ whose intersection with $H_{j,i}$ is zero. Then the orthogonal complement of U in $W_{j,i}$ would be a closed invariant subspace of $W_{j,i}$ containing $H_{j,i}$ and hence must coincide with $W_{j,i}$ by the minimality of $W_{j,i}$. Hence $U = \{0\}$. □

Consider all the intersections $U \cap H_{j,i}$ as U ranges over all closed invariant subspaces. Since $H_{j,i}$ is finite dimensional, the spaces $U \cap H_{j,i}$ are all finite dimensional and by the lemma, non-zero. Pick one such subspace $H^1_{i,j}$ of minimal dimension, and now set

$$W^1_{j,i} := \bigcap U$$

where U now ranges over all closed invariant subspaces satisfying

$$U \subset W_{j,i} \quad \text{and} \quad U \cap H_{j,i} = H^1_{j,i}.$$

Thus $W^1_{j,i}$ is the smallest closed invariant subspace of $W_{j,i}$ whose intersection with $H_{j,i}$ is $H^1_{j,i}$.

Lemma 3.3. *$W^1_{j,i}$ is irreducible.*

Proof. Any proper closed invariant subspace of $W^1_{j,i}$ would have to intersect $H^1_{j,i}$ in a proper subspace of $H^1_{j,i}$ contradicting our choice of $H^1_{j,i}$. □

We have found one finite dimensional irreducible subspace of $L_2(G)$.

Let us now replace $H_{j,i}$ by $H_{j,i} \cap (H^1_{j,i})^\perp$ and $W_{j,i}$ by $W_{j,i} \cap (W^1_{j,i})^\perp$ in the above argument. Proceeding as before, we will find a collection $H^1_{j,i}, H^2_{j,i}, \ldots$ of mutually orthogonal subspaces of $H_{j,i}$ and a collection of $W^1_{j,i}, W^2_{j,i}, \ldots$ of irreducible mutually orthogonal subspaces with $W^k_{j,i} \cap H_{j,i} = H^k_{j,i}$. Since $H_{j,i}$ is finite dimensional, there will be only finitely many such subspaces $H^k_{j,i}$ and $W^k_{j,i}$.

So far we have dealt with a fixed j and i. Let us now see what happens when we replace (j,i) with (s,r):

Lemma 3.4. *Suppose that the spaces $W^k_{j,i}$ and $W^\ell_{s,r}$ are not orthogonal. Then $W^\ell_{s,r} \subset W_{j,i}$.*

Proof. If $W^k_{j,i}$ and $W^\ell_{s,r}$ are not orthogonal, then the operator of orthogonal projection onto $W^k_{j,i}$ is non-trivial when restricted to $W^\ell_{s,r}$. But this operator commutes with action of G, and since $W^k_{j,i}$ and $W^\ell_{s,r}$ are irreducible, Schur's lemma tells us that these irreducible representations are equivalent. Therefore the operator $R(f_i) = \int_G f_i(a)a da$ has the same eigenvalue on these two subspaces, and so $W^\ell_{s,r}$ has non-zero intersection with $H_{j,i}$, and being irreducible, must be contained in any closed invariant subspace containing $H_{j,i}$. □

Let us relabel the $W_{j,i}$ as W_1, W_2, etc., and set

$$U_1 = W_1, \quad U_2 = W_2 \cap U_1^\perp, \ldots, U_{j+1} = W_j \cap (U_1 \oplus \cdots \oplus U_j)^\perp, \ldots.$$

Thus each U_j is the direct sum of finitely many irreducibles (which we know to be finite dimensional) and

$$W_1 + \cdots + W_j = U_1 \oplus \cdots \oplus U_j.$$

By Lemma 3.1 we conclude that $L_2(G)$ is the Hilbert space direct sum of the U_j which completes the proof of part 2 of the Peter-Weyl theorem.

As a corollary to the proof observe that there can only finitely many irreducible subspaces of a given type. Indeed, on equivalent irreducibles the operators R_i must have the same eigenvalues, and so correspond to the same $H_{j,i}$. Statement 4 in the Peter-Weyl theorem (which we have yet to prove) gives precise information as to how many times a given irreducible occurs in $L_2(G)$.

3.4.2 An extension of the argument used in the proof of Part 2 of the Peter-Weyl theorem.

Let G be a topological group and dg a left invariant measure on G. Let r be a unitary representation of G on some Hilbert space. For any continuous function

f of compact support on G we can form the operator

$$R_f = \int_G f(g)r(g)dg.$$

We will say that the representation r is **completely continuous** if each of the operators R_f is a compact operator. An examination of the argument given above shows that it proves

Proposition 3.3. *Let r be a completely continuous representation of the topological group G on a Hlbert space $\mathfrak{H}$. Then $\mathfrak{H}$ decomposes into a Hilbert space direct sum of finite dimensional irreducible subspaces where there are only finitely many irreducible subpaces in any given equivalence class of irreducible representations.*

The importance of this extension for us is the fact that if W is a closed invariant subspace of an $\mathfrak{H}$ as in the proposition, then the restriction of r to W is again completely continuous.

3.4.3 The action of $G \times G$ on $L_2(G)$.

Back to the case where G is a compact group: We have a map

$$\theta : L_2(G) \to L_2(G \times G), \quad (\theta f)(a,b) := f(ab^{-1}).$$

Notice that

$$\|\theta f\|^2_{G\times G} = \int_G \int_G |f(ab^{-1})|^2 da db = \int_G |f(a)|^2 da = \|f\|^2_G.$$

In other words, θ is an isometry.

Also

$$\theta(\ell_g f)(a,b) = f(g^{-1}ab^{-1}) = \ell_{(g,e)}(\theta f)(a,b).$$

In other words, if we regard G as the subgroup of $G \times G$ consisting of all (g,e) then the map θ intertwines the action of G with the action of $G \times \{e\}$.

If F is a function on $G \times G$ which satisfies $F(ag, bg) = F(a,b)$ for all $g \in G$, then taking $g = b^{-1}$ we see that $F(a,b) = F(ab^{-1}, e)$. So if $F \in L_2(G \times G)$ is invariant under the right action of the diagonal subgroup $\Delta = \{g, g\} \subset G \times G$, the $F = \theta f$ where $f(c) = F(c,e)$. The converse is obvious. So we can characterize the image of θ as being the subspace of $L_2(G \times G)$ consisting of vectors which are invariant under the right action of the diagonal subgroup G.

Now it is a fundamental principle of mathematics that (whenever the associative law holds) right action commutes with left action. In our case this says that

$$\left(\ell_{(c,d)}\left(r_{(g,g)}F\right)\right)(a,b) = F(c^{-1}ag, d^{-1}bg) = (r_{(g,g)}(\ell_{(c,d)}F))(a,b).$$

This shows that the image of θ is an invariant subspace of $L_2(G \times G)$ under the (left) action of $G \times G$, i.e. under the regular representation of $G \times G$.

So by Proposition 3.3 we know that $L_2(G)$ (identified with its image under θ) decomposes into a direct sum of irreductibles under $G \times G$ each occurring a finite number of times, and each finite dimensional.

Proposition 3.4. *Every finite dimensional irreducible representation of $G \times G$ that occurs in the decomposition of $L_2(G)$ into irreducibles under $G \times G$ occurs exactly once.*

Proof. Let W_1 and W_2 be two irreducible subspaces of $L_2(G)$ under $G \times G$ which define equivalent representations. Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be bases of W_1 and W_2 such that $G \times G$ has the same unitary matrix representation relative to these bases. Let

$$F(a, b) := \sum_i u_i(a)\overline{v_i(b)}.$$

Then

$$\begin{aligned} F(g^{-1}ah, g^{-1}bh) &= \sum_i u_i(h^{-1}ag)\overline{v_i(g^{-1}bh)} \\ &= \left(\sum_j r_{ij}(g, h)\overline{r_{ij}(g, h)}\right) \sum_i u_i(a)\overline{v_i(b)} \\ &= F(a, b) \end{aligned}$$

since the matrix (r_{ij}) is unitary. Taking $h = e = b$ and $a = g$ we obtain

$$\sum_i u_i(g)\overline{v_i(e)} = F(g, e) = F(e, g^{-1}) = \sum_i u_i(e)\overline{v_i(g^{-1})} = \sum_i u_i(e)\overline{\check{v}_i(g)}.$$

If we set

$$v = \sum_i u_i(e)v_i \in W_2$$

the preceding equation implies that $\overline{\check{v}} \in W_1$. The space of functions of the form $\overline{\check{v}}$, $V \in W_2$ is invariant under the action of $G \times G$ and hence must coincide with W_1. If we take $W_2 = W_1$ in the above argument, we see that $\overline{\check{u}} \in W_1$ if and only if $u \in W_1$. So we have proved that every irreducible representation that occurs in the decomposition of the representation of $G \times G$ on $L_2(G)$ occurs exactly once.

∎

Remark. In the course of that above proof, we had occasion, starting with the function v, to introduce the function $g \mapsto \overline{v(g^{-1})}$. So let us introduce the notation $\tilde{v}$ for this function. In other words, we define

$$\tilde{v}(g) := \overline{v(g^{-1})}.$$

Remember that we introduced this notation in Lecture 5 (on the Fourier transform) in conjunction with Plancherel's theorem.

We will now show that every finite dimensional irreducible representation of G gives rise to an irreducible occurring in the decomposition of the representation of $G \times G$ on $L_2(G)$. Suppose that we have an irreducible representation R_V on a finite dimensional space. Let $R_V^\sharp$ be the representation of G on the dual space V^* given by

$$R_V^\sharp(g) := \left(R_V(g^{-1})\right)^* .$$

There can not be a non-trivial invariant subspace of V^* under this representation, since its null space would be a non-trivial invariant subspace of V. So the representation $R_V^\sharp$ is irreducible, and hence the representation

$$R_V \otimes R_V^\sharp \quad \text{of} \quad G \times G \ \text{ on } V \otimes V^*$$

is irreducible.

Given $u \otimes v^* \in V \otimes V^*$ define the function $f_{u\otimes v^*}$ on G by

$$f_{u\otimes v^*}(g) := v^*(gu).$$

This clearly extends to a linear map

$$V \otimes V^* \to L_2(G).$$

Now

$$f_{(R_V(g)\times R_V^\sharp(h))(u\otimes v^*)}(a) = f_{gu\otimes h^{*-1}v^*}(a) = v^*(h^{-1}agu) = f_{u\otimes v^*}(h^{-1}ag).$$

In other words, the map that we have constructed from $V \otimes V^* \to L_2(G)$ is equivariant (and non-zero).

So each $V \otimes V^*$ occurs exactly once in the decomposition of $L_2(G)$ under $G \times G$.

We will have completed the proof of item 4 in the Peter-Weyl theorem if we could prove that these are all the irreducibles which occur in the decomposition of $L_2(G)$ under $G \times G$. By Proposition 3.1, we know that *any* irreducible representation of $G\times G$ is of the form $R_V \otimes R_W$ where R_V and R_W are irreducible reprsentations of G. We must show that if such a representation occurs in the decomposition of $L_2(G)$, then we must have $R_W = R_V^\sharp$. We can certainly write $R_W = R_Y^\sharp$ for some irreducible representation R_Y of G, so we must prove that R_Y is (equivalent to) R_V. Now $V \otimes Y^* = \mathrm{Hom}(Y, V)$ (as representations of G). By Schur's lemma, we must have $Y \sim V$ if there is a non-zero element of this space invariant under all $(g, g) \in G \times G$. So we will be done once we prove

Lemma 3.5. *Let Z be an irreducible subspace of $L_2(G)$. Then there exists an element of Z invariant under the action of all $(g, g) \in G \times G$.*

Proof. Let $z_1, \ldots, z_n$ be an orthonormal basis of Z and define the map $B : G \times G \to Z$ by

$$B(a, b) := \overline{z_1(ab^{-1})}z_1 + \cdots + \overline{z_n(ab^{-1})}z_n.$$

Then

$$B(ha,gb) = \overline{z_1(hab^{-1}g^{-1})}z_1 + \cdots + \overline{z_n(ab^{-1}g^{-1})}z_n. \qquad (*)$$

But

$$z_j(hab^{-1}g^{-1}) = \sum r_{ij}(h^{-1},g^{-1})z_i(ab^{-1})$$

where (r_{ij}) is the matrix of the representation of $G \times G$ on W, and this matrix is unitary, so the above equation becomes

$$B(ha,gb) = ((h,g)B)(a,b).$$

Now B does not map all of $G \times G$ into 0, so this last equation implies that $B(e,e) \neq 0$. But then taking $a = b = e$ and $h = g$ in (*) shows that $B(e,e)$ is invariant under all (g,g). This completes the proof of the lemma, and with it the proof of part 4 of the Peter-Weyl theorem.

□

It is easy to see that the $f_{u\otimes v^*}$ are precisely the matrix elements for the representation of G on $V \otimes V^*$ if we take u to be part of a basis of V and v^* to be part of a basis of V^*. Thus the irreducible subspaces of $L_2(G)$ consist precisely of the representative functions for the various irreducible finite dimensional representations of G. This proves 1.

Statement 3 is a consequence of Statement 5. So we must prove 5. Before doing so, let us state:

3.4.4 The analogue of Parseval's relation.

For each irreducible representation r^k occurring in the decomposition of $L_2(G)$ choose an orthonormal basis. We then know that matrix elements r^k_{ij} form an orthogonal basis of $L_2(G)$ and that $(r^k_{ij}, r^k_{ij}) = 1/n_k$ where n_k is the dimension of the corresponding vector space. So if we set

$$c^k_{ij}(f) := (f, r^k_{ij})$$

and

$$s^k_{ij} := n_k^{\frac{1}{2}} r^k_{ij}$$

then the s^k_{ij} form an orthonormal basis of $L_2(G)$ so we have the expansion

$$f = \sum_{k,i,j} n_k^{\frac{1}{2}} c^k_{ij} s^k_{ij}$$

and hence "Parseval's relation"

$$\|f\|_2^2 = \sum_k n_k \left(\sum_{ij} |c^k_{ij}(f)|^2 \right).$$

3.4.5 Convolution.

For any pair of continuous functions f_1 and f_2 on G, define their **convolution** $f_1 \star f_2$ by

$$(f_1 \star f_2)(a) := \int_G f_1(ag^{-1}) f_2(g) dg.$$

If r is any unitary representation of G and f is a continuous function define the operator r_f by

$$r_f := \int_G f(a) r(a) da.$$

11. Show that $r_{f_1} \circ r_{f_2} = r_{f_1 \star f_2}$.

12. Let r be a unitary representation of G. Show that

$$(r_f)^* = r_{\tilde{f}}.$$

Let r^k be an irreducible finite dimensional representation of G and (r^k_{ij}) its matrix relative to an orthonormal basis. We have

$$\operatorname{tr} r^k_f (r^k_f)^* = \sum_{ij} \int_G f(z) r^k_{ij}(a) \int_G \overline{f(b) r_{ij}(b)} db$$

$$= \sum_{ij} |c^k_{ij}(\overline{f})|^2.$$

Since $\|f\|_2 = \|\overline{f}\|_2$ we can write our Parseval relation as

$$\|f\|_2^2 = \sum_k n_k \operatorname{tr} r^k_f (r^k_f)^* = \sum_k n_k \operatorname{tr} r^k_{\tilde{f} \star f}$$

by Problem **12**.

Let r^1 and r^2 be irreducible finite dimensional representations of G. Then taking the convolution of their matrix elements gives

$$(r^1_{ij} \star r^2_{k\ell})(g) = \sum_p r^1_{ip}(g) \int_G r^1_{pj}(b^{-1}) r^2_{k\ell}(b) db.$$

If $r^1 \not\sim r^2$ all these last integrals vanish.

If $r^1 = r^2 = r_V$ (say) then these integrals vanish unless $j = k$ and $p = \ell$ in which case the integral equals $1/n$ where $n = \dim V$. So

$$\chi^i \star \chi^j = 0 \quad \text{if} \quad r^i \not\sim r^j$$

while for any irreducible, finite dimensional representation with character χ we have

$$\chi^i \star \chi^i = \frac{1}{\chi^i(e)} \chi^i.$$

To get feeling for the operators r_f let us consider the case where $G = \mathbb{T}$, the unit circle, and its regular representation r , i.e. acting on $L_2(\mathbb{T})$ (using additive notation by)

$$c : f \to cf, \qquad (cf)(x) = f(x - c).$$

If $\chi_n(x) = e^{inx}$ then

$$(r_{\chi_n} f) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{inc} f(x - c) dc = e^{inx} \frac{1}{2\pi} e^{-inu} f(u) du$$

by the change of variables $u = x - c$. In other words, the operator r_{χ_n} projects f onto the one dimensional space spanned by χ_n.

13. Let r be a unitary representation of a compact group G on a Hilbert space $\mathfrak{H}$. Let

$$P_\chi := \chi(e) R_\chi$$

where χ is the character of a finite dimensional irreducible representation of G. Show that P_χ is a projection operator. [Hint: use Problem **11** and the above formula for $\chi \star \chi$.]

Let r be a unitary representation of a compact group G on a Hilbert space $\mathfrak{H}$, and let

$$P_i = \chi^i(e) r_{\chi^i}$$

as the χ^i range over all the finite dimensional irreducible representations of G. I propose to show that the sum of all of these projections is the identity operator. Since we know that $P_i P_j = 0$ if $i \neq j$, this amounts to showing that

Proposition 3.5. *If $v \in \mathfrak{H}$ is such that $P_i v = 0$ for all i, then $v = 0$.*

Proof. Suppose that $P_i v = 0$ for all i. Define the function f by

$$f(g) := (v, r_g v)_{\mathfrak{H}}.$$

Then f is continuous and

$$\overline{(f \star \chi^i)(a)} = \int_G (v, \chi^i(b) r_{ab^{-1}} v)_{\mathfrak{H}} db$$

$$= \frac{1}{\chi^i(e)} (v, r_a P_i v)_{\mathfrak{H}} = 0.$$

So

$$f \star \chi^i \equiv 0$$

for all i. Hence

$$\tilde{f} \star f \star \chi^i = 0$$

for all i. Now for any function ϕ and any finite dimensional (irreducible) representation $r^i = (r^i_{k\ell})$ we have

$$(r^i_\phi)_{kk} = \phi \star r^i_{kk}$$

and hence

$$\phi \star \chi^i = \operatorname{tr} r^i_\phi.$$

Applied to $\phi := \tilde{f} \star f$ this tells us that

$$\operatorname{tr} r^i_{(\tilde{f}\star f)} = 0$$

for all i. Our last version of Parseval's relation, namely

$$\|f\|_2^2 = \sum_k n_k \operatorname{tr} r^k_{\tilde{f}\star f},$$

tells us that $\|f\| = 0$. But f is continuous, so this implies that $f(e) = 0$, so $(v, v)_{\mathfrak{H}} = f(e) = 0$ and hence $v = 0$ as was to be proved. □

To complete the proof of the Peter-Weyl theorem, it suffices to show that if $v = P_i v$ for some i, then v lies in a finite dimensional subspace of $\mathfrak{H}$.

Let $v = P_i v$ and let W be the space spanned by all the vectors $r_a v$. Let n_i be the dimension of the space of the irreducible representation r^i. In other words $n_i = \chi^i(e)$.

Proposition 3.6. $\dim W \leq n_i^2$.

Proof. It suffices to show that any collection of more than n_i^2 vectors of the form $r_{a_i} v$ must be linearly dependent, i.e. that the matrix

$$\left((r_{a_k} v, r_{a_j} v)_{\mathfrak{H}}\right)$$

is singular. Recalling our definition of the function f given by $f(b) = (v, r_a v)_{\mathfrak{H}}$, the k, j entry of the above matrix is $f(a_k^{-1} a_j)$. So it will more that enough to prove that the functions

$$f_j(g) := f(g a_j)$$

are linearly dependent if there are more that n_i^2 values of j. Now we have already verified that

$$n_i f(b) = f \star \overline{\chi^i}(b)$$

so

$$\begin{aligned} f(ga) &= n_i \int_G f(gab^{-1} \chi(b^{-1}) db \\ &= n_i \int_G f(gab) \chi(b) db \\ &= n_i \int_G f(b) \chi^i(a^{-1} g^{-1} b) db \end{aligned}$$

is a linear combination of the functions r^i_{jk} and there are only n_i^2 linearly independent such functions. This proves te proposition, and with it completes the proof of the Peter-Weyl theorem. □

3.5 The representative functions are dense among the continuous functions in the uniform topology.

Recall that we proved in class via Fejer's theorem that every continuous funtion on the circle can be approximated in the uniform topology by a trigonometric polynomial (but not necessarily by its Fourier series). We will now prove an analogous result for the representative functions on a compact group.

Let r denote the regular representation. Let h be a continuous function on G so that

$$(r_h v)(a) = \int_G v(g^{-1}a)h(g)dg$$

for any continuous function v. In particular, by Cauchy-Schwarz,

$$\sup_{a\in G} |(r_h v)(a)| \leq \|h\|_2 \|v\|_2.$$

On the other hand, suppose that f is (uniformly) continuous. Let us choose a neighborhood U of e such that

$$|f(x) - f(y)| < \frac{\epsilon}{2} \quad \forall\, x, y \;\text{ with } xy^{-1} \in U$$

and choose h to be a non-negative continuous function with support in U and with $\int_G h(g)dg = 1$. Then

$$\sup_{a\in G} |(r_h(f))(a) - a| < \frac{\epsilon}{2}.$$

Now the operator r_h preserves each of the subspaces $V_i \otimes V_i^*$ as i ranges over the irreducible representations of G, and hence any finite direct sum of such spaces. So if p belongs to such a space so does $r_h p$. Now choose p (by Peter-Weyl) so that

$$\|f - p\|_2 \leq \frac{\epsilon}{2\|h\|_2}.$$

Then

$$\begin{aligned}
\sup_{a\in G} |f(a) - (r_h p)(a)| &\leq \sup_{a\in G} (|f(a) - (r_h f)(a)| + |(r_h(f-p))(a)|) \\
&\leq \frac{\epsilon}{2} + \|f-p\|_2 \|h\|_2 \\
&\leq \epsilon. \qquad \square
\end{aligned}$$

Here is an amusing consequence:

Proposition 3.7. *A compact group is commutative if and only if all its irreducible represenations are one dimensional.*

Proof. If G is compact, we know that all its irreducible representations are finite dimensional. If it is commutative, Schur's lemma implies that any finite dimensional irreducible representation is one dimensional.

If all irreducibles are one dimensional, then any representative function f must satisfy $f(ab) = f(ba)$. But if G is not commutative, i.e. $ab \neq ba$ for some $a, b \in G$, then we can find a continuous function f with $f(ab) \neq f(ba)$ so f can not be approximated by representative functions. $\square$