

# Math 212a - Problem set 3

Convexity, arbitrage, and probability.

Tuesday, September 23, 2014, Due Sept. 30

The purpose of this problem set is to describe a formula in the pricing of options which was involved in the big financial crisis of 1998 (which pales in comparison to the crisis of 2008 but with similar government bail-out). I mean the infamous Black-Scholes formula. In the course of doing so we will encounter the “free market based” foundations for probability theory due to Bruno De Finetti “La prevision: ses lois logiques, ses sources subjectives” (1937) *Annales de l’Institut Henri Poincaré* **7** 1-68. We will start with a beautiful and seemingly harmless theorem of Caratheodory.

At the end, I will append some historical comments by Scott Kominers to this problem set when he took 212a in 2008.

Horse racing terminology: A bet on a horse to place, means that you are betting that your horse will finish first or second. If you bet on a horse to show, means that your are betting that your horse will finish first, second or third. I will not allow such bets.

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# 1 Farkas' lemma.

## 1.1 Caratheodory's theorem.

There are several variants of this theorem. Here is the one we will use:

**Theorem 1 [Caratheodory.]** *Let  $\mathbf{a}_1, \dots, \mathbf{a}_r$  be vectors in a real vector space  $V$ . Let  $\mathbf{x}$  be a linear combination of the  $\mathbf{a}_i$  with non-negative coefficients. Then  $\mathbf{x}$  is a linear combination with non-negative coefficients of a linearly independent subset of the  $\mathbf{a}_i$ .*

1. Prove this theorem. [Hint: Write

$$\mathbf{x} = \sum_{i=1}^k \lambda_i \mathbf{x}_i, \quad \lambda_i > 0$$

where the  $\mathbf{x}_i$  range over a subset of cardinality  $k$  of the  $\mathbf{a}_i$  and we have chosen  $k$  minimal with this property. If  $k = 0$  then  $\mathbf{x} = 0$  and there is nothing to prove. Otherwise, we wish to show that the  $\mathbf{x}_i$  are linearly independent. Suppose not. Then there exist  $\alpha_i \in \mathbb{R}$  not all zero with

$$\sum_i \alpha_i \mathbf{x}_i = 0.$$

We may assume that at least one of the  $\alpha_i$  is positive. (Otherwise multiply them all by  $-1$ .) So  $\alpha_i > 0$  for at least one  $i$ , and choose  $m$  so that  $\lambda_m/\alpha_m$  is minimal among the positive  $\alpha_i$ . Show that then we can write  $x$  as a combination with non-negative coefficients of the  $x_i$ ,  $i \neq m$ .]

The set of all linear combinations of the  $\mathbf{a}_i$  with non-negative coefficients is called the **cone** generated by the  $\mathbf{a}_i$ .

**Corollary 1** *The cone generated by a finite number of vectors in a real (topological) vector space is closed.*

**Proof.** By Caratheodory's theorem, this cone is the union of the cones generated by the linearly independent subsets, and there are finitely many of these. So it is enough to prove the corollary under the additional assumption that the vectors in question are linearly independent. The subspace spanned by these vectors is closed (being finite dimensional) and we may identify this subspace with  $\mathbb{R}^k$  with the vectors identified with the standard basis. If we do this, the cone becomes the closed (first) orthant consisting of all vectors with non-negative coordinates.  $\square$

Since the cone generated by a finite number of vectors is convex, the corollary implies that the cone generated by a finite number of vectors is a closed convex set. So if  $A$  is an  $m \times n$  matrix, then the set

$$\{Ax\} \quad x \in \mathbb{R}^n, \quad x \geq 0$$

is a closed convex subset of  $\mathbb{R}^m$ . Here we are using the notation  $x \geq 0$  for  $x \in \mathbb{R}^n$  to denote the assertion that all the coordinates of  $x$  are non-negative.

## 1.2 The separation theorem for closed convex bodies.

Let  $S$  be a closed convex non-empty subset of a real Banach space  $V$ . Let  $\mathbf{x} \in V$ ,  $\mathbf{x} \notin S$ . Then the separation theorem asserts that

**Theorem 2 [Separation.]** *There exists a continuous linear function  $\ell$  on  $V$  and a real number  $c$  such that  $\ell(\mathbf{y}) \geq c$  for all  $\mathbf{y} \in S$  and  $\ell(\mathbf{x}) < c$ .*

We will only need this theorem for finite dimensional vector spaces so we will only prove it in this case. We will therefore take  $V = \mathbb{R}^n$  with its standard metric, and we will write vectors as  $x, y$  etc.

**Lemma 1** *For any  $x \in V$  there is a unique point  $p(x, S) \in S$  which is closest to  $x$ , i.e. such that*

$$\|x - p(x, S)\| \leq \|x - y\| \quad \forall y \in S.$$

Indeed, if  $B(x, r)$  denotes the closed ball centered at  $x$  then  $B(x, r) \cap S$  is compact and is non-empty for large enough  $r$ . The function  $y \mapsto \|x - y\|$  is continuous, and hence attains a minimum on such a non-empty  $B(x, r) \cap S$ , say at  $y_0$ , and clearly

$$\|x - y_0\| \leq \|x - y\| \quad \forall y \in S.$$

We must show that  $y_0$  is unique. Suppose that there is also a  $y_1 \in S$  with

$$\|x - y_1\| \leq \|x - y\| \quad \forall y \in S.$$

Then  $x, y_0, y_1$  form an isosceles triangle with vertex at  $x$  and hence if we take  $z = \frac{1}{2}(y_0 + y_1)$  to be the midpoint of the base then  $z \in S$  since  $S$  is convex and

$$\|x - z\| < \|x - y_0\|$$

unless  $y_1 = y_0$ .  $\square$

Now suppose that  $x \notin S$  so  $p(x, S) \neq x$  so that we may form the unit vector

$$u(x, S) := \frac{1}{d(x, S)}(x - p(x, S))$$

where  $d(x, S)$  denotes the distance from  $x$  to  $S$  so  $d(x, S) = \|x - p(x, S)\|$ . Let  $H$  denote the hyperplane through  $p(x, S)$  orthogonal to  $u(x, S)$ . So  $H$  is defined by

$$H = \{z \mid u(x, S) \cdot (x - z) = u(x, S) \cdot (x - p(x, S)) = d(x, S)\}.$$

The hyperplane  $H$  is a **support** hyperplane in the sense that  $y = p(x, S) \in S \cap H$  and  $S$  lies entirely in one of the half spaces defined by  $H$ . Indeed, let  $H^-$  denote the closed halfspace bounded by  $H$  which does not contain  $x$ . We claim that  $S$  is completely contained in this halfspace. Indeed, suppose that there is some  $y \in S$  not in  $H^-$ . Consider the line segment  $[p(x, S), y]$  and let  $z$  be the point of this line segment closest to  $x$ . Then  $z \in S$  and  $\|x - z\| < \|x - p(x, S)\|$  contradicting the definition of  $p(x, S)$ . This shows that  $H$  is a support hyperplane and completes the proof of Theorem 2.  $\square$

Notice that a support hyperplane for a cone is a codimension one subspace (passing through the origin). We will actually need the existence of these support hyperplanes.

### 1.3 Farkas' lemma.

This says that for  $A$  an  $n \times m$  matrix

$$(\exists x \in \mathbb{R}^m, x \geq 0, Ax = b) \Leftrightarrow (\forall y \in \mathbb{R}^n (y^\dagger A \geq 0 \Rightarrow y^\dagger b \geq 0)). \quad (1)$$

2. Prove Farkas' lemma. [Hint: Use the result of the preceding section about support hyperplanes for cones.]

The Farkas Lemma is sometimes formulated as an alternative:

**Theorem 3 [The Farkas Lemma.]** *Either  $\exists x \geq 0$  with  $Ax = b$  or  $\exists y \in \mathbb{R}^n$  with  $y^\dagger A \geq 0$  and  $y^\dagger b < 0$  but not both.*

## 2 De Finetti's arbitrage theorem.

Suppose that there is an experiment having  $m$  possible outcomes for which there are  $n$  possible wagers. That is, if you bet the amount  $y$  on wager  $i$  you win the amount  $yr_i(j)$  if the outcome of the experiment is  $j$ . Here  $y$  can be positive, zero, or negative. A **betting strategy** is a vector  $\mathbf{y} = (y_1, \dots, y_n)^\dagger$  which means that you simultaneously bet the amount  $y_i$  on wager  $i$  for  $i = 1, \dots, n$ . So if the outcome of the experiment is  $j$ , your gain (or loss) from the strategy  $\mathbf{y}$  is

$$\sum_{i=1}^n y_i r_i(j).$$

**Theorem 4 [De Finetti's arbitrage theorem.]** *Exactly one of the following is true: Either*

- *there exists a probability vector  $\mathbf{p} = (p_1, \dots, p_m)$  so  $p_j \geq 0 \forall j$ ,  $\sum_j p_j = 1$  such that*

$$\sum_{j=1}^m p_j r_i(j) = 0 \quad \forall i = 1, \dots, n,$$

*or*

- *there exists a betting strategy  $\mathbf{y}$  such that*

$$\sum_{i=1}^n y_i r_i(j) > 0 \quad \forall j = 1, \dots, m.$$

*In other words either there exists a probability distribution on the outcome under which all bets have expected gain equal to zero, or else there is a betting strategy which always results in a positive win.*

3. Prove De Finetti's theorem. [Hint: Consider the matrix

$$A := \begin{pmatrix} r_1(1) & \cdots & r_1(m) \\ \vdots & \cdots & \vdots \\ r_n(1) & \cdots & r_n(m) \\ -1 & \cdots & -1 \end{pmatrix}$$

and vector

$$b := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}.$$

Use Farkas.]

It is important to observe that De Finetti's theorem does *not* say that  $\mathbf{p}$  is unique. But there are special circumstances in which uniqueness is obvious.

## 2.1 The price of a one period European call option.

Suppose that  $m = 2$  and that there is exactly one wager. So the matrix  $A$  is given by

$$A = \begin{pmatrix} r(1) & r(2) \\ -1 & -1 \end{pmatrix}.$$

If  $r(1) \neq r(2)$  this matrix is non-singular with inverse

$$A^{-1} = \frac{1}{r(2) - r(1)} \begin{pmatrix} -1 & -r(2) \\ 1 & r(1) \end{pmatrix}$$

so

$$A^{-1}b = \frac{1}{r(2) - r(1)} \begin{pmatrix} -1 & -r(2) \\ 1 & r(1) \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \frac{1}{r(2) - r(1)} \begin{pmatrix} r(2) \\ -r(1) \end{pmatrix}.$$

So if  $r(2) > 0$  and  $r(1) < 0$  both entries are positive and yield the unique probability vector

$$\mathbf{p} = \begin{pmatrix} 1 - p \\ p \end{pmatrix} = \frac{1}{r(2) - r(1)} \begin{pmatrix} r(2) \\ -r(1) \end{pmatrix}.$$

Of course, if both  $r(1)$  and  $r(2)$  are positive any wager which assigns a positive bet to both is a guaranteed win, and if both are negative then any wager which assigns a negative bet to both is a guaranteed win.

Now suppose that another wager is allowed. Suppose this bet has the return  $a$  if 1 occurs and the return  $b$  if 2 occurs. Then according to De Finetti's theorem, unless

$$a(1 - p) + bp = 0,$$

there will some combination of the two wagers that has a guaranteed profit.

As an illustration, suppose that an asset (say a stock) has value  $S(0)$  at the present time, and has only two possible values at time 1: Either

$$S(1) = uS(0) \quad \text{or} \quad S(1) = dS(0) \quad u > 1 > d.$$

In other words, either the stock can go up by a factor of  $u$  or down by a factor of  $d$ . Suppose also that if money is kept in the bank for this period it increases by a factor of  $1 + r$ . So the current value of  $M$  future dollars is  $M(1 + r)^{-1}$ .

So

$$r(2) = \frac{u}{1+r} - 1, \quad r(1) = \frac{d}{1+r} - 1$$

and

$$\begin{aligned} 1 - p &= \frac{u - 1 - r}{u - d}, \\ p &= \frac{1 + r - d}{u - d}. \end{aligned} \tag{2}$$

Notice that these two values,  $p$  and  $1 - p$  have nothing to do directly with any intuitive idea of how “probable” the stock is to go up or down. Of course, the current market price will be influenced by what people believe the stock will do, so there is an indirect relation between  $p$  and intuitive probability. This is De Finetti’s “market based” approach to the foundations of probability.

A **European call option** is the right to buy a number of shares of stock at time 1 at a specified **strike price**  $K$ . Let  $C$  be the (current) price of the call option. Let

$$K = kS(0).$$

Thus if the stock goes up by a factor of  $u$  then the gain per unit purchased is

$$\frac{u - k}{1 + r} - C$$

since you can buy the stock at time 1 for a price of  $kS(0)$  and sell it immediately at the price  $uS(0)$  and the option costs you  $C$  dollars today. If the stock goes down by a factor of  $d$  you lose  $C$  dollars. (I am assuming that  $d \leq k \leq u$ .) So unless

$$0 = -(1 - p)C + p \left( \frac{u - k}{1 + r} - C \right) = p \cdot \frac{u - k}{1 + r} - C = \frac{1 + r - d}{u - d} \cdot \frac{u - k}{1 + r} - C$$

De Finetti’s theorem guarantees the existence of a mixed strategy of buying or selling the stock and buying or selling the option with a sure profit. Since a fundamental law of economics says that “there is no free lunch” we must have

$$C = \frac{1 + r - d}{u - d} \cdot \frac{u - k}{1 + r}. \tag{3}$$

This is the “fair price” of the option in the sense that if the option were priced differently, an arbitrageur could make a guaranteed profit.

## 2.2 Odds.

In some situations the only type of wagers allowed are ones that choose one of the outcomes  $i = 1, \dots, m$  and then bet that  $i$  is the outcome. An example: a horse race where each bet is on a single horse and one can only bet that horse wins or does not win. The return on such a bet is usually quoted in term of **odds**. If the odds against outcome  $i$  are  $o_i$  to 1 then a one unit bet will return  $o_i$  if  $i$  occurs and  $-1$  if  $i$  is not the outcome.

4. Show that unless

$$\sum_{i=1}^m \frac{1}{1 + o_i} = 1$$

you can make a sure profit at the race track. Remember that at my race track you can bet a positive or negative amount on any horse to win or any combination of such bets, but there is no bet on a horse to place or to show.

5. Suppose that there are three horses and the odds are 1,2, and 3. By the previous problem a strategy exists for a sure win. Find such a strategy.

## 2.3 A multiperiod stock model.

Suppose that there are  $n$  consecutive periods and the bank interest rate is  $r$  per period.  $S(0)$  denotes the initial price of the stock and  $S(i)$  its price at time  $i$ . Suppose that  $S(i)$  is either  $uS(i-1)$  or  $dS(i-1)$  where

$$d < 1 + r < u.$$

Stock may be purchased or sold at any one of the times  $i = 0, 1, \dots, n$ . Let  $X_i = 1$  if the price goes up by the factor  $u$  from time  $i-1$  to time  $i$  and  $X_i = 0$  if it goes down. The succession of stock prices can be regarded as the outcome of an experiment whose possible values are given by the vector  $X = (X_1, \dots, X_n)$ . According to De Finetti's theorem, there must be probabilities on these outcomes which make all bets fair. Otherwise there is a way of making a sure profit.

6. Show that to satisfy the De Finetti no sure profit condition, the  $X_i$  must be independent and have probability  $p$  for  $X_i = 1$  and  $1 - p$  for  $X_i = 0$  where  $p$  is given by (2). [Hint: Consider the following bet: Choose  $i$  and a specific vector  $(x_1, \dots, x_{i-1})$  of zeros and ones. Observe the stock market. If  $X_j = x_j$  for every  $j = 1, \dots, i-1$  buy a unit of stock at time  $i-1$  and sell it at time  $i$ .]

So

$$W = \sum_{i=1}^n X_i$$

is binomial with parameters  $n$  and  $p$  and so

$$S(n) = u^W d^{n-W} S(0). \quad (4)$$

Now suppose that you could also buy or sell an option to buy stock at time  $n$  at a price equal to  $kS(0)$ . The present value of the option is given by the random variable

$$(1+r)^{-n} S(0) (u^W d^{n-W} - k)^+$$

per unit stock and so the fair price of the option per unit stock is

$$C = (1+r)^{-n} E [(u^W d^{n-W} - k)^+] S(0). \quad (5)$$

[I am using the notation  $f^+$  to denote the function  $x \mapsto \max(f(x), 0)$ .]

### 3 The Black-Scholes formula.

#### 3.1 Warm up.

To give an illustration of equations (2), (4), and (5), suppose that each period is an interval of length  $t/n$  and for each period we have  $u = e^{\sigma\sqrt{t/n}}$  and  $d = e^{-\sigma\sqrt{t/n}}$  for some positive number  $\sigma$ . Also, the interest rate  $r$  per period is now taken to be  $r \cdot \frac{t}{n}$  (so that the (uncompounded) rate over the interval of length  $t$  is  $r$ ). Then (2) says that

$$p = \frac{1 + rt/n - e^{-\sigma\sqrt{t/n}}}{e^{\sigma\sqrt{t/n}} - e^{-\sigma\sqrt{t/n}}}.$$

Using the Taylor series approximations to second order

$$\begin{aligned} e^{\sigma\sqrt{t/n}} &\doteq 1 + \sigma\sqrt{t/n} + \frac{\sigma^2 t}{2n} \\ e^{-\sigma\sqrt{t/n}} &\doteq 1 - \sigma\sqrt{t/n} + \frac{\sigma^2 t}{2n} \end{aligned}$$

Then we get

$$\begin{aligned} p &\doteq \frac{\sigma\sqrt{t/n} - \sigma^2 t/2n + rt/n}{2\sigma\sqrt{t/n}} = \frac{1}{2} + \frac{t\sqrt{t/n}}{2\sigma} - \frac{\sigma\sqrt{t/n}}{4} \\ &= \frac{1}{2} \left( 1 + \frac{r - \sigma^2/2}{\sigma} \sqrt{t/n} \right). \end{aligned}$$

So  $S(t) = d^n \left(\frac{u}{d}\right)^W S(0)$  or

$$\log \left( \frac{S(t)}{S(0)} \right) = n \log d + \log \left( \frac{u}{d} \right) W$$

$$= -n\sigma\sqrt{t/n} + 2\sigma\sqrt{t/n}W.$$

Recall that  $W = \sum_1^n X_i$  and the  $X_i$  are independent random variables each taking on the value 1 with probability  $p$  and 0 with probability  $1 - p$ . So

$$E(W) = np \quad \text{and} \quad \text{Var}(W) = np(1 - p).$$

Thus

$$E \left[ \log \left( \frac{S(t)}{S(0)} \right) \right] = -n\sigma\sqrt{t/n} + 2\sigma np\sqrt{t/n}.$$

In the above expression for  $p$  let us write  $\mu := r - \sigma^2$  so we have

$$p = \frac{1}{2} \left( 1 + \frac{\mu}{\sigma} \sqrt{t/n} \right)$$

and we obtain

$$E \left[ \log \left( \frac{S(t)}{S(0)} \right) \right] = \mu t.$$

We have

$$p(1 - p) = \frac{1}{4} \left( 1 - \frac{\mu^2 t}{\sigma^2 n} \right) \doteq \frac{1}{4}$$

for large  $n$ . Using  $\frac{1}{4}$  for the variance of the  $X_i$  we get

$$\text{Var} \left[ \log \left( \frac{S(t)}{S(0)} \right) \right] \doteq \sigma^2 t.$$

The central limit theorem says that  $\left[ \log \left( \frac{S(t)}{S(0)} \right) \right]$  is (approximately) normally distributed with mean  $\mu t$  and variance  $\sigma^2 t$ .

The compounded interest for the entire period is  $(1 + \frac{r}{n})^n$  which, according to Euler, tends to  $e^{rt}$ . To summarize: Under the above assumptions and approximations the price  $S(t)$  of a stock at time  $t$  is a random variable of the form  $e^X S(0)$  where  $X$  is a Gaussian with mean  $\mu t$  and variance  $\sigma^2 t$ . Furthermore, the compounded interest rate  $r$  is related to  $\mu$  and  $\sigma$  by  $r = \mu + \frac{1}{2}\sigma^2$ .

Of course the central limit theorem holds under much more general hypotheses, so this model of the stock price ratio is more general than just taking the limit of (4).

In the literature, a random variable of the form  $e^X$  where  $X$  is normal is called “log normal” meaning that its logarithm has a normal distribution.

We begin with some facts about the exponential of a normal distribution.

### 3.2 The mean and variance of $Y = e^X$ , $X$ normal.

Let  $X$  be a normal (=Gaussian) random variable with mean  $\mu$  and variance  $\sigma^2$ . Let  $Y$  be the random variable  $Y = e^X$ . We claim that

$$E(Y) = e^{\mu + \frac{1}{2}\sigma^2}, \tag{6}$$

and

$$\text{Var}(Y) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1). \tag{7}$$

7. Prove these formulas, first for  $\mu = 0$  and then for the general case.

### 3.3 Determining the drift from De Finetti.

Let us give the following formulation of De Finetti's idea in more abstract form, and motivated by the preceding discussion: According to De Finetti, if  $A$  is any asset, whose value at time  $t = 0$  is the number  $V(A, 0)$  and whose value at time  $t$  is the random variable  $V(A, t)$ , and if  $r$  is the continuously compounded interest rate, then

$$V(A, 0) = e^{-rt} E(V(A, t)) \quad (8)$$

A universe where this holds is called a **risk neutral** universe. (Perhaps it should be called a De Finetti universe but I am following standard usage.)

Now suppose that

$$V(A, t) = V(A, 0)e^{\sigma t^{\frac{1}{2}}N + \nu t} \quad (9)$$

where  $N$  is the unit normal.

**Lemma 2** *If  $V$  satisfies (8) and (9) then*

$$r = \nu + \frac{1}{2}\sigma^2. \quad (10)$$

8. Prove the lemma.

### 3.4 Black-Scholes

Let  $\mathcal{E}$  denote the Gauss error function so

$$\mathcal{E}(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-x^2/2} dx.$$

Let  $s(t) = V(A, t)$  as above and let  $S = s(0)$ . Let  $C$  be the value of a European call option with strike price  $K$  and time to expiration  $t$ . (**European** means that the call can only be exercised at the expiration date.) Let  $r$  be the continuously compounded bank (risk free) rate. The Black-Scholes equation asserts that

$$C = S\mathcal{E}\left(\frac{\log \frac{S}{K} + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) - e^{-rt}K\mathcal{E}\left(\frac{\log \frac{S}{K} + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right). \quad (11)$$

**Proof.** We know that

$$s(t) = Se^{\sigma t^{\frac{1}{2}}N + \nu t}$$

and that the value at time  $t$  is

$$\max(s(t) - K, 0).$$

So

$$C = e^{-rt}E(\max(s(t) - K, 0)).$$

Now

$$E(\max(s(t) - K, 0)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \max(Se^{\sigma t^{\frac{1}{2}}y + \nu t} - K, 0)e^{-y^2/2} dy.$$

The maximum in the integrand equals 0 if

$$\sigma t^{\frac{1}{2}}y + \nu t \leq \log \frac{K}{S}$$

i.e. if

$$y \leq \frac{\log \frac{K}{S} - \nu t}{\sigma \sqrt{t}}.$$

So the expectation in question is

$$\frac{1}{\sqrt{2\pi}} \int_{\frac{\log \frac{K}{S} - \nu t}{\sigma \sqrt{t}}}^{\infty} Se^{\sigma t^{\frac{1}{2}}y + \nu t} e^{-y^2/2} dy - \frac{1}{\sqrt{2\pi}} \int_{\frac{\log \frac{K}{S} - \nu t}{\sigma \sqrt{t}}}^{\infty} Ke^{-y^2/2} dy \quad (12)$$

The second integral is

$$K \left( 1 - \mathcal{E} \left( \frac{\log \frac{K}{S} - \nu t}{\sigma \sqrt{t}} \right) \right) = K \mathcal{E} \left( -\frac{\log \frac{K}{S} - \nu t}{\sigma \sqrt{t}} \right) = K \mathcal{E} \left( \frac{\log \frac{S}{K} + \nu t}{\sigma \sqrt{t}} \right).$$

This accounts for the second term in (11). We can write the exponential in the first integral as

$$\sigma t^{\frac{1}{2}}y + \nu t - y^2/2 = -(y - \sigma t^{\frac{1}{2}})^2/2 + (\nu + \frac{1}{2}\sigma^2)t$$

so the first integral in (12) becomes

$$\frac{1}{\sqrt{2\pi}} \int_{\frac{\log \frac{K}{S} - \nu t}{\sigma \sqrt{t}}}^{\infty} Se^{\sigma t^{\frac{1}{2}}y + \nu t} e^{-y^2/2} dy = Se^{(\nu + \frac{1}{2}\sigma^2)t} \frac{1}{\sqrt{2\pi}} \int_{\frac{\log \frac{K}{S} - \nu t}{\sigma \sqrt{t}} - \sigma \sqrt{t}}^{\infty} e^{-z^2/2} dz$$

as we make the change of variables  $z = y - \sigma t^{\frac{1}{2}}$ . The integral on the right of this equation evaluates as

$$\begin{aligned} \mathcal{E} \left( -\frac{\log \frac{K}{S} - \nu t}{\sigma \sqrt{t}} - \sigma \sqrt{t} \right) &= \mathcal{E} \left( \frac{\log \frac{S}{K} + \nu t}{\sigma \sqrt{t}} + \sigma \sqrt{t} \right) \\ \mathcal{E} \left( \log \frac{\frac{S}{K} + (\nu + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2)t}{\sigma \sqrt{t}} \right) &= \mathcal{E} \left( \log \frac{\frac{S}{K} + (r + \frac{1}{2}\sigma^2)t}{\sigma \sqrt{t}} \right). \end{aligned}$$

We must multiply this last expression by

$$e^{-rt} Se^{(\nu + \frac{1}{2}\sigma^2)t} = S.$$

This accounts for the first term in (11).  $\square$

### Editorial comments by Scott Kominers.

As with any economic model, the Black-Scholes formula makes a few simplifying assumptions which do not hold up in practice. Asset values do not actually follow a strict, stationary log-normal process.<sup>1</sup> Similarly, the Black-Scholes model assumes that returns in different markets are uncorrelated.<sup>2</sup> However, these assumptions are approximately accurate [most of the time].

Since Black-Scholes pricing is clearly and simply calculated from model parameters, the Black-Scholes model was revolutionary when it was announced. It appeared in the one of the top economics journals, the *Journal of Political Economy*.<sup>3</sup> It also led (with related work) to Robert Merton's and Myron Scholes's 1973 Economics Nobel Prize award.<sup>4</sup>

In a somewhat parallel series of events, Merton, Scholes and others founded *Long-Term Capital Management* (LTCM), a quantitative hedge fund based upon the theory underlying the Black-Scholes model. The fund was successful for several years. However, after Russia defaulted on its debt in 1998, several basic market regularities failed. This temporarily rendered the functional form assumptions of the Black-Scholes model *extraordinarily* inaccurate. Consequently, LTCM nearly went bankrupt. The Federal Reserve Bank of New York was forced to orchestrate a bailout, against fears that the collapse would precipitate a global financial crisis.

For a fascinating non-technical account of how the Black Scholes equation, or rather the use of this equation together with a certain amount of hubris by its practitioners almost led to the collapse of the American financial system in September 1998 see the book *When Genius Failed* by Roger Lowenstein.

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<sup>1</sup>Indeed, if asset values were that well-controlled, the market would be far less volatile.

<sup>2</sup>A look at the current financial crisis should convince you that this is definitely not the case in reality.

<sup>3</sup>Fischer Black and Myron Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* **81**(3)L637–654.

<sup>4</sup>Sadly, Fischer Black died in 1995 and was therefore ineligible to receive the award.