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The L^2 spectral representation
The Lax-Milgram theorem
The Friedrichs extension

Math 212a

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Contents

Recall:A sufficient condition for a symmetric operator to be self-adjoint.

Let A be a symmetric operator on a Hilbert space $\mathfrak{H}$. The following theorem has been found to be very useful:

Theorem

If there is a complex number z such that $A + zI$ and $A + \overline{z}I$ both map $D(A)$ surjectively onto $\mathfrak{H}$ then A is self-adjoint.

Proof.

We must show that if ψ and f are such that

$$(f, \phi) = (\psi, A\phi) \quad \forall\, \phi \in D(A)$$

$$\text{then} \qquad \psi \in D(A) \;\; \text{and} \quad A\psi = f.$$

Choose $w \in D(A)$ such that $(A + \bar{z}I)w = f + \bar{z}\psi$. Then for any $\phi \in D(A)$

$$(\psi, (A + zI)\phi) = (f + \bar{z}\psi, \phi) = (Aw + \bar{z}w, \phi) = (w, A\phi + z\phi).$$

Then choose $\phi \in D(A)$ such that $(A + zI)\phi = \psi - w$. So $(\psi, \psi - w) = (w, \psi - w)$ and hence $\|\psi - w\|^2 = 0$, i.e $\psi = w$, so

$$\psi \in D(A) \;\; \text{and} \quad A\psi = f.$$

Multiplication operators.

Here was an important application of this theorem:
Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\mathfrak{H} := L_2(X, \mu)$. Since we have not yet done measure theory, take $\mathfrak{H} := L_2(\mathbb{R}^n)$.
Let a be a real valued $\mathcal{F}$ measurable function (say continuous) on X (on $\mathbb{R}^n$) with the property that a is bounded on any measurable subset of X of finite measure (on bounded subsets of $\mathbb{R}^n$). Let

$$\mathcal{D} := \{u \in \mathfrak{H} | \int_X (1 + a^2)|u|^2 d\mu < \infty.$$

Notice that $\mathcal{D}$ is dense in $\mathfrak{H}$. Let S be the linear operator

$$u \mapsto au$$

defined on the domain $\mathcal{D}$. Notice that S is symmetric.

Proposition

The operator S with domain $\mathcal{D}$ is self-adjoint.

Proof.

The operator consisting of multiplication by

$$\frac{1}{i+a}$$

is bounded since $\left|\frac{1}{i+a}\right| \leq 1$ and clearly maps $\mathfrak{H}$ to $\mathcal{D}$. Its inverse is multiplication by $i+a$. Similarly multiplication by $-i+a$ maps $\mathcal{D}$ onto $\mathfrak{H}$. So we may take $z=i$ in the previous Theorem. □

The spectrum of a multiplication operator

Let A be the multiplication operator associated to the function a as above. The proof of the following proposition is obvious:

Proposition

The spectrum of A equals the **essential range** *of a, that is, the set of all $\lambda \in \mathbb{R}$ such that*

$$\mu\{x|\ |a(x) - \lambda| < \epsilon\} > 0$$

for all $\epsilon > 0$. If $\lambda \notin \mathrm{Spec}(A)$ then $(\lambda I - A)^{-1}$ is given by

$$((\lambda I - A)^{-1})f)((x) = (\lambda - a(x))^{-1} f(x)$$

and

$$\|(\lambda I - A)^{-1}\| = [\mathrm{dist}\,(\lambda, \mathrm{Spec}(A))]^{-1}\,.$$

Review: the spectral theorem that we proved via Hille-Yosida-Stone.

Theorem

Let A be a self-adjoint operator on a Hilbert space, $\mathfrak{H}$. There exists a unique linear map $f \mapsto f(A)$ from $\mathcal{C}_0(\mathbb{R})$ to $\mathcal{L}(\mathfrak{H})$ such that

- *The map is multiplicative, i.e. $(f_1 f_2)(A) = f_1(A) f_2(A)$,*

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- $(\overline{f})(A) = f(A)^*$,
- $\|f(A)\| \leq \|f\|_\infty$,
- *if $w \notin \mathbb{R}$ and r_w is the function $r_w(x) = 1/(w - x)$ then*

$$r_w(A) = R(w, A),$$

Review: the spectral theorem that we proved via Hille-Yosida-Stone.

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- $\|f(A)\| \leq \|f\|_\infty$,
- *if* $w \notin \mathbb{R}$ *and* r_w *is the function* $r_w(x) = 1/(w-x)$ *then*

$$r_w(A) = R(w, A),$$

- *if the support of f is disjoint from* $\mathrm{Spec}(A)$ *then* $f(A) = 0$.

Extension to Borel functions.

We then extended the homomorphism $f \mapsto f(A)$ from the space $C_0(\mathbb{R}$ to the space of bounded Borel functions using two Riesz representations theorems. All of this was completely canonical.

I will now give a different version of the spectral theorem. This version asserts that any self-adjoint operator on a separable Hilbert space is unitarily equivalent to a multiplication operator on a suitable measure space. This version is not canonical but is extremely useful.

The multiplication version of the spectral theorem.

I now want to turn to the multiplication form of the spectral theorem. This says that if A is a self-adjoint operator on a separable Hilbert space $\mathfrak{H}$, then there is a unitary isomorphism from $\mathfrak{H}$ to $L_2(M,\mu)$ (where (M,μ) is a measure space) which takes A into a multiplication operator.

The space (M,μ) is far from unique. We will find that we can take M to be the union of finite or countable copies of $\mathbb{R}$, each with its own measure. We will start with the case where there is a cyclic vector for A (see the next slide for the definition) in which case we can take M to be $\mathbb{R}$ with an appropriate measure.

Cyclic vectors.

A vector $v \in \mathfrak{H}$ is called **cyclic** for A if the linear combinations of all the vectors $R(z, A)v$ as z ranges over all non-real complex numbers is dense in $\mathfrak{H}$. Of course there might not be any cyclic vectors.

But suppose that v is a cyclic vector. Consider, as before, the continuous linear function ℓ on $C_0(\mathbb{R})$ given by

$$\ell(f) := (f(A)v, v).$$

If f is real valued and non-negative, then
$\ell(f) = (f^{\frac{1}{2}}(A)v, f^{\frac{1}{2}}(A)v) \geq 0.$
In other words, ℓ is a non-negative continuous linear functional.

As we have seen, the Riesz representation theorem then says that there is a non-negative, finite, countably additive measure $\mu = \mu_{v,v}$ on $\mathbb{R}$ such that

$$\ell(f) = \int_{\mathbb{R}} f d\mu.$$

In fact, from its definition, the total measure $\mu(\mathbb{R}) \leq \|v\|^2$.

Let us consider $C_0(\mathbb{R})$ as a (dense) subset of $L_2(\mathbb{R},\mu)$, and let $(\cdot,\cdot)_2$ denote the scalar product on this L_2 space. Then for $f,g \in C_0(\mathbb{R})$ we have

$$(f,g)_2 = \ell(\overline{g}f) = (g(A)^* f(A)v, v) = (f(A)v, g(A)v),$$

(where the last two scalar products are in $\mathfrak{H}$).

This shows that the map

$$f \mapsto f(A)v$$

is an isometry from $C_0(\mathbb{R})$ to the subspace of $\mathfrak{H}$ consisting of vectors of the form $f(A)v$. The space of vectors of the form $f(A)v$ is dense in $\mathfrak{H}$ by our assumption of cyclicity (since already the linear combinations of vectors of the form $r_z(A), \quad z \notin \mathbb{R}$ are dense). The space $C_0(\mathbb{R})$ is dense in $L_2(\mathbb{R})$. So the map above extends to a unitary map from $L_2(\mathbb{R}, \mu)$ to $\mathfrak{H}$ whose inverse we will denote by U.

So $U:\mathfrak{H}\to L_2(\mathbb{R},\mu)$ is a unitary isomorphism such that

$$U(f(A)v)=f,\quad \forall\ f\in C_0(\mathbb{R}).$$

Now let $f,g,h\in C_0(\mathbb{R})$ and set

$$\phi:=g(A)v,\qquad \psi:=h(A)v.$$

Then

$$(f(A)\phi,\psi)=\int_{\mathbb{R}} fg\overline{h}d\mu=(fU(\phi),U(\psi))_2$$

where, in this last term, the f denotes the operator of multiplication by f.

In other words,

$$Uf(A)U^{-1}$$

is the operator of multiplication by f on $L_2(\mathbb{R},\mu)$. In particular, U of the image of the operator $f(A)$ is the image of multiplication by f in L_2.

Let us apply this last fact to the function $f = r_z,\ z \notin \mathbb{R}$, i.e.

$$r_z(x) = \frac{1}{z-x}.$$

We know that the resolvent $r_z(A)$ maps $\mathfrak{H}$ onto the domain $D(A)$, and that multiplication by r_z, which is the resolvent of the operator on L_2 maps L_2 to the domain of the operator of multiplication by x. This latter domain is the set of $k \in L_2$ such that $xk(x) \in L_2$.

Now $(zI - A)r_z(A) = I$, so

$$Ar_z(A) = zr_z(A) - I.$$

Applied to $U^{-1}g, \quad g \in L_2(\mathbb{R},\mu)$ this gives

$$Ar_z(A)U^{-1}g = zr_z(A)U^{-1}g - U^{-1}g.$$

So

$$AU^{-1}Ur_z(A)U^{-1}g = zU^{-1}Ur_z(A)U^{-1}g - U^{-1}g,$$

and multiplying by U gives

$$UAU^{-1}r_z \cdot g = zr_z \cdot g - g.$$

So if we set $h = r_z \cdot g$ so $zr_z \cdot g - g = xh$ we see that

$$UAU^{-1}h = x \cdot h. \tag{1}$$

If $y \notin \mathrm{supp}(\mu)$ then multiplication by r_y is bounded on $L_2(\mathbb{R}, \mu)$ and conversely. So the support of μ is exactly the spectrum of A.

Now for a general separable Hilbert space $\mathfrak{H}$ with a self-adjoint operator A we can decompose $\mathfrak{H}$ into a direct sum of Hilbert spaces each of which has a cyclic vector. Here is a sketch of how this goes. Start with a countable dense subset $\{x_1, x_2, \dots\}$ of $\mathfrak{H}$. Let $\mathfrak{L}_1$ be the cyclic subspace generated by x_1, i.e. $\mathfrak{L}_1$ is the smallest (closed) cyclic subspace containing x_1. Let $m(1)$ be the smallest integer such that $x_{m(1)} \notin \mathfrak{L}_1$. Let $y_{m(1)}$ be the component of $x_{m(1)}$ orthogonal to $\mathfrak{L}_1$, and let $\mathfrak{L}_2$ be the cyclic subspace generated by $y_{m(1)}$.

Proceeding inductively, suppose that we have constructed the cyclic subspaces $\mathfrak{L}_i, \quad i = 1, \dots, n$ and let $m(n)$ be the smallest integer for which $x_{m(n)}$ does not belong to the (Hilbert space direct sum) $\mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \cdots \oplus \mathfrak{L}_n$. Let $y_{m(n)}$ be the component of $x_{m(n)}$ orthogonal to this direct sum and let $\mathfrak{L}_{n+1}$ by the cyclic subpace generated by $y(m)$. At each stage of the induction there are two possibilities: If no $m(n)$ exists, the $\mathfrak{H}$ is the finite direct sum $\mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \cdots \oplus \mathfrak{L}_n$. If the induction continues indefinitely, then the closure of the infinite Hilbert space direct sum $\mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \cdots \oplus \mathfrak{L}_n \oplus \cdots$ contains all the x_i and so coincides with $\mathfrak{H}$.

By construction, each of the spaces $\mathfrak{L}_i$ is invariant under all the $R(z, A)$ so we can apply the results of the cyclic case to each of the $\mathfrak{L}_i$. Let us choose the cyclic vector $v_i \in \mathfrak{L}_i$ to have norm 2^{-n} so that the total measure of $\mathbb{R}$ under the corresponding measure μ_i is 2^{-2n}. Recall that S denotes the spectrum of A and each of the measures μ_i is supported on S. So we put a measure μ on $S \times \mathbb{N}$ so that the restriction of μ to $S \times \{n\}$ is μ_n. Then combine the U_n given above in the obvious way.

We obtain the following theorem:

Let A be a self-adjoint operator on a separable Hilbert space $\mathfrak{H}$ and let $S = \mathrm{Spec}(A)$.

Theorem

There exists a finite measure μ on $S \times \mathbb{N}$ and a unitary isomorphism

$$U : \mathfrak{H} \to L_2(S \times \mathbb{N}, \mu)$$

such that UAU^{-1} is multiplication by the function $a(s,n) = s$. More precisely, U takes the domain of A to the set of functions $h \in L_2$ such that $ah \in L_2$ and for for all such functions h we have

$$UAU^{-1}h = ah.$$

For any $f \in C_0(\mathbb{R})$ we have

$$Uf(A)U^{-1} = \text{multiplication by } f.$$

In particular, if $supp(f) \cap S = \emptyset$ then $f(A) = 0$.

Non-negative operators

Definition

A symmetric operator H with domain $\mathcal{D}$ is **non-negative** if

$$(Hf, f) \geq 0 \quad \forall\, f \in \mathcal{D}.$$

More generally, we say that $H \geq c$ if $(Hf, f) \geq c\|f\|^2$ for all $f \in \mathcal{D}$.

The following theorem is an immediate consequence of our L^2 version of the spectral theorem and the spectral properties of multiplication operators:

Theorem

The following are equivalent:

- $H \geq c$.
- *The spectrum of* H *is contained in* $[c, \infty)$
- *In a spectral* L^2 *representation the function* $(s, n) \mapsto h(s, n) - c$ *is non-negative except on a set of measure zero.*

The case $c = 0$ state that a non.negative self-adjoint operator H is unitarily equivalent to a non-negative multiplication operator h on an L^2 space.

For any $\lambda \in \mathbb{R}$ we would like to define the operator H^λ so that it is unitarily equivalent to the multiplication operator h^λ. Since the L^2 version of the spectral theorem is not canonical, we must check that this is well defined.

H^λ is well defined.

For this, consider the function f on the real line given by

$$f(x) := (|x|^\lambda + 1)^{-1}.$$

f is a Borel function (actually continuous) so our functional calculus version of the spectral theorem tells us that there is a unique self-adjoint operator G such that $G = f(H)$. In any L^2 spectral representation G corresponds to the multiplication operator of the form $(h^\lambda + 1)^{-1}$ and so there is a (generally unbounded) canonical self-adjoint operator K such that $(K+1)^{-1} = f(H)$, and the uniqueness of the map $H \mapsto f(H)$ tells us that K is canonically defined, so H^λ is well defined.

The case $0 < \lambda < 1$.

Theorem

If H is a non-negative self-adjoint operator and $0 < \lambda < 1$, then

$$f \in \mathrm{Dom}(H) \;\Leftrightarrow f \in \mathrm{Dom}(H^\lambda) \text{ and } \; f \in \mathrm{Dom}(H^{1-\lambda})$$

in which case

$$Hf = H^{1-\lambda}(H^\lambda f).$$

Proof of the theorem

Proof.

It is enough to prove this in an L^2 spectral representation in case the theorem becomes the obvious statement that $\int(1+|h|^2)|f|^2 dmu < \infty$ if and only if both

$$\int(1+|h|^{2\lambda})|f|^2 d\mu < \infty \quad \text{and} \quad \int(1+|h|^{2-2\lambda})|h^\lambda f|^2 d\mu < \infty.$$

□

The case $\lambda = \frac{1}{2}$, the quadratic form associated a non-negative self-adjoint operator

Let H be a non-negative self-adjoint operator. For $f, g \in \mathrm{Dom}(H^{\frac{1}{2}})$ define

$$Q(f,g) := \left(H^{\frac{1}{2}}f, H^{\frac{1}{2}}g\right).$$

Proposition

$f \in \mathrm{Dom}(H)$ if and only if $f \in \mathrm{Dom}(H^{\frac{1}{2}})$ and there exists a $k \in \mathfrak{H}$ such that

$$Q(f,g) = (k,g)$$

for all $g \in \mathrm{Dom}(H^{\frac{1}{2}})$. If this happens then $Hf = k$.

Proof

Proof.

The equation in the proposition asserts that $f \in \mathrm{Dom}((H^{\frac{1}{2}})^*)$ with $(H^{\frac{1}{2}})^* H^{\frac{1}{2}} f = k$. Since $H^{\frac{1}{2}}$ is self-adjoint, the proposition is the special $\lambda = \frac{1}{2}$ of the preceding theorem. □

The Lax-Milgram theorem

Over the next few slides I follow the treatment in Helffer *Spectral theory and applications*. Let $\mathfrak{H}$ be a Hilbert space, and a a continuous sesquilinear form on $\mathfrak{H}$ meaning that there is a constant C such that

$$|a(u,v)| \leq C\|u\| \cdot \|v\| \quad \forall\ u,v\ \in \mathfrak{H}. \tag{2}$$

(Here the norms are those of $\mathfrak{H}$.) Since a in anti-linear and continuous in v we know by the Riesz representation theorem for Hilbert spaces that there is a bounded linear map $A : \mathfrak{H} \to \mathfrak{H}$ such that

$$a(u,v) = (Au, v)_{\mathfrak{H}}. \tag{3}$$

The Lax-Milgram theorem

$\mathfrak{H}$-ellipticity

Statement of the theorem

Definition

a is **$\mathfrak{H}$-elliptic** if there exists an $\alpha > 0$ such that

$$|a(u,u)| \geq \alpha \|u\|^2. \tag{4}$$

Theorem

[Lax-Milgram] *If a is $\mathfrak{H}$-elliptic, then the associated operator A is an isomorphism.*

The Lax-Milgram theorem

Proof, 1

From $a(u,v) = (Au,v)_{\mathfrak{H}}$ and the ellipticity and Cauchy-Schwarz we conclude that

$$\|Au\|\|u\| \geq |a(u,u)| \geq \alpha\|u\|^2.$$

So

$$\|Au\| \geq \alpha\|u\| \tag{5}$$

so A is injective and has an inverse bounded by α^{-1} on its image.

The Lax-Milgram theorem

Proof, 2

Claim: the image of A is dense. Indeed if u is orthogonal to the image of A, then $0 = (Au, u)_{\mathfrak{H}} = a(u, u)$ implying that $u = 0$ by the ellipticity.

Claim: The image of A is closed: Let $\{v_n\}$ be a Cauchy sequence in the image of A, and $\{u_n\}$ a sequence such that $Au_n = v_n$. From

$$\|Au\| \geq \alpha\|u\| \qquad (5)$$

we conclude that the u_n converge to some w and the continuity says that $v_n \to v = Aw$.

The above two facts show that A is surjective and (5) implies that A^{-1} is bounded with inverse α^{-1}, completing the proof of the Lax-Milgram theorem.

Gelfand's rigged Hilbert spaces

Let $\mathfrak{H}$ and $\mathfrak{M}$ be two Hilbert spaces together with a continuous injection of $\mathfrak{H}$ into $\mathfrak{M}$ which we write sloppily as

$$\mathfrak{H} \subset \mathfrak{M} \tag{6}$$

with the continuity of the injection meaning, as usual, that there is some constant $C > 0$ such that for any $u \in \mathfrak{H}$

$$\|u\|_{\mathfrak{M}} \leq C\|u\|_{\mathfrak{H}}.$$

We also assume that

$$\mathfrak{H} \text{ is dense in } \quad \mathfrak{M}. \tag{7}$$

The injection of $\mathfrak{M}$ into $\mathfrak{H}'$

If $h \in \mathfrak{M}$, then the linear function on $\mathfrak{H}$ given by

$$\ell_h(u) = (u, h)_{\mathfrak{M}}$$

is continuous. So $h \mapsto \ell_h$ gives a map of $\mathfrak{M} \to \mathfrak{H}'$, the dual space of $\mathfrak{H}$. This map is injective, for the density of $\mathfrak{H}$ in $\mathfrak{M}$ tells us that $0 = \ell_h(u) = (u, h)_{\mathfrak{M}}$ for all $u \in \mathfrak{H}$ implies that $h = 0$. So we have

$$\mathfrak{H} \subset \mathfrak{M} \subset \mathfrak{H}'.$$

Now let a be a continuous (with respect to $\mathfrak{H}$) elliptic sesquilinear form as above. We will associate to a an unbounded operator S on $\mathfrak{M}$ as follows: We define the domain $D(S)$ of S to consist of all $u \in \mathfrak{H}$ such that the map $v \mapsto a(u,v)$ is continuous on $\mathfrak{H}$ for the topology induced by $\mathfrak{M}$. Since we are assuming that $\mathfrak{H}$ is dense in $\mathfrak{M}$, this extends to a unique continuous anti-linear function on $\mathfrak{M}$, and hence by the Riesz representation theorem applied to $\mathfrak{M}$, we conclude that we get an $Su \in \mathfrak{M}$ such that

$$a(u,v) = (Su, v)_{\mathfrak{M}}. \tag{8}$$

Theorem

[Lax-Milgram-II] *S is a bijective map from $D(S)$ onto $\mathfrak{M}$ with S^{-1} a bounded operator on $\mathfrak{M}$. Furthermore, $D(S)$ is dense in $\mathfrak{M}$.*

S is injective

Proof.

By the continuity of the injection $\mathfrak{H} \to \mathfrak{M}$ and the ellipticity of a we have, for $u \in D(S)$,

$$\alpha \|u\|_{\mathfrak{M}}^2 \le C\alpha \|u\|_{\mathfrak{H}}^2 \le C|a(u,u)| = C\,|(Su,u)_{\mathfrak{M}}| \le \|Su\|_{\mathfrak{M}} \cdot \|u\|_{\mathfrak{M}}.$$

So

$$\alpha \|u\|_{\mathfrak{M}} \le C\|Su\|_{\mathfrak{M}}. \tag{9}$$

□

Notice that (9) implies that S^{-1} is bounded on its image.

S is surjective

Proof.

Let $h \in \mathfrak{M}$. Then $v \mapsto (h, v)_{\mathfrak{M}}$ is a continuous anti-linear function on $\mathfrak{H}$ so by the Riesz representation theorem for $\mathfrak{H}$ there is a $w \in \mathfrak{H}$ such that

$$(h, v)_{\mathfrak{M}} = (w, v)_{\mathfrak{H}}.$$

Let A be the (invertible) linear operator on $\mathfrak{H}$ given by the Lax-Milgram theorem. Let $u = A^{-1}w$ so

$$a(u, v) = (w, v)_{\mathfrak{H}} = (h, v)_{\mathfrak{M}}.$$

Therefore $u \in D(S)$ and $Su = h$, proving that $S : D(S) \to \mathfrak{M}$ is surjective, and hence bijective. □

$D(S)$ is dense in $\mathfrak{M}$,

We must show that $(u,h)_{\mathfrak{M}} = 0 \ \forall \ u \in D(S)$ implies that $h = 0$.

Proof.

By the surjectivity of S we can find a $v \in D(S)$ such that $Sv = h$. So $(Sv,u)_{\mathfrak{M}} = 0$ for all $u \in mathfrakM$. In particular $(Sv,v)_{\mathfrak{M}} = 0$. But $(Sv,v)_{\mathfrak{M}} = a(v,v) \geq \alpha \|v\|^2_{\mathfrak{H}}$ by our ellipticity assumption, so $v = 0$. □

This completes the proof of the theorem.

The Hermitian case - Lax-Milgram-III

We now specialize to the case that a is Hermitian, i,e, that

$$a(u,v) = \overline{a(v,u)}, \tag{10}$$

in addition to the previous assumptions. Then

Theorem

1. *S is closed,*
2. $S = S^*$,
3. *$D(S)$ is dense in $\mathfrak{H}$.*

Notice that 2) $\Rightarrow$ 1) since adjoints are closed.

Proof that $S = S^*$.

Proof.

Hermiticity implies that for $u, v \in D(S)$ we have $(Su, v)_{\mathfrak{M}} = (u, Sv)_{\mathfrak{M}}$, i.e. that the operator S on $\mathfrak{M}$ is symmetric. In particular,

$$D(S) \subset D(S^*).$$

We must show that $D(S) = D(S^*)$. So let $v \in D(S^*)$. The surjectivity of S tells us that there is a $w \in D(S)$ such that

$$Sw = S^* v.$$

So for all $u \in D(S)$ we have

$$(Su, w)_{\mathfrak{M}} = (y, Sw)_{\mathfrak{M}} = (u, S^* v)_{\mathfrak{M}} = (Su, v)_{\mathfrak{M}}.$$

The surjectivity of S now implies that $v = w \in D(S)$ and $Sv = S^* v$.

Proof that $D(S)$ is dense in $\mathfrak{H}$.

Proof.

Suppose that $h \in \mathfrak{H}$ is orthogonal to $D(S)$ in the $\mathfrak{H}$ metric. Choose $f \in \mathfrak{H}$ such that $Af = h$, where $A : \mathfrak{H} \to \mathfrak{H}$ is the isomorphism given by Lax-Milgram. So for all $u \in D(S)$ we have

$$0 = (u, h)_{\mathfrak{H}} = (u, Af)_{\mathfrak{H}} = \overline{(Af, u)_{\mathfrak{H}}} = \overline{a(f, u)} = a(u, f).$$

But $a(u, f) = (Su, f)_{\mathfrak{M}}$ and the surjectivity of S implies that $f = 0$ and hence $h = 0$. □

Semi-bounded operators

Definition

Let T_0 be a symmetric unbounded operator on a Hilbert space $\mathfrak{M}$ with domain D. We say that T_0 is **semibounded** (from below) if there exists a constant C such that

$$(T_0 u, u)_{\mathfrak{M}} \geq -C\|u\|^2_{\mathfrak{M}} \ \ \forall\ u \in D.$$

The Friedrichs extension

Theorem

A symmetric semibounded operator T_0 whose domain $D = D(T_0)$ is dense in $\mathfrak{M}$ has (at least one) self adjoint extension.

Replacing T_0 by $T_0 + cI$ for a suitable c we may assume that the operator T_0 satisfies

$$(T_0 u, u)_{\mathfrak{M}} \geq \|u\|^2_{\mathfrak{M}} \quad \forall\ u \in D. \tag{11}$$

We have the Hermitian form a_0 defined on $D \times D$ by

$$a_0(u, v) := (T_0 u, v)_{\mathfrak{M}},$$

and (11) says that

$$a_0(u, u) \geq \|u\|^2_{\mathfrak{M}}. \tag{12}$$

The space $\mathfrak{H}$

Consider the norm p_0 on D given by

$$p_0(u) := \sqrt{a_0(u, u)}$$

and let $\mathfrak{H}$ denote the completion of D relative to this norm. $\mathfrak{H}$ inherits a scalar product given by

$$(u, v)_{\mathfrak{H}} = \lim_{n \to \infty} a_0(u_n, v_n),$$

where $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in D for p_0 tending respectively to u and v in $\mathfrak{M}$.

The inequality (12) implies that

$$\|u\|_{\mathfrak{H}} \geq \|u\|_{\mathfrak{H}}$$

so the injection of $\mathfrak{H}$ into $\mathfrak{M}$ is continuous, and since $\mathfrak{H} \supset D$ and D is dense in $\mathfrak{M}$ so is $\mathfrak{H}$.

Applying Lax-Milgram

We are now in the context $\mathfrak{H} \subset \mathfrak{M} \subset \mathfrak{H}'$ of Lax-Milgram-III with

$$a(u, v) = (u, v)_{\mathfrak{H}}$$

so we get an unbounded self-adjoint operator S on $\mathfrak{M}$ extending T_0 whose domain $D(S)$ is contained in $\mathfrak{H}$. This completes the proof of the existence of the Friedrichs extension.

The hydrogen Hamiltonian

Up to various parameters (which I will absorb into the choice of units so as not to complicate the appearance of the formulas), this is the operator (initially on $C_0^\infty(\mathbb{R}^3)$) given by the operator

$$S_Z := H_0 - \frac{Z}{r}$$

where H_0 is the "free Hamiltonian" $H_0 = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)$ and $Z > 0$.. We will show that

$$(S_Z u, u)_{L_2} \geq -Z^2 \|u\|^2_{L_2} \tag{13}$$

so that S_Z is semi-bounded and hence has a Friedrichs extension. The proof of (13) hinges on an inequality that goes back to G.H.Hardy.

Hardy's inequality

This says that for $u \in C_0(\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} \|x\|^{-2}|u(x)|dx \leq 4(H_0 u, u)_{L_2} = 4\int_{\mathbb{R}^3} \|p\|^2|\hat{u}(p)|^2 dp. \qquad (14)$$

For the proof of this inequality, observe that

$$\int_{\mathbb{R}^3} \left\| \nabla u + \frac{1}{2}\frac{x}{\|x\|^2}u \right\|^2 dx \geq 0 \qquad (15)$$

so that

$$\int_{\mathbb{R}^3} \left(\|\nabla u\|^2 + \frac{1}{4}\frac{1}{\|x\|^2}|u|^2 \right) dx \geq -\int_{\mathbb{R}^3} \nabla u \cdot \frac{x}{\|x\|^2} u dx. \qquad (16)$$

I will now massage the right hand side of (16).

Integrating by parts

We have (since u is assumed to have compact support)

$$-2\int_{\mathbb{R}^3} u\frac{\partial u}{\partial x_i}\frac{x}{\|x\|^2}dx = \int_{\mathbb{R}^3} |u(x)|^2\frac{\partial}{\partial x_i}\left(\frac{x}{\|x\|^2}\right)dx$$

so summing over i gives

$$-2\int_{\mathbb{R}^3} \nabla u\cdot\frac{x}{\|x\|^2}udx = \int_{\mathbb{R}^3} |u(x)|^2\cdot\frac{1}{\|x\|^2}dx.$$

Substituting this into (16) gives Hardy's inequality.

Using Cauchy-Schwarz and Hardy

By Cauchy-Schwarz and Hardy (and $r = \|x\|$) we have

$$\int_{\mathbb{R}^3} \frac{1}{r}|u^2|dx \leq \left(\int_{\mathbb{R}^3} \frac{1}{r^2}|u|^2 dx\right)^{\frac{1}{2}} \|u\|_{L_2} \leq 2(H_0 u, u)_{L_2}^{\frac{1}{2}} \|u\|_{L_2}.$$

We now use our old trick $2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2$ to conclude that

$$(S_Z u, u) \geq (1 - \epsilon Z)(H_0 u, u) - \frac{Z}{\epsilon}\|u\|^2.$$

Taking $\epsilon = \frac{1}{Z}$ gives

$$(S_Z u, u)_{L_2} \geq -Z^2 \|u\|_{L_2}^2.$$