212a1415
Radon Nikodym.
Dualities and Riesz representation theorems.
The spectral theorem, improved.
Riesz redux

Math 212a

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Integration according to Daniel

Daniel's theory starts with a space of bounded real valued
functions on a set S which was closed under the operations $\wedge$ and
$\vee$ and a real valued function I on L which satisfies

1. I is linear: $I(af + bg) = aI(f) + bI(g)$
2. I is non-negative: $f \geq 0 \Rightarrow I(f) \geq 0$ or equivalently
   $f \geq g \Rightarrow I(f) \geq I(g)$.
3. $f_n \searrow 0 \Rightarrow I(f_n) \searrow 0$.

By using monotone limits the space L and the function I on it is
enlarged to a space L^1, and if $\mathcal{B}$ denotes the smallest monotone
class containing L then

Theorem

[12J] If $f \in \mathcal{B}$ then $f \in L^1 \Leftrightarrow \exists g \in L^1$ with $|f| \leq g$.

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Integrable functions

A function $f \in \mathcal{B}^+$ is called **integrable** if either f^+ or f^- is summable. Then

$$I(f) := I(f^+) - I(f^-)$$

is unambiguously defined, but may $= +\infty$ or $-\infty$. So a function f is summable iff it is integrable and $|I(f)| < \infty$.
To summarize:

Theorem

[12K] *If f and g are integrable then $f + g$ is integrable with $I(f + g) = I(f) + I(g)$ provided that $I(f)$ and $I(g)$ are not oppositely infinite. If f_n is integrable, $I(f_1) > -\infty$, and $f_n \nearrow f$ then f is integrable and $I(f_n) \to I(f)$.*

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Integrable sets.

Loomis calls a set A **integrable** if $\mathbf{1}_A \in \mathcal{B}$. The monotone class properties of $\mathcal{B}$ imply that the integrable sets form a σ-field. Then define

$$\mu(A) := I(\mathbf{1}_A)$$

and the monotone convergence theorem guarantees that μ is a measure. Loomis calls a set A **summable** if it is integrable and $\mu(A) < \infty$.

Notice that if S (the entire space) is summable then I is bounded with respect to the uniform norm.

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Using Stone's axiom

If we add Stone's axiom which says that

$$f \in L \Rightarrow f \wedge \mathbf{1} \in L$$

then the Lebesgue integral of $f \in L^1$ with respect to the measure μ coming from I is the same as I:

$$\int_S f d\mu = I(f).$$

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Integration on locally compact Hausdorff spaces.

Integration on locally compact Hausdorff spaces.

We now suppose that S is a locally compact Hausdorff space. As in the case of $\mathbb{R}^n$, we can (and will) take L to be the space of continuous functions of compact support. Dini's lemma then says that if $f_n \in L \searrow 0$ then $f_n \to 0$ in the uniform topology.

If A is any subset of S we will denote the set of $f \in L$ whose support is contained in A by L_A.

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Non-negative linear functions

Lemma

A non-negative linear function I is bounded in the uniform norm on L_C whenever C is compact.

Proof.

Choose $g \geq 0 \in L$ so that $g(x) \geq 1$ for $x \in C$. If $f \in L_C$ then

$$|f| \leq \|f\|_\infty g$$

so

$$|I(f)| \leq I(|f|) \leq I(g) \cdot \|f\|_\infty.$$

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Integration on locally compact Hausdorff spaces.

A Riesz representation theorem

Theorem

Every non-negative linear functional I on L is an integral.

Proof.

This is Dini's lemma together with the preceding lemma. Indeed, by Dini we know that $f_n \in L \searrow 0$ implies that $\|f_n\|_\infty \searrow 0$. Since f_1 has compact support, let C be its support, a compact set. All the succeeding f_n are then also supported in C and so by the preceding lemma $I(f_n) \searrow 0$. □

I now want to replace the "non negative" assumption in the theorem by a boundedness assumption. I want to prove

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Integration on locally compact Hausdorff spaces.

A better Riesz representation theorem

Theorem

*Let F be a bounded linear function on L (with respect to the
uniform norm). Then there are two integrals I^+ and I^- such that*

$$F(f) = I^+(f) - I^-(f).$$

Furthermore, if A is any integrable set with respect to I^+ then

$$\mu_{I^+}(A) \leq \|F\|_\infty$$

and similarly for I^-.

For this I need some facts about the "variations" of bounded linear
functions.

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Variations of a bounded linear function

The "positive part" of a bounded linear functional.

Suppose we start with an arbitrary L and I. For each $1 \leq p \leq \infty$ we have the norm $\| \cdot \|_p$ on L which makes L into a real normed linear space. Let F be a linear function on L which is bounded with respect to this norm, so that

$$|F(f)| \leq C\|f\|_p$$

for all $f \in L$ where C is some non-negative constant. The least upper bound of the set of C which is called $\|F\|_p$ as usual. If $f \geq 0 \in L$, define

$$F^+(f) := \mathrm{lub}\{F(g) : 0 \leq g \leq f, \ \ g \in L\}.$$

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Integration on locally compact Hausdorff spaces.

Variations of a bounded linear function
The "positive part" of a bounded linear functional, continued.

Then

$$F^+(f) \geq 0$$

and

$$F^+(f) \leq \|F\|_p \|f\|_p$$

since $F(g) \leq |F(g)| \leq \|F\|_p \|g\|_p \leq \|F\|_p \|f\|_p$ for all $0 \leq g \leq f$, $g \in L$, since $0 \leq g \leq f$ implies $|g|^p \leq |f|^p$ for $1 \leq p < \infty$ and also implies $\|g\|_\infty \leq \|f\|_\infty$. Also

$$F^+(cf) = cF^+(f) \quad \forall \ c \geq 0$$

as follows directly from the definition.

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Integration on locally compact Hausdorff spaces.

Variations of a bounded linear function

The "positive part" of a bounded linear functional, continued.

Suppose that f_1 and f_2 are both non-negative elements of L. If $g_1, g_2 \in L$ with

$$0 \le g_1 \le f_1 \text{ and } 0 \le g_2 \le f_2$$

then

$$F^+(f_1+f_2) \ge \mathrm{lub}\, F(g_1+g_2) = \mathrm{lub}\, F(g_1)+\mathrm{lub}\, F(g_2) = F^+(f_1)+F^+(f_2).$$

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Integration on locally compact Hausdorff spaces.

Variations of a bounded linear function

The "positive part" of a bounded linear functional, continued.

On the other hand, if $g \in L$ satisfies $0 \le g \le (f_1 + f_2)$ then $0 \le g \wedge f_1 \le f_1$, and $g \wedge f_1 \in L$. Also $g - g \wedge f_1 \in L$ and vanishes at points x where $g(x) \le f_1(x)$ while at points where $g(x) > f_1(x)$ we have $g(x) - g \wedge f_1(x) = g(x) - f_1(x) \le f_2(x)$. So

$$g - g \wedge f_1 \le f_2$$

and so

$$F^+(f_1+f_2) = \operatorname{lub} F(g) \le \operatorname{lub} F(g \wedge f_1) + \operatorname{lub} F(g - g \wedge f_1) \le F^+(f_1) + F^+(f_2)$$

So

$$F^+(f_1 + f_2) = F^+(f_1) + F^+(f_2)$$

if both f_1 and f_2 are non-negative elements of L.

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Integration on locally compact Hausdorff spaces.

Variations of a bounded linear function
The "positive part" of a bounded linear functional, continued.

Now write any $f \in L$ as $f = f_1 - g_1$ where f_1 and g_1 are non-negative. (For example we could take $f_1 = f^+$ and $g_1 = f^-$.)
Define: $F^+(f) = F^+(f_1) - F^+(g_1)$.
This is well defined, for if we also had $f = f_2 - g_2$ then $f_1 + g_2 = f_2 + g_1$ so

$$F^+(f_1) + F^+(g_2) = F^+(f_1 + g_2) = F^+(f_2 + g_1) = F^+(f_2) + F^+(g_1)$$

so

$$F^+(f_1) - F^+(g_1) = F^+(f_2) - F^+(g_2).$$

From this it follows that F^+ so extended is linear, and

$$|F^+(f)| \leq F^+(|f|) \leq \|F\|_p \|f_p\|$$

so F^+ is bounded.

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Integration on locally compact Hausdorff spaces.

Variations of a bounded linear function

The "negative part" of a bounded linear functional.

Define F^- by

$$F^-(f) := F^+(f) - F(f).$$

As F^- is the difference of two linear functions it is linear. Since by its definition, $F^+(f) \geq F(f)$ if $f \geq 0$, we see that $F^-(f) \geq 0$ if $f \geq 0$. Clearly $\|F^-\|_p \leq \|F^+\|_p + \|F\| \leq 2\|F\|_p$. We have proved:

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Integration on locally compact Hausdorff spaces.

Variations of a bounded linear function

Proposition

*Every linear function on L which is bounded with respect to the
$\| \cdot \|_p$ norm can be written as the difference $F = F^+ - F^-$ of two
linear functions which are bounded with respect to the $\| \cdot \|_p$ norm
and take non-negative values on non-negative functions.*

In fact, we could formulate this proposition more abstractly as dealing
with a normed vector space which has an order relation consistent with
its metric but we shall refrain from this more abstract formulation.
Together with the theorem about non-negative linear functions, the
proposition allows us to conclude the desired

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Integration on locally compact Hausdorff spaces.

The Riesz representation for measures

Theorem

Let F be a bounded linear function on L (with respect to the uniform norm). Then there are two integrals I^+ and I^- such that

$$F(f) = I^+(f) - I^-(f).$$

Furthermore, if A is any integrable set with respect to I^+ then

$$\mu_{I^+}(A) \le \|F\|_\infty$$

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Integration on locally compact Hausdorff spaces.

The complex case

So far, we have been dealing with vector spaces and linear
functions over the real numbers, because we used monotonicity to
develop the theory. But any vector space over the complex
numbers can be considered as a vector space over the real
numbers, and any (complex) linear function G can be written as
$G = F_1 + iF_2$ where F_1 and F_2 are real linear functions. We can
similarly consider complex valued measures. So, if L now denotes
the space of complex valued functions of compact support on a
locally compact Hausdorff space S we have:

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Integration on locally compact Hausdorff spaces.

The Riesz representation theorem for measures

Theorem

*Let F be a bounded linear function on L (with respect to the
uniform norm). Then there is a complex valued measure μ on S
such that*

$$F(f) = \int_S f d\mu.$$

I will later need an improved version of this theorem which gives
more details about μ. I will present this at the end of these slides.
But I will now pause in the treatment of measure and integration
to give a remarkable application of this Riesz representation
theorem, together with the Riesz representation theorem on Hilbert
spaces to get an improved version of the spectral theorem.

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Notational reminder

I am now going to use these Riesz representation theorems to
extend the functional calculus version of the spectral theorem that
we have already proved from continuous functions vanishing at
infinity to bounded Borel functions. I remind you of our notation
and of the theorem:

$\mathcal{C}_0(\mathbb{R})$ denotes the space of continuous functions vanishing at
infinty, and for $f \in \mathcal{C}_0(\mathbb{R})$, $\|f\|_\infty$ denotes its sup norm, i.e.

$$\|f\|_\infty = \sup_{t \in \mathbb{R}} |f(t)|.$$

$\mathfrak{H}$ denotes a (separable) Hilbert space and $\mathcal{L}(\mathfrak{H})$ denotes the space
of bounded linear operators on $\mathfrak{H}$.

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That part of the spectral theorem that we have already proved.

Theorem

Let A be a self-adjoint operator on a Hilbert space, $\mathfrak{H}$. There exists a unique linear map $f \mapsto f(A)$ from $\mathcal{C}_0(\mathbb{R})$ to $\mathcal{L}(\mathfrak{H})$ such that

- *The map is multiplicative, i.e. $(f_1 f_2)(A) = f_1(A) f_2(A)$,*

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- $(\overline{f})(A) = f(A)^*$,

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- *The map is multiplicative, i.e. $(f_1 f_2)(A) = f_1(A) f_2(A)$,*
- *$(\overline{f})(A) = f(A)^*$,*
- *$\|f(A)\| \leq \|f\|_\infty$,*

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That part of the spectral theorem that we have already proved.

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- *The map is multiplicative, i.e. $(f_1 f_2)(A) = f_1(A) f_2(A)$,*
- *$(\overline{f})(A) = f(A)^*$,*
- *$\|f(A)\| \leq \|f\|_\infty$,*
- *if $w \notin \mathbb{R}$ and r_w is the function $r_w(x) = 1/(w - x)$ then*

$$r_w(A) = R(w, A),$$

$\curvearrowright \curvearrowleft \curvearrowleft$

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That part of the spectral theorem that we have already proved.

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- *The map is multiplicative, i.e. $(f_1 f_2)(A) = f_1(A) f_2(A)$,*
- $(\overline{f})(A) = f(A)^*$,
- $\|f(A)\| \leq \|f\|_\infty$,
- *if $w \notin \mathbb{R}$ and r_w is the function $r_w(x) = 1/(w-x)$ then*

$$r_w(A) = R(w, A),$$

- *if the support of f is disjoint from $\mathrm{Spec}(A)$ then $f(A) = 0$.*

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That part of the spectral theorem that we have already proved.

Theorem

Let A be a self-adjoint operator on a Hilbert space, $\mathfrak{H}$. There exists a unique linear map $f \mapsto f(A)$ from $\mathcal{C}_0(\mathbb{R})$ to $\mathcal{L}(\mathfrak{H})$ such that

- *The map is multiplicative, i.e. $(f_1 f_2)(A) = f_1(A) f_2(A)$,*
- $(\overline{f})(A) = f(A)^*$,
- $\|f(A)\| \leq \|f\|_\infty$,
- *if $w \notin \mathbb{R}$ and r_w is the function $r_w(x) = 1/(w - x)$ then*

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Formulas for $f(A)$

If $f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t) e^{ixt} dt$ and $U(t)$ is the one parameter group generated by iA (guaranteed to exist by Stone's theorem) then

$$f(A) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t) U(t) dt.$$

If $f \in C_0^\infty(\mathbb{R})$ has compact support and $\tilde{f}$ is a almost holomorphic extension of f then

$$f(A) = -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial} \tilde{f}(z) R(z, A) dz.$$

I now embark on extending the class of functions for which $f(A)$ is defined using our Riesz representation theorems:

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I will temporarily use the notation $O(f)$ (the operator corresponding to f) instead of $f(A)$.

Fix vectors x, y in our Hilbert space $\mathfrak{H}$. If $f \in \mathcal{C}_0(\mathbb{R})$ then

$$|(O(f)x, y)| \leq \|O(f)\|\|x\|\|y\| \leq \|f\|_\infty \|x\|\|y\|.$$

So the map

$$f \mapsto (O(f)x, y)$$

is a continuous linear function on $\mathcal{C}_0(\mathbb{R})$ bounded in norm by $\|f\|_\infty \|x\|\|y\|$. By the Riesz representation theorem for measures, this means that there is a **unique** complex valued bounded measure

$$\mu_{x,y}$$

on $\mathbb{R}$ such that

$$(O(f)x, y) = \int_{\mathbb{R}} f d\mu_{x,y}.$$

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Now

$$(O(f)x, y) = (x, O(f)^*y) = (x, O(\overline{f})y) = \overline{(O(\overline{f})y, x)}$$

$$\overline{\int_{\mathbb{R}} \overline{f} d\mu_{y,x}} = \int_{\mathbb{R}} f d\overline{\mu_{y.x}}.$$

The uniqueness of the measure implies that

$$\mu_{y,x} = \overline{\mu_{x,y}}.$$

Thus, for each fixed Borel set $U \subset \mathbb{R}$ its measure $\mu_{x,y}(U)$ depends linearly on x and anti-linearly on y.

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We can choose a sequence of $f_n \in \mathcal{C}_0(\mathbb{R})$ which satisfy $0 \leq f_n \leq 1$
and which are monotone increasing to the constant function $\mathbf{1}_{\mathbb{R}}$. It
then follows from the monotone convergence theorem and
$|(O(f)x, x)| \leq \|f\|_\infty \|x\|^2$ that $\mu_{x,x}(\mathbb{R}) \leq \|x\|^2$ and hence that
$\mu_{x,y}$ defines a bounded measure on $\mathbb{R}$..

So if f is now any bounded Borel function on $\mathbb{R}$, the integral

$$\int_{\mathbb{R}} f d\mu_{x,y}$$

is well defined, and is bounded in absolute value by some constant
times $\|f\|_\infty \|x\| \|y\|$. If we hold f and x fixed, this integral is a
bounded anti-linear function of y, and hence by the Riesz
representation theorem (for Hilbert space) there exists a $w \in \mathfrak{H}$
such that this integral is given by (w, y). The w in question
depends linearly on f and on x because the integral does, and so:

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We have defined a linear map O from bounded Borel functions on $\mathbb{R}$ to bounded operators on $\mathfrak{H}$ such that

$$(O(f)x, y) = \int_{\mathbb{R}} f d\mu_{x,y}.$$

We now want to verify that this map $f \mapsto O(f)$ satisfies the properties listed in the theorem above, and so gives an extension of that theorem from continuous functions vanishing at infinity to bounded Borel functions.

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Verifying that $O(\overline{f}) = O(f)^*$.

Recall that
$$(O(f)x, y) = \int_{\mathbb{R}} f d\mu_{x,y}.$$

So $(O(\overline{f})x, y) = \int_{\mathbb{R}} \overline{f} d\mu_{x,y} = \overline{\int_{\mathbb{R}} f d\mu_{y,x}}$

$\qquad = \overline{(O(f)y, x)} = (x, O(f)y) = (O(f)^*x, y) \qquad \square$

We now turn to the proof of the multiplicativity which we know to be true for $f, g \in \mathcal{C}_0(\mathbb{R})$.

for $f, g \in \mathcal{C}_0(\mathbb{R})$ we have:

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$$\int_{\mathbb{R}} fg d\mu_{x,y} = (O(fg)x, y) = (O(f)O(g)x, y) = \int_{\mathbb{R}} f d\mu_{O(g)x,y}.$$

Since this holds for all $f \in C_0(\mathbb{R})$ (for fixed g, x, y) we conclude by the uniqueness of the measure that

$$\mu_{O(g)x,y} = g\mu_{x,y}.$$

Therefore, for any bounded Borel function f we have

$$(O(g)x, O(f)^*y) = (O(f)O(g)x, y)$$

$$= \int_{\mathbb{R}} f d\mu_{O(g)x,y} = \int_{\mathbb{R}} fg d\mu_{x,y}.$$

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$$(O(g)x, O(f)^*y) = \int_{\mathbb{R}} fg d\mu_{x,y}.$$

This holds for all $g \in \mathcal{C}_0(\mathbb{R})$ and so by the uniqueness of the measure again, we conclude that $\mu_{x,O(f)^*y} = f\mu_{x,y}$ and hence

$$(O(fg)x, y) = \int_{\mathbb{R}} gf d\mu_{x,y}$$

$$= \int_{\mathbb{R}} g d\mu_{x,O(f)^*y} = (O(g)x, O(f)^*y) = (O(f)O(g)x, y)$$

or

$$O(fg) = O(f)O(g)$$

as desired. $\square$

We have extended the homomorphism from $C_0(\mathbb{R})$ to a homomorphism from the bounded Borel functions on $\mathbb{R}$ to bounded operators on $\mathfrak{H}$.

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Resolutions of the identity.

Partitions of unity aka resolutions of the identity.

Define: $P(U) := O(\mathbf{1}_U)$ for any Borel set U. The following
facts are immediate:

1 $P(\emptyset) = 0$

Such a P is called a **resolution of the identity**. It follows from
the last item that for any fixed $x \in \mathfrak{H}$, the map $U \mapsto P(U)x$ is an
$\mathfrak{H}$ valued measure.

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3 $P(U \cap V) = P(U)P(V)$ and $P(U)^* = P(U)$. In particular, $P(U)$ is a self-adjoint projection operator.

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4. If $U \cap V = \emptyset$ then $P(U \cup V) = P(U) + P(V)$.

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4. If $U \cap V = \emptyset$ then $P(U \cup V) = P(U) + P(V)$.
5. For each fixed $x, y \in \mathfrak{H}$ the set function $P_{x,y} : U \mapsto (P(U)x, y)$ is a complex valued measure.

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Resolutions of the identity.

Actually, given any resolution of the identity we can give a meaning to the integral

$$\int_{\mathbb{R}} f dP$$

for any bounded Borel function f in the strong sense as follows: If $s = \sum \alpha_i \mathbf{1}_{U_i}$ is a simple function where

$$\mathbb{R} = U_1 \cup \cdots \cup U_n, \quad U_i \cap U_j = \emptyset, \quad i \neq j$$

and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, define

$$O(s) := \sum \alpha_i P(U_i) =: \int_{\mathbb{C}} s dP.$$

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and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, define

$$O(s) := \sum \alpha_i P(U_i) =: \int_{\mathbb{C}} s dP.$$

This is well defined on simple functions (is independent of the expression) and is multiplicative

$$O(st) = O(s)O(t).$$

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Resolutions of the identity.

Also, since the $P(U)$ are self adjoint,

$$O(\bar{s}) = O(s)^*.$$

It is also clear that O is linear and

$$(O(s)x, y) = \int_{\mathbb{R}} s dP_{x,y}.$$

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Resolutions of the identity.

$$(O(s)x, y) = \int_{\mathbb{C}} s \, dP_{x,y}.$$

As a consequence, we get

$$\|O(s)x\|^2 = (O(s)^*O(s)x, x) = \int_{\mathbb{C}} |s|^2 dP_{x,x}$$

so

$$\|O(s)x)\|^2 \leq \|s\|_\infty \|x\|^2.$$

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so

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If we choose i such that $|\alpha_i| = \|s\|_\infty$ and take $x = P(U_i)y \neq 0$, then we see that

$$\|O(s)\| = \|s\|_\infty$$

provided we now take $\|f\|_\infty$ to denote the **essential supremum** which (I recall) means the following:

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Resolutions of the identity.

Recall: the definition of the essential supremum

It follows from the properties of a resolution of the identity that if U_n is a sequence of Borel sets such that $P(U_n) = 0$, then $P(U) = 0$ if $U = \bigcup U_n$. So if f is any complex valued Borel function on $\mathbb{R}$, there will exist a largest open subset $V \subset \mathbb{R}$ such that $P(f^{-1}(V)) = 0$.

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Resolutions of the identity.

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It follows from the properties of a resolution of the identity that if U_n is a sequence of Borel sets such that $P(U_n) = 0$, then $P(U) = 0$ if $U = \bigcup U_n$. So if f is any complex valued Borel function on $\mathbb{R}$, there will exist a largest open subset $V \subset \mathbb{R}$ such that $P(f^{-1}(V)) = 0$. We define the **essential range** of f to be the complement of V, say that f is **essentially bounded** if its essential range is compact,

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Resolutions of the identity.

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It follows from the properties of a resolution of the identity that if U_n is a sequence of Borel sets such that $P(U_n) = 0$, then $P(U) = 0$ if $U = \bigcup U_n$. So if f is any complex valued Borel function on $\mathbb{R}$, there will exist a largest open subset $V \subset \mathbb{R}$ such that $P(f^{-1}(V)) = 0$. We define the **essential range** of f to be the complement of V, say that f is **essentially bounded** if its essential range is compact, and then define its essential supremum $\|f\|_\infty$ to be the supremum of $|\lambda|$ for λ in the essential range of f. Furthermore we identify two essentially bounded functions f and g if $\|f - g\|_\infty = 0$ and call the corresponding space $L^\infty(P)$.

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Resolutions of the identity.

Every element of $L^\infty(P)$ can be approximated in the $\|\ \|_\infty$ norm by simple functions, and hence the integral

$$O(f) = \int_{\mathbb{R}} f\, dP$$

is defined as the strong limit of the integrals of the corresponding simple functions. The map $f \mapsto O(f)$ is linear, multiplicative, and satisfies

$$O(\overline{f}) = O(f)^*$$

and

$$\|O(f)\| \le \|f\|_\infty$$

as before.

I now return to the general study of measures and integration.

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Absolute continuity.

Suppose we are given two integrals, I and J on the same space L. That is, both I and J satisfy the three conditions of linearity, positivity, and the monotone limit property that went into our definition of the term "integral". We say that J is **absolutely continuous** with respect to I if every set which is I null (i.e. has measure zero with respect to the measure associated to I) is J null.

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Bounded integrals or measures.

The integral I is said to be **bounded** if

$$I(\mathbf{1}) < \infty,$$

or, what amounts to the same thing, that

$$\mu_I(S) < \infty$$

where μ_I is the measure associated to I.

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The Radon-Nikodym theorem.

We will first formulate the Radon-Nikodym theorem for the case of bounded integrals, where there is a very clever proof due to von-Neumman which reduces it to the Riesz representation theorem in Hilbert space theory.

Theorem

[Radon-Nikodym] *Let I and J be bounded integrals, and suppose that J is absolutely continuous with respect to I. Then there exists an element $f_0 \in \mathcal{L}^1(I)$ such that*

$$J(f) = I(ff_0) \quad \forall \ f \in \mathcal{L}^1(J). \tag{1}$$

The element f_0 is unique up to equality almost everywhere (with respect to μ_I).

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Proof, after von-Neumann.

Consider the linear function

$$K := I + J$$

on L. Then K satisfies all three conditions in our definition of an integral, and in addition is bounded. We know from the case $p = 2$ of the completeness of L^p that $L^2(K)$ is a (real) Hilbert space. (Assume for this argument that we have passed to the quotient space so an element of $L^2(K)$ is an equivalence class of of functions.) The fact that K is bounded, says that $\mathbf{1} := \mathbf{1}_S \in L^2(K)$. If $f \in L^2(K)$ then the Cauchy-Schwartz inequality says that

$$K(|f|) = K(|f| \cdot \mathbf{1}) = (|f|, \mathbf{1})_{2,K} \leq \|f\|_{2,K} \|\mathbf{1}\|_{2,K} < \infty$$

so $|f|$ and hence f are elements of $L^1(K)$.

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Furthermore, if $f \in L^2(K)$,

$$|J(f)| \leq J(|f|) \leq K(|f|) \leq \|f\|_{2,K}\|\mathbf{1}\|_{2,K}.$$

So $f \mapsto J(f)$ is a bounded linear function on $L^2(K)$. We conclude
from the real version of the Riesz representation theorem, that
there exists a unique $g \in L^2(K)$ such that

$$J(f) = (f, g)_{2,K} = K(fg).$$

If A is any K- measurable subset of S , then $0 \leq J(\mathbf{1}_A) = K(\mathbf{1}_A g)$
so g is non-negative. (More precisely, g is equivalent K-almost
everywhere to a function which is non-negative.)

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We obtain inductively

$$
\begin{aligned}
J(f) &= K(fg) = \\
I(fg) + J(fg) &= I(fg) + I(fg^2) + J(fg^2) = \\
&\vdots \\
&= I\left(f \cdot \sum_{i=1}^{n} g^i\right) + J(fg^n).
\end{aligned}
$$

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$$
\begin{aligned}
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&\vdots \\
&= I\left(f \cdot \sum_{i=1}^{n} g^i \right) + J(fg^n).
\end{aligned}
$$

Let N be the set of all x where $g(x) \geq 1$. Taking $f = \mathbf{1}_N$ in the preceding string of equalities shows that

$$
J(\mathbf{1}_N) \geq nI(\mathbf{1}_N).
$$

Since n is arbitrary, we have proved

Lemma

The set where $g \geq 1$ has I measure zero.

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$$
\begin{aligned}
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&\vdots \\
&= I\left(f \cdot \sum_{i=1}^{n} g^i \right) + J(fg^n).
\end{aligned}
$$

We have not yet used the assumption that J is absolutely
continuous with respect to I. We now use this assumption to
conclude that N is also J-null. This implies that if $f \geq 0$ and
$f \in L^1(J)$ then $fg^n \searrow 0$ almost everywhere (J), and hence by the
dominated convergence theorem

$$
J(fg^n) \searrow 0.
$$

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&\vdots \\
&= I\left(f \cdot \sum_{i=1}^{n} g^i\right) + J(fg^n).
\end{aligned}
$$

We have proved that

$$
J(fg^n) \searrow 0.
$$

Plugging this back into the above string of equalities shows (by the monotone convergence theorem for I) that $f \sum_{i=1}^{\infty} g^n$ converges in the $L^1(I)$ norm to $J(f)$.

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$$f \sum_{i=1}^{\infty} g^n$$

converges in the $L^1(I)$ norm to $J(f)$.
In particular, since $J(\mathbf{1}) < \infty$, we may take $f = \mathbf{1}$ and conclude
that $\sum_{i=1}^{\infty} g^i$ converges in $L^1(I)$. So set

$$f_0 := \sum_{i=1}^{\infty} g^i \in L^1(I).$$

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$$f_0 := \sum_{i=1}^{\infty} g^i \in L^1(I).$$

We have

$$f_0 = \frac{1}{1-g} \quad \text{almost everywhere}$$

so

$$g = \frac{f_0 - 1}{f_0} \quad \text{almost everywhere}$$

and

$$J(f) = I(ff_0)$$

for $f \geq 0$, $f \in L^1(J)$. By breaking any $f \in L^1(J)$ into the difference of its positive and negative parts, we conclude that (1) holds for all $f \in L^1(J)$. The uniqueness of f_0 (almost everywhere (I)) follows from the uniqueness of g in $L^2(K)$. $\square$

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Extensions, I.

The Radon Nikodym theorem can be extended in two directions.
First of all, let us continue with our assumption that I and J are
bounded, but drop the absolute continuity requirement. Let us say
that an integral H is **absolutely singular** with respect to I if there
is a set N of I-measure zero such that $H(h) = 0$ for any h
vanishing on N.
Let us now go back to Lemma 10:

Lemma

The set where $g \geq 1$ has I measure zero.

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Define J_{sing} by

$$J_{sing}(f) = J(\mathbf{1}_N f).$$

Then J_{sing} is singular with respect to I, and we can write

$$J = J_{cont} + J_{sing}$$

where

$$J_{cont} = J - J_{sing} = J(\mathbf{1}_{N^c} \cdot).$$

Then we can apply the rest of the proof of the Radon Nikodym theorem to J_{cont} to conclude that

$$J_{cont}(f) = I(ff_0)$$

where $f_0 = \sum_{i=1}^{\infty}(\mathbf{1}_{N^c} g)^i$ is an element of $L^1(I)$ as before. In particular, J_{cont} is absolutely continuous with respect to I.

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Extensions, II.

A second extension is to certain situations where S is not of finite
measure. We say that a function f is **locally** L^1 if $f\mathbf{1}_A \in L^1$ for
every set A with $\mu(A) < \infty$. We say that S is σ-**finite** with respect
to μ if S is a countable union of sets of finite μ measure. This is
the same as saying that $\mathbf{1} = \mathbf{1}_S \in \mathcal{B}$. If S is σ-finite then it can be
written as a disjoint union of sets of finite measure. If S is σ-finite
with respect to both I and J it can be written as the disjoint union
of countably many sets which are both I and J finite. So if J is
absolutely continuous with respect I, we can apply the
Radon-Nikodym theorem to each of these sets of finite measure,
and conclude that there is an f_0 which is locally L^1 with respect to
I, such that $J(f) = I(ff_0)$ for all $f \in L^1(J)$, and f_0 is unique up to
almost everywhere equality. Read the details in Loomis.

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The map from $L^q \to (L^p)^*$

Recall that Hölder's inequality says that

$$\left| \int fg d\mu \right| \leq \|f\|_p \|g\|_q$$

if $f \in L^p$ and $g \in L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. For the rest of this lecture we will assume without further mention that this relation between p and q holds. Hölder's inequality implies that we have a map from

$$L^q \to (L^p)^*$$

sending $g \in L^p$ to the continuous linear function on L^p which sends

$$f \mapsto I(fg) = \int fg d\mu.$$

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The map from $L^q \rightarrow (L^p)^*$ is injective

Furthermore, Hölder's inequality says that the norm of this map
from $L^q \rightarrow (L^p)^*$ is ≤ 1. In particular, this map is injective.

The theorem we want to prove is that under suitable conditions on
S and I (which are more general even that σ-finiteness) this map is
surjective for $1 \leq p < \infty$

We will first prove the theorem in the case where $\mu(S) < \infty$, that
is when I is a bounded integral. For this we will will use our results
about the "variations of a of a bounded linear functional":

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Duality of L^p and L^q when $\mu(S) < \infty$.

Duality of L^p and L^q when $\mu(S) < \infty$.

Theorem

Suppose that $\mu(S) < \infty$ and that F is a bounded linear function on L^p with $1 \le p < \infty$. Then there exists a unique $g \in L^q$ such that

$$F(f) = (f, g) = I(fg).$$

Here $q = p/(p - 1)$ if $p > 1$ and $q = \infty$ if $p = 1$.

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Duality of L^p and L^q when $\mu(S) < \infty$.

Proof, 1.

Consider the restriction of F to L. We know that $F = F^+ - F^-$
where both F^+ and F^- are linear and non-negative and are
bounded with respect to the $\| \cdot \|_p$ norm on L. The monotone
convergence theorem implies that if $f_n \searrow 0$ then $\|f_n\|_p \to 0$ and
the boundedness of F^+ with respect to the $\| \cdot \|_p$ says that

$$\|f_n\|_p \to 0 \quad \Rightarrow \quad F^+(f_n) \to 0.$$

So F^+ satisfies all the axioms for an integral, and so does F^-. If f
vanishes outside a set of I measure zero, then $\|f\|_p = 0$. Applied
to a function of the form $f = \mathbf{1}_A$ we conclude that if A has $\mu = \mu_I$
measure zero, then A has measure zero with respect to the
measures determined by F^+ or F^-.

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Duality of L^p and L^q when $\mu(S) < \infty$.

Proof, 2- Using Radon-Nikodym.

We can apply the Radon-Nikodym theorem to conclude that there
are functions g^+ and g^- which belong to $L^1(I)$ and such that

$$F^{\pm}(f) = I(fg^{\pm})$$

for every f which belongs to $L^1(F^{\pm})$. In particular, if we set
$g := g^+ - g^-$ then

$$F(f) = I(fg)$$

for every function f which is integrable with respect to both F^+
and F^-, in particular for any $f \in L^p(I)$. We must show that
$g \in L^q$.

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Duality of L^p and L^q when $\mu(S) < \infty$.

Proof, 3.

We first treat the case where $p > 1$. Suppose that $0 \le f \le |g|$ and that f is bounded. Then

$$I(f^q) \le I(f^{q-1} \cdot \operatorname{sgn}(g)g) = F(f^{q-1} \cdot \operatorname{sgn}(g)) \le \|F\|_p \|f^{q-1}\|_p.$$

So

$$I(f^q) \le \|F\|_p (I(f^{(q-1)p}))^{\frac{1}{p}}.$$

Now $(q-1)p = q$ so we have

$$I(f^q) \le \|F\|_p I(f^q)^{\frac{1}{p}} = \|F\|_p I(f^q)^{1-\frac{1}{q}}.$$

This gives

$$\|f\|_q \le \|F\|_p$$

for all $0 \le f \le |g|$ with f bounded. Choose such functions f_n with $f_n \nearrow |g|$. It follows from the monotone convergence theorem that $|g|$ and hence $g \in L^q(I)$ proving the theorem for $p > 1$.

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Duality of L^p and L^q when $\mu(S) < \infty$.

The case $p = 1$

Let us now give the argument for $p = 1$. We want to show that $\|g\|_\infty \leq \|F\|_1$. Suppose that $\|g\|_\infty \geq \|F\|_1 + \epsilon$ where $\epsilon > 0$. Consider the function $\mathbf{1}_A$ where

$$A := \{x : |g(x)| \geq \|F\|_1 + \frac{\epsilon}{2}\}.$$

Then

$$(\|F\|_1 + \frac{\epsilon}{2})\mu(A) \leq I(\mathbf{1}_A|g|) = I(\mathbf{1}_A \mathrm{sgn}(g)g) = F(\mathbf{1}_A \mathrm{sgn}(g))$$

$$\leq \|F\|_1 \|\mathbf{1}_A \mathrm{sgn}(g)|_1 = \|F\|_1 \mu(A)$$

which is impossible unless $\mu(A) = 0$, contrary to our assumption. $\square$

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The case where $\mu(S) = \infty$.

The case where $\mu(S) = \infty$, $p > 1$.

Let us first consider the case where $p > 1$. If we restrict the functional F to any subspace of L^p its norm can only decrease. Consider a subspace consisting of all functions which vanish outside a subset S_1 where $\mu(S_1) < \infty$. We get a corresponding function g_1 defined on S_1 (and set equal to zero off S_1) with $\|g_1\|_q \leq \|F\|_p$ and $F(f) = I(fg_1)$ for all f belonging to this subspace. If (S_2, g_2) is a second such pair, then the uniqueness part of the theorem shows that $g_1 = g_2$ almost everywhere on $S_1 \cap S_2$. Thus we can consistently define g_{12} on $S_1 \cup S_2$. Let

$$b := \mathrm{lub}\{\|g_\alpha\|_q\}$$

taken over all such g_α. Since this set of numbers is bounded by $\|F\|_p$ this least upper bound is finite.

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The case where $\mu(S) = \infty$.

We can therefore find a nested sequence of sets S_n and
corresponding functions g_n such that

$$\|g_n\|_q \nearrow b.$$

By the triangle inequality, if $n > m$ then

$$\|g_n - g_m\|_q \leq \|g_n\|_q - \|g_m\|_q$$

and so, as in your proof of the L^2 Martingale convergence theorem
in Problem set 1, this sequence is Cauchy in the $\|\cdot\|_q$ norm.
Hence there is a limit $g \in L^q$ and g is supported on

$$S_0 := \bigcup S_n.$$

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The case where $\mu(S) = \infty$.

There can be no pair (S', g') with S disjoint from S_0 and $g' \neq 0$
on a subset of positive measure of S'. Indeed, if this were the case,
then we could consider $g + g'$ on $S \cup S'$ and this would have a
strictly larger $\| \cdot \|_q$ norm than $\|g\|_q = b$, contradicting the
definition of b. (It is at this point in the argument that we use
$q < \infty$ which is the same as $p > 1$.) Thus F vanishes on any
function which is supported outside S_0. We have thus reduced the
theorem to the case where S is σ-finite.

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The case where $\mu(S) = \infty$.

If S is σ-finite, decompose S into a disjoint union of sets A_i of finite measure. Let f_m denote the restriction of $f \in L^p$ to A_m and let h_m denote the restriction of g to A_m. Then

$$\sum_{m=1}^{\infty} f_m = f$$

as a convergent series in L^p and so

$$F(f) = \sum_m F(f_m) = \sum_m \int_{A_m} f_m h_m$$

and this last series converges to $I(fg)$ in L^1.

So we have proved that $(L^p)^* = L^q$ in complete generality when $p > 1$, and for σ-finite S when $p = 1$.

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The case where $\mu(S) = \infty$.

It may happen (and will happen for the Haar integral on the most general locally compact group) that we don't even have σ-finiteness. But we will have the following more complicated condition: Recall that a set A is called **integrable** (by Loomis) if $\mathbf{1}_A \in \mathcal{B}$. Now suppose that

$$S = \bigcup_\alpha S_\alpha$$

where this union is disjoint, but possibly uncountable, of integrable sets, and with the property that every integrable set is contained in at most a countable union of the S_α. A set A is called **measurable** if the intersections $A \cap S_\alpha$ are all integrable, and a function is called **measurable** if its restriction to each S_α has the property that the restriction of f to each S_α belongs to $\mathcal{B}$, and further, that either the restriction of f^+ to every S_α or the restriction of f^- to every S_α belongs to L^1.

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The case where $\mu(S) = \infty$.

If we find ourselves in this situation, then we can find a g_α on each
S_α since S_α is σ-finite, and piece these all together to get a g
defined on all of S. If $f \in L^1$ then the set where $f \neq 0$ can have
intersections with positive measure with only countably many of
the S_α and so we can apply the result for the σ-finite case for
$p = 1$ to this more general case as well.

Our proof of Radon-Nikodym and the duality between L^p and L^q
works in generality. Our proof of the Fubini theorem is much
simpler in the Hausdorff setting than in general, so we will restrict
our attention for the rest of this lecture to the Hausdorff case.

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Fubini's theorem.

Fubini's theorem in the Hausdorff setting.

Theorem

Let S_1 and S_2 be locally compact Hausdorff spaces and let I and J be non-negative linear functionals on $L(S_1)$ and $L(S_2)$ respectively. Then

$$I_x(J_y h(x, y)) = J_y(I_x(h(x, y))$$

for every $h \in L(S_1 \times S_2)$ in the obvious notation, and this common value is an integral on $L(S_1 \times S_2)$.

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Proof via Stone-Weierstrass. The equation in the theorem is clearly true if $h(x, y) = f(x)g(y)$ where $f \in L(S_1)$ and $g \in L(S_2)$ and so it is true for any h which can be written as a finite sum of such functions. Let h be a general element of $L(S_1 \times S_2)$. Then we can find compact subsets $C_1 \subset S_1$ and $C_2 \subset S_2$ such that h is supported in the compact set $C_1 \times C_2$. The functions of the form

$$\sum f_i(x)g_i(y)$$

where the f_i are all supported in C_1 and the g_i in C_2, and the sum is finite, form an algebra which separates points.

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Fubini's theorem.

So for any $\epsilon > 0$ we can find a k of the above form with

$$\|h - k\|_\infty < \epsilon.$$

Let B_1 and B_2 be bounds for I on $L(C_1)$ and J on $L(C_2)$ as provided by our lemma about non-negative functionals on compact sets. Then

$$|J_y h(x, y) - \sum J(g_i) f_i(x)| = |J_y(f - k)| < \epsilon B_2.$$

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Fubini's theorem.

$$|J_y h(x, y) - \sum J(g_i) f_i(x)| = |J_y(f - k)| < \epsilon B_2.$$

This shows that $J_y h(x, y)$ is the uniform limit of continuous functions supported in C_1 and so $J_y h(x, y)$ is itself continuous and supported in C_1. It then follows that $I_x(J_y(h)$ is defined, and that

$$|I_x(J_y h(x, y) - \sum I(f)_i J(g_i)| \leq \epsilon B_1 B_2.$$

Doing things in the reverse order shows that

$$|I_x(J_y h(x, y)) - J_y(I_x(h(x, y))| \leq 2\epsilon B_1 B_2.$$

Since ϵ is arbitrary, this gives the equality in the theorem. Since this (same) functional is non-negative, it is an integral by the first of the Riesz representation theorems above. $\quad\square$

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So far, in our treatment of the Daniell-Stone integral and the Riesz representation theorems, I have been following Loomis rather slavishly. But for certain applications, for example to Wiener measure, a more precise version of the Riesz representation theorem is needed. So I want to give an alternative proof of the Riesz representation theorem which will give some information about the possible σ-fields on which μ is defined. In particular, I want to show that we can find a μ (which is possibly an extension of the μ given by our previous proof of the Riesz representation theorem) which is defined on a σ-field which contains the Borel field $\mathcal{B}(X)$. Recall that $\mathcal{B}(X)$ is the smallest σ-field which contains the open sets.

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Regular measures.

Let $\mathcal{F}$ be a σ-field which contains $\mathcal{B}(X)$. A (non-negative valued) measure μ on $\mathcal{F}$ is called **regular** if

1 $\mu(K) < \infty$ for any compact subset $K \subset X$.

2 For any $A \in \mathcal{F}$

$$\mu(A) = \inf\{\mu(U): \ A \subset U, \ U \text{ open}\}$$

3 If $U \subset X$ is open then

$$\mu(U) = \sup\{\mu(K): K \subset U, \ K \text{ compact }\}.$$

The second condition is called **outer regularity** and the third condition is called **inner regularity**.

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The improved version of the Riesz representation theorem.

Theorem

Let X be a locally compact Hausdorff space, L the space of continuous functions of compact support on X, and I a non-negative linear functional on L. Then there exists a σ-field $\mathcal{F}$ containing $\mathcal{B}(X)$ and a non-negative regular measure μ on $\mathcal{F}$ such that

$$I(f) = \int f d\mu \qquad (2)$$

for all $f \in L$. Furthermore, the restriction of μ to $\mathcal{B}(X)$ is unique.

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The proof of this theorem hinges on some topological facts whose true place is in the chapter on metric spaces, but I will prove them here. The importance of the theorem is that it will allow us to derive some conclusions about spaces which are very huge (such as the space of "all" paths in $\mathbb{R}^n$) but are nevertheless locally compact (in fact compact) Hausdorff spaces. It is because we want to consider such spaces, that the earlier proof, which hinged on taking limits of *sequences* in the very definition of the Daniell integral, is insufficient to get at the results we want.

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Propositions in topology.

Proposition

Let X be a Hausdorff space, and let H and K be disjoint compact subsets of X. Then there exist disjoint open subsets U and V of X such that $H \subset U$ and $K \subset V$.

This we actually isprove in the chapter on metric spaces in my real variable notes.

Proposition

Let X be a locally compact Hausdorff space, $x \in X$, and U an open set containing x. Then there exists an open set O such that

- $x \in O$
- $\overline{O}$ *is compact, and*
- $\overline{O} \subset U$.

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Propositions in topology.

Proof.

Choose an open neighborhood W of x whose closure is compact, which is possible since we are assuming that X is locally compact. Let $Z = U \cap W$ so that $\overline{Z}$ is compact and hence so is $H := \overline{Z} \setminus Z$. Take $K := \{x\}$ in the preceding proposition. We then get an open set V containing x which is disjoint from an open set G containing $\overline{Z} \setminus Z$. Take $O := V \cap Z$. Then $x \in O$ and $\overline{O} \subset \overline{Z}$ is compact and O has empty intersection with $\overline{Z} \setminus Z$, and hence is is contained in $Z \subset U$. $\qquad\square$

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Propositions in topology.

Proposition

Let X be a locally compact Hausdorff space, $K \subset U$ with K compact and U open subsets of X. Then there exists a V with

$$K \subset V \subset \overline{V} \subset U$$

with V open and $\overline{V}$ compact.

Proof.

Each $x \in K$ has a neighborhood O with compact closure contained in U, by the preceding proposition. The set of these O cover K, so a finite subcollection of them cover K and the union of this finite subcollection gives the desired V. □

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Proposition

Let X be a locally compact Hausdorff space, $K \subset U$ with K compact and U open. Then there exists a continuous function h with compact support such that

$$\mathbf{1}_K \leq h \leq \mathbf{1}_U$$

and

$$\mathrm{supp}(h) \subset U.$$

Proof.

Choose V as in preceding Proposition. By Urysohn's lemma applied to the compact space $\overline{V}$ we can find a function $h : \overline{V} \to [0,1]$ such that $h = 1$ on K and $f = 0$ on $\overline{V} \setminus V$. Extend h to be zero on the complement of $\overline{V}$. Then h does the trick. □

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Proposition

Let X be a locally compact Hausdorff space, $f \in L$, i.e. f is a continuous function of compact support on X. Suppose that there are open subsets $U_1, \ldots U_n$ such that

$$\text{supp}(f) \subset \bigcup_{i=1}^{n} U_i.$$

Then there are $f_1, \ldots, f_n \in L$ such that

$$\text{supp}(f_i) \subset U_i$$

and

$$f = f_1 + \cdots + f_n.$$

If f is non-negative, the f_i can be chosen so as to be non-negative.

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Proof.

By induction, it is enough to consider the case $n = 2$. Let
$K := \operatorname{supp}(f)$, so $K \subset U_1 \cup U_2$.

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Propositions in topology.
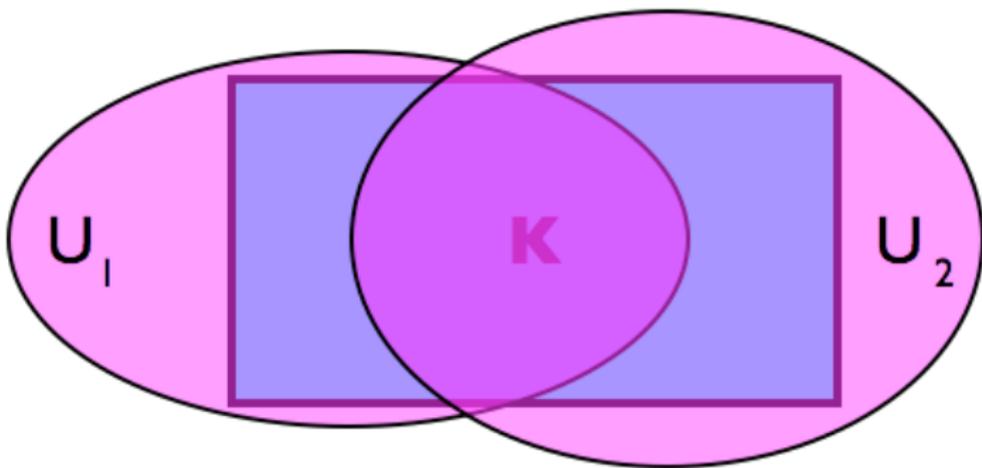
Let

$$L_1 := K \setminus U_1, \quad L_2 := K \setminus U_2.$$

So L_1 and L_2 are disjoint compact sets.

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Propositions in topology.
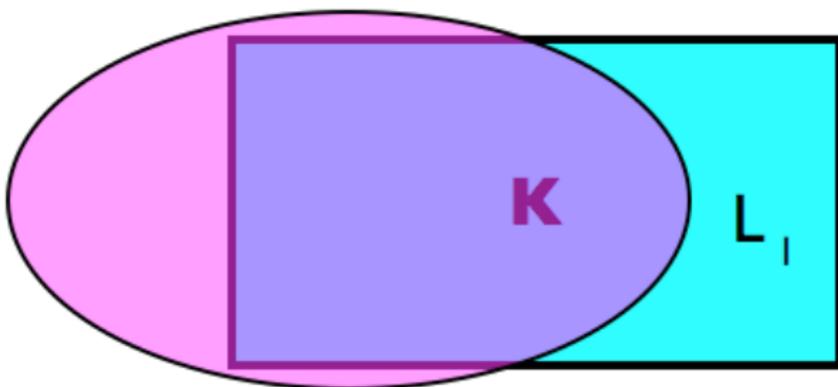
By Proposition 6.1 we can find disjoint open sets V_1, V_2 with

$$L_1 \subset V_1, \quad L_2 \subset V_2.$$

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Propositions in topology.

$$L_1 \subset V_1, \quad L_2 \subset V_2.$$

Set $\quad K_1 := K \setminus V_1, \quad K_2 := K \setminus V_2.$ $\quad$ Then K_1 and K_2 are compact, and

$$K = K_1 \cup K_2, \quad K_1 \subset U_1, \ K_2 \subset U_2.$$

Choose h_1 and h_2 as in Proposition 6.4. Then set

$$\phi_1 := h_1, \quad \phi_2 := h_2 - h_1 \wedge h_2.$$

Then $\operatorname{supp}(\phi_1) = \operatorname{supp}(h_1) \subset U_1$ by construction, and $\operatorname{supp}(\phi_2) \subset \operatorname{supp}(h_2) \subset U_2$, the ϕ_i take values in $[0,1]$, and, if $x \in K = \operatorname{supp}(f)$

$$\phi_1(x) + \phi_2(x) = (h_1 \vee h_2)(x) = 1.$$

Then set

$$f_1 := \phi_1 f, \quad f_2 := \phi_2 f. \quad \square$$

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Proof of the uniqueness and regularity of the measure.

Proof of the uniqueness of the μ restricted to $\mathcal{B}(X)$.

It is enough to prove that

$$
\begin{align}
\mu(U) &= \sup\{I(f) : f \in L, \ 0 \le f \le \mathbf{1}_U\} \tag{3}\\
&= \sup\{I(f) : f \in L, \ 0 \le f \le \mathbf{1}_U, \ \operatorname{supp}(f) \subset U\} \tag{4}
\end{align}
$$

for any open set U, since either of these equations determines μ on any open set U and hence for the Borel field. Since $f \le \mathbf{1}_U$ and both are measurable functions, it is clear that $\mu(U) = \int \mathbf{1}_U$ is at least as large as the expression on the right hand side of (3).
This in turn is as least as large as the right hand side of (4) since the supremum in (2) is taken over as smaller set of functions that that of (3). So it is enough to prove that $\mu(U)$ is $\le$ the right hand side of (4).

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Proof of the uniqueness and regularity of the measure.

To prove: $\mu(U) \leq \sup\{I(f): f \in L, \ 0 \leq f \leq \mathbf{1}_U, \ \operatorname{supp}(f) \subset U\}$. $(*)$

Proof.

Let $a < \mu(U)$. Interior regularity implies that we can find a compact set $K \subset U$ with

$$a < \mu(K).$$

Recall that we proved that there exists a continuous function f with compact support such that

$$\mathbf{1}_K \leq f \leq \mathbf{1}_U$$

Then $a < I(f)$, and so the right hand side of $(*)$ is $\geq a$. Since a was any number $< \mu(U)$, we conclude that $\mu(U)$ is $\leq$ the right hand side of $(*)$.

$\square$

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Proof of the uniqueness and regularity of the measure.

Outline of the proof of existence

We have proved the uniqueness part of our improved Riesz representation theorem. We now turn to the more difficult existence part. We will proceed as follows: We will

- define a function m^* defined on all subsets,
- show that it is an outer measure,
- show that the set of measurable sets in the sense of Caratheodory include all the Borel sets, and that
- integration with respect to the associated measure μ assigns $I(f)$ to every $f \in L$.

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Proof of the uniqueness and regularity of the measure.

Definition of m^*.

Define m^* on open sets by

$$m^*(U) = \sup\{I(f) : f \in L, 0 \leq f \leq \mathbf{1}_U, \ \operatorname{supp}(f) \subset U\}. \quad (5)$$

Clearly, if $U \subset V$ are open subsets, $m^*(U) \leq m^*(V)$. Next define m^* on an arbitrary subset by

$$m^*(A) = \inf\{m^*(U) : A \subset U, \ U \text{ open}\}. \quad (6)$$

Since U is contained in itself, this does not change the definition on open sets. It is clear that $m^*(\emptyset) = 0$ and that $A \subset B$ implies that $m^*(A) \leq m^*(B)$. So to prove that m^* is an outer measure we must prove countable subadditivity.

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Proof of the uniqueness and regularity of the measure.

Counable additivity of m^* on open sets

We will first prove countable subadditivity on open sets, and then use the $\epsilon/2^n$ argument to conclude countable subadditivity on all sets:

Suppose $\{U_n\}$ is a sequence of open sets. We wish to prove that

$$m^* \left(\bigcup_n U_n \right) \leq \sum_n m^*(U_n). \tag{7}$$

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Proof of the uniqueness and regularity of the measure.

Counable additivity of m^* on open sets, 2

Set

$$U := \bigcup_n U_n,$$

and suppose that

$$f \in L, 0 \le f \le \mathbf{1}_U, \ \mathrm{supp}(f) \subset U.$$

Since $\mathrm{supp}(f)$ is compact and contained in U, it is covered by finitely many of the U_i. In other words, there is some finite integer N such that

$$\mathrm{supp}(f) \subset \bigcup_{n=1}^N U_n.$$

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Counable additivity of m^* on open sets, 3

There is some finite integer N such that

$$\text{supp}(f) \subset \bigcup_{n=1}^{N} U_n.$$

By our "partition of unity" Proposition we can write

$$f = f_1 + \cdots + f_N, \quad \text{supp}(f_i) \subset U_i, \quad i = 1, \ldots, N.$$

Then

$$I(f) = \sum I(f_i) \leq \sum m^*(U_i),$$

using the original definition of m^* on an open set for each U_i. .
Replacing the finite sum on the right hand side of this inequality
by the infinite sum, and then taking the supremum over f proves
(7), where we use the original definition (5) of m^* once again.

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Proof of the uniqueness and regularity of the measure.

Countable subadditivity in general.

Let $\{A_n\}$ be any sequence of subsets of X. We wish to prove that

$$m^* \left(\bigcup_n A_n \right) \leq \sum_n m^*(A_n).$$

This is automatic if the right hand side is infinite. So assume that

$$\sum_n m^*(A_n) < \infty$$

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Proof of the uniqueness and regularity of the measure.

Countable subadditivity in general, 2.

and choose open sets $U_n \supset A_n$ so that

$$m^*(U_n) \leq m^*(A_n) + \frac{\epsilon}{2^n}.$$

Then $U := \bigcup U_n$ is an open set containing $A := \bigcup A_n$ and

$$m^*(A) \leq m^*(U) \leq \sum m^*(U)_i \leq \sum_n m^*(A_n) + \epsilon.$$

Since ϵ is arbitrary, we have proved countable subadditivity.

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Measurability of the Borel sets.

Let $\mathcal{F}$ denote the collection of subsets which are measurable in the
sense of Caratheodory for the outer measure m^*. We wish to prove
that $\mathcal{F} \supset \mathcal{B}(X)$. Since $\mathcal{B}(X)$ is the σ-field generated by the open
sets, it is enough to show that every open set is measurable in the
sense of Caratheodory, i.e. that

$$m^*(A) \geq m^*(A \cap U) + m^*(A \cap U^c) \qquad (8)$$

for any open set U and any set A with $m^*(A) < \infty$: If $\epsilon > 0$,
choose an open set $V \supset A$ with

$$m^*(V) \leq m^*(A) + \epsilon$$

which is possible by the definition (6).

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Proof of the uniqueness and regularity of the measure.

Measurability of the Borel sets, 2.

We will show that

$$m^*(V) \geq m^*(V \cap U) + m^*(V \cap U^c) - 2\epsilon. \qquad (9)$$

This will then imply that

$$m^*(A) \geq m^*(A \cap U) + m^*(A \cap U^c) - 3\epsilon$$

and since $\epsilon > 0$ is arbitrary, this will imply (8).

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Proof of the uniqueness and regularity of the measure.

Measurability of the Borel sets, 3.

Choose an open set $V \supset A$ with

$$m^*(V) \leq m^*(A) + \epsilon$$

To prove:

$$m^*(V) \geq m^*(V \cap U) + m^*(V \cap U^c) - 2\epsilon. \quad (9)$$

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Proof of the uniqueness and regularity of the measure.

Measurability of the Borel sets, 4.

Using the definition (5), we can find an $f_1 \in L$ such that

$$f_1 \leq \mathbf{1}_{V \cap U} \quad \text{and} \quad \text{supp}(f_1) \subset V \cap U$$

with

$$I(f_1) \geq m^*(V \cap U) - \epsilon.$$

So:

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$f_1 \in L$, $f_1 \leq \mathbf{1}_{V \cap U}$ and $\operatorname{supp}(f_1) \subset V \cap U$ with

$$I(f_1) \geq m^*(V \cap U) - \epsilon.$$

Let $K := \operatorname{supp}(f_1)$. Then $K \subset U$ and so $K^c \supset U^c$ and K^c is open.
Hence $V \cap K^c$ is an open set and

$$V \cap K^c \supset V \cap U^c.$$

Using the definition (5), we can find an $f_2 \in L$ such that

$$f_2 \leq \mathbf{1}_{V \cap K^c} \quad \text{and} \quad \operatorname{supp}(f_2) \subset V \cap K^c$$

with

$$I(f_2) \geq m^*(V \cap K^c) - \epsilon.$$

But $m^*(V \cap K^c) \geq m^*(V \cap U^c)$ since $V \cap K^c \supset V \cap U^c$. So

$$I(f_2) \geq m^*(V \cap U^c) - \epsilon.$$

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Measurability of the Borel sets, concluded.

$f_1 \in L$, $f_1 \leq \mathbf{1}_{V \cap U}$ and $\text{supp}(f_1) \subset V \cap U$ with
$$I(f_1) \geq m^*(V \cap U) - \epsilon.$$

$f_2 \in L$ with
$$I(f_2) \geq m^*(V \cap U^c) - \epsilon.$$

So
$$f_1 + f_2 \leq \mathbf{1}_K + \mathbf{1}_{V \cap K^c} \leq \mathbf{1}_V$$

since $K = \text{supp}(f_1) \subset V$ and $\text{supp}(f_2) \subset V \cap K^c$. Also
$$\text{supp}(f_1 + f_2) \subset (K \cup V \cap K^c) = V.$$

Thus $f = f_1 + f_2 \in L$ and so by (5),
$$I(f_1 + f_2) \leq m^*(V).$$

This proves (9) and hence that all Borel sets are measurable.

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Proof of the uniqueness and regularity of the measure.

Regularity of the measure.

Let μ be the measure associated to m on the σ-field $\mathcal{F}$ of measurable sets. We will now prove that μ is regular. The condition of outer regularity is automatic, since this was how we defined $\mu(A) = m^*(A)$ for a general set $A \in \mathcal{F}$.

So we must prove inner regularity. We first show that compact sets have finite measure, which was our first condition for regularity.

Compact sets have finite measure.

If K is a compact subset of X, we can find an $f \in L$ such that
$\mathbf{1}_K \leq f$ by an earlie Proposition . Let $0 < \epsilon < 1$ and set

$$U_\epsilon := \{x : f(x) > 1 - \epsilon\}.$$

Then U_ϵ is an open set containing K. If $0 \leq g \in L$ satisfies
$g \leq \mathbf{1}_{U_\epsilon}$, then $g = 0$ on U_ϵ^c, and for $x \in U_\epsilon$, $g(x) \leq 1$ while
$f(x) > 1 - \epsilon$. So

$$g \leq \frac{1}{1 - \epsilon} f$$

and hence $m^*(U_\epsilon) \leq \frac{1}{1-\epsilon} I(f)$.   So,

$$\mu(K) \leq m^*(U_\epsilon) \leq \frac{1}{1 - \epsilon} I(f) < \infty.$$

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Proof of the uniqueness and regularity of the measure.

Reviewing the preceding argument, we see that we have in fact
proved the more general statement

Proposition

If A is any subset of X and $f \in L$ is such that

$$\mathbf{1}_A \leq f$$

then

$$m^*(A) \leq I(f).$$

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Proof of the uniqueness and regularity of the measure.

Interior regularity.

We now prove interior regularity, which is very important in certain applications. We wish to prove that

$$\mu(U) = \sup\{\mu(K) : K \subset U, \ \ K \text{ compact }\},$$

for any open set U, where, according to (5),

$$m^*(U) = \sup\{I(f) : \ f \in L, 0 \leq f \leq \mathbf{1}_U, \ \text{supp}(f) \subset U\}.$$

Since $\text{supp}(f)$ is compact, and contained in U, we will be done if we show that

$$f \in L, \ \ 0 \leq f \leq 1 \Rightarrow I(f) \leq \mu(\text{supp}(f)). \tag{10}$$

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Proof of the uniqueness and regularity of the measure.

To show:

$$f \in L, \ \ 0 \le f \le 1 \Rightarrow I(f) \le \mu(\mathrm{supp}(f)). \qquad (10)$$

Let V be an open set containing $\mathrm{supp}(f)$. By definition (5),

$$\mu(V) \ge I(f)$$

and, since V is an arbitrary open set containing $\mathrm{supp}(f)$, we have

$$\mu(\mathrm{supp}(f)) \ge I(f)$$

using the definition (6) of $m^*(\mathrm{supp}(f))$. $\square$

In the course of this argument we have proved

Proposition

If $g \in L, 0 \le g \le \mathbf{1}_K$ where K is compact, then

$$I(g) \le \mu(K).$$

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Conclusion of the proof of the improved Riesz representation theorem.

Finally, we must show that all the elements of L are integrable with respect to μ and

$$I(f) = \int f d\mu. \tag{11}$$

Since the elements of L are continuous, they are Borel measurable. As every $f \in L$ can be written as the difference of two non-negative elements of L, and as both sides of (11) are linear in f, it is enough to prove (11) for non-negative functions.

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Proof of the uniqueness and regularity of the measure.

Following Lebesgue, divide the "y-axis" up into intervals of size ϵ. That is, let ϵ be a positive number, and, for every positive integer n set

$$f_n(x) := \begin{cases} 0 & \text{if} & f(x) \leq (n-1)\epsilon \\ f(x) - (n-1)\epsilon & \text{if} & (n-1)\epsilon < f(x) \leq n\epsilon \\ \epsilon & \text{if} & n\epsilon < f(x) \end{cases}$$

If $(n-1)\epsilon \geq \|f\|_\infty$ only the first alternative can occur, so all but finitely many of the f_n vanish, and they all are continuous and have compact support so belong to L. Also

$$f = \sum f_n$$

this sum being finite, as we have observed, and so

$$I(f) = \sum I(f_n).$$

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Proof of the uniqueness and regularity of the measure.

$$I(f) = \sum I(f_n).$$

Set $K_0 := \mathrm{supp}(f)$ and

$$K_n := \{x : f(x) \geq n\epsilon\} \quad n = 1, 2, \ldots.$$

Then the K_i are a nested decreasing collection of compact sets, and

$$\epsilon \mathbf{1}_{K_n} \leq f_n \leq \epsilon \mathbf{1}_{K_{n-1}}.$$

By the two preceding propositions we have

$$\epsilon\mu(K_n) \leq I(f_n) \leq \epsilon\mu(K_{n-1}).$$

On the other hand, the monotonicity of the integral (and its definition) imply that

$$\epsilon\mu(K_n) \leq \int f_n d\mu \leq \epsilon\mu(K_{n-1}).$$

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Proof of the uniqueness and regularity of the measure.

$$\epsilon\mu(K_n) \leq I(f_n) \leq \epsilon\mu(K_{n-1})$$

and

$$\epsilon\mu(K_n) \leq \int f_n d\mu \leq \epsilon\mu(K_{n-1}).$$

Summing these inequalities gives

$$\epsilon \sum_{i=1}^{N} \mu(K_n) \quad \leq I(f) \quad \leq \epsilon \sum_{i=0}^{N-1} \mu(K_n)$$

$$\epsilon \sum_{i=1}^{N} \mu(K_n) \quad \leq \int f d\mu \quad \leq \epsilon \sum_{i=0}^{N-1} \mu(K_n)$$

where N is sufficiently large.

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Proof of the uniqueness and regularity of the measure.

Thus $I(f)$ and $\int f d\mu$ lie within a distance

$$\epsilon \sum_{i=0}^{N-1} \mu(K_n) - \epsilon \sum_{i=1}^{N} \mu(K_n) = \epsilon \mu(K_0) - \epsilon \mu(K_N) \leq \epsilon \mu(\operatorname{supp}(f))$$

of one another. Since ϵ is arbitrary, we have proved (11) and
completed the proof of the improved Riesz representation theorem.

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