212a1214
Daniell's integration theory.

Shlomo Sternberg

October 30, 2014

Daniell's idea was to take the axiomatic properties of the integral as the starting point and develop integration for broader and broader classes of functions. Then derive measure theory as a consequence.

A crucial step in relating the Daniel integral to the Lebesgue integral is the introduction of an axiom due to Stone. Much of the presentation here is taken from the book *Abstract Harmonic Analysis* by Lynn Loomis, available on the web site of this course. Some of the lemmas, propositions and theorems stated here indicate the corresponding sections in Loomis's book.

The space L.

Let L be a vector space of **bounded** real valued functions on a set S closed under $\wedge$ and $\vee$.

Recall that the operation $\vee$ on functions is given by

$$(f \vee g)(x) = \max(f(x), g(x))$$

and the operation $\wedge$ is given by

$$(f \wedge g)(x) = \min(f(x), g(x)).$$

For example, S might be a complete metric space, and L might be the space of continuous functions of compact support on S.

Integral.

A map $I : L \to \mathbb{R}$ is called an **Integral** if

1. I is linear: $I(af + bg) = aI(f) + bI(g)$
2. I is non-negative: $f \geq 0 \Rightarrow I(f) \geq 0$ or equivalently $f \geq g \Rightarrow I(f) \geq I(g)$.
3. $f_n \searrow 0 \Rightarrow I(f_n) \searrow 0$.

For example, we might take $S = \mathbb{R}^n$, $L =$ the space of continuous functions of compact support on $\mathbb{R}^n$, and I to be the Riemann integral. The first two items on the above list are clearly satisfied. As to the third, we recall Dini's lemma from the notes on metric spaces, which says that a sequence of continuous functions of compact support $\{f_n\}$ on a (complete) metric space which satisfies $f_n \searrow 0$ actually converges uniformly to 0. See the next slide. Furthermore the supports of the f_n are all contained in a fixed compact set - for example the support of f_1. This establishes the third item.

Dini's lemma

Let X be a metric space and L the space of real valued continuous functions of compact support on X. So if $f \in L$ then the closure of the set where $|f(x)| > 0$ is compact. So $f \in L \Rightarrow |f| \in L$ and if $f, \ g \in L$ so are $f+g, f \wedge g$ and $f \vee g$.

Theorem

[Dini's lemma.] *Ig $f_n \in L$ and $f_n \searrow 0$ then the f_n converge uniformly to zero.*

Proof.

Given $\epsilon > 0$ let $C_N = \{x|f_n(x) \geq \epsilon\|$. The C_n are compact, $C_n \supset C_{n+1}$ and $\bigcap C_n = \emptyset$. So a finite intersection is already empty by compactness. Hence there is some n such that $f_n(x) \leq \epsilon$ for all x. □

The space U.

The plan is now to successively increase the class of functions on which the integral is defined:

Every monotone non-decreasing sequence of real valued functions has a limit, provided that $+\infty$ is allowed as a possible value of the limit function.

Define

$U :=$ {limits of monotone non-decreasing sequences of elements of L}.

We will use the word "increasing" as synonymous with "monotone non-decreasing" so as to simplify the language.

Extending I to U.

Lemma

If f_n is an increasing sequence of elements of L and if $k \in L$ satisfies $k \leq \lim f_n$ then $\lim I(f_n) \geq I(k)$.

Proof.

If $k \in L$ and $\lim f_n \geq k$, then

$$f_n \wedge k \leq k \quad \text{and } f_n \geq f_n \wedge k$$

so $I(f_n) \geq I(f_n \wedge k)$ while $\quad [k - (f_n \wedge k)] \searrow 0 \quad$ so

$$I([k - f_n \wedge k]) \searrow 0 \qquad \text{by 3) or}$$

$I(f_n \wedge k) \nearrow I(k)$. Hence $\lim I(f_n) \geq \lim I(f_n \wedge k) = I(k)$. □

Extending I to U, continued.

Lemma

[12C] *If* $\{f_n\}$ *and* $\{g_n\}$ *are increasing sequences of elements of* L *and* $\lim g_n \leq \lim f_n$ *then* $\lim I(g_n) \leq \lim I(f_n)$*.*

Proof.

Fix m and take $k = g_m$ in the previous lemma. Then $I(g_m) \leq \lim I(f_n)$. Now let $m \to \infty$. □

Extending I to U, concluded.

Thus

$$f_n \nearrow f \text{ and } g_n \nearrow f \quad \Rightarrow \lim I(f_n) = \lim I(g_n)$$

so we may extend I to U by setting

$$I(f) := \lim I(f_n) \quad \text{for} \quad f_n \nearrow f.$$

If $f \in L$, this coincides with our original I, since we can take $g_n = f$ for all n in the preceding lemma.

Doing it again yields nothing new.

We have now extended I from L to U. The next lemma shows that if we now start with I on U and apply the same procedure again, we do not get any further.

Lemma

[12D] *If* $f_n \in U$ *and* $f_n \nearrow f$ *then* $f \in U$ *and* $I(f_n) \nearrow I(f)$.

Proof.

For each fixed n choose $g_n^m \nearrow_m f_n$. Set

$$h_n := g_1^n \vee \cdots \vee g_n^n$$

so $h_n \in L$ and h_n is increasing with

$$g_i^n \leq h_n \leq f_n \quad \text{for } i \leq n.$$

Let $n \to \infty$. Then $f_i \leq \lim h_n \leq f$. Now let $i \to \infty$. We get

$$f \leq \lim h_n \leq f.$$

So we have written f as a limit of an increasing sequence of elements of L, So $f \in U$. Also $I(g_i^n) \leq I(h_n) \leq I(f_n)$ so letting $n \to \infty$ we get $I(f_i) \leq I(f) \leq \lim I(f_n)$ so passing to the limits gives $I(f) = \lim I(f_n)$. □

Extending to $-U$.

We have

$$I(f+g) = I(f) + I(g) \quad \text{for} \quad f, g \in U.$$

Define

$$-U := \{-f \mid f \in U\}$$

and

$$I(f) := -I(-f) \qquad f \in -U.$$

If $f \in U$ and $-f \in U$ then $I(f) + I(-f) = I(f - f) = I(0) = 0$ so $I(-f) = -I(f)$ in this case. So the definition is consistent.

$-U$ is closed under monotone decreasing limits. etc.

An important remark

If $g \in -U$ and $h \in U$ with $g \leq h$ then $-g \in U$ so $h - g \in U$ and $h - g \geq 0$ so $I(h) - I(g) = I(h + (-g)) = I(h - g) \geq 0$.

I-summable functions.

A function f is called I-**summable** if for every $\epsilon > 0$, $\exists\ g \in -U,\ h \in U$ with

$$g \leq f \leq h, \quad |I(g)| < \infty,\ |I(h)| < \infty \ \text{ and } I(h-g) \leq \epsilon.$$

For such f define

$$I(f) = \text{glb } I(h) = \text{lub } I(g).$$

If $f \in U$ take $h = f$ and $f_n \in L$ with $f_n \nearrow f$. Then $-f_n \in L \subset U$ so $f_n \in -U$. If $I(f) < \infty$ then we can choose n sufficiently large so that $I(f) - I(f_n) < \epsilon$.

So we have extended I in a consistent manner to the space of I-summable functions.

The space of I-summable functions is denoted by $\overline{L}_1$. It is clearly a vector space, and I satisfies conditions 1) and 2) above, i.e. is linear and non-negative.

The monotone convergence theorem.

Theorem

[12G] Monotone convergence theorem. $f_n \in \overline{L}_1$, $f_n \nearrow f$ *and* $\lim I(f_n) < \infty \ \Rightarrow f \in \overline{L}_1$ *and* $I(f) = \lim I(f_n)$.

Proof.

Replacing f_n by $f_n - f_0$ we may assume that $f_0 = 0$. Choose

$$h_n \in U, \quad \text{such that } f_n - f_{n-1} \leq h_n \text{ and } I(h_n) \leq I(f_n - f_{n-1}) + \frac{\epsilon}{2^n}.$$

Then

$$f_n \leq \sum_1^n h_i \quad \text{and} \quad \sum_{i=1}^n I(h_i) \leq I(f_n) + \epsilon.$$

Since U is closed under monotone increasing limits,

$$h := \sum_{i=1}^\infty h_i \in U, \quad f \leq h \quad \text{and } I(h) \leq \lim I(f_n) + \epsilon.$$

Since $f_m \in \overline{L}_1$ we can find a $g_m \in -U$ with $I(f_m) - I(g_m) < \epsilon$ and hence for m large enough $I(h) - I(g_m) < 2\epsilon$. So $f \in \overline{L}_1$ and $I(f) = \lim I(f_n)$. □

We can summarize the preceding results as saying that

Theorem

12F *The space* $\overline{L}_1$ *and the integral* I *extended as above to* $\overline{L}_1$ *have all the properties of our original* L *and* I.

Monotone classes.

A collection of functions which is closed under monotone increasing and monotone decreasing limits is called a **monotone class**. $\mathcal{B}$ is defined to be the smallest monotone class containing L.

Lemma

Let $h \leq k$. If $\mathcal{M}$ is a monotone class which contains $(g \vee h) \wedge k$ for every $g \in L$, then $\mathcal{M}$ contains all $(f \vee h) \wedge k$ for all $f \in \mathcal{B}$.

Proof.

The set of f such that $(f \vee h) \wedge k \in \mathcal{M}$ is a monotone class containing L by the distributive laws. □

Taking $h = 0$, $k = \infty$ this says that the smallest monotone class containing L^+, the set of non-negative functions in L, is the set $\mathcal{B}^+$, the set of non-negative functions in $\mathcal{B}$.

Here is a series of monotone class theorem style arguments:

Theorem

$f, g \in \mathcal{B} \Rightarrow af + bg \in \mathcal{B}, f \vee g \in \mathcal{B}$ *and* $f \wedge g \in \mathcal{B}$.

Proof.

For $f \in \mathcal{B}$, let

$$\mathcal{M}(f) := \{g \in \mathcal{B} | f + g, f \vee g, f \wedge g \in \mathcal{B}\}.$$

$\mathcal{M}(f)$ is a monotone class. If $f \in L$ it includes all of L, hence all of $\mathcal{B}$. But

$$g \in \mathcal{M}(f) \;\Leftrightarrow\; f \in \mathcal{M}(g).$$

So $L \subset \mathcal{M}(g)$ for any $g \in \mathcal{B}$, and since it is a monotone class $\mathcal{B} \subset \mathcal{M}(g)$. This says that $f, g \in \mathcal{B} \Rightarrow f + g \in \mathcal{B},\ f \wedge g \in \mathcal{B}$ and $f \vee g \in \mathcal{B}$. Similarly, let $\mathcal{M}$ be the class of functions for which $cf \in \mathcal{B}$ for all real c. This is a monotone class containing L hence contains $\mathcal{B}$. □

Lemma

[12I] *If* $f \in \mathcal{B}$ *there exists a* $g \in U$ *such that* $f \leq g$.

Proof.

The limit of a monotone increasing sequence of functions in U belongs to U by Lemma **12D**. So the set of functions for which the lemma holds is closed under increasing limits. It is trivially closed under decreasing limiits. Hence the set of f for which the lemma is true is a monotone class which contains L. Hence it contains $\mathcal{B}$. □

L -bounded functions, L-monotone classes.

Definition

A function f is L-**bounded** if there exists a $g \in L^+$ with $|f| \leq g$.

Definition

A class $\mathcal{F}$ of functions is said to be L-**monotone** if $\mathcal{F}$ is closed under monotone limits of L-bounded functions, that is, whenever $f_n \in \mathcal{F}$ is a sequence of L-bounded functions and $f_n \nearrow f$ or $f_n \searrow f$ then $f \in \mathcal{F}$.

Theorem

The smallest L-monotone class including L^+ is $\mathcal{B}^+$.

Proof.

Call this smallest family $\mathcal{F}$. If $g \in L^+$, the set of all $f \in \mathcal{B}^+$ such that $f \wedge g \in \mathcal{F}$ form a monotone class containing L^+, hence containing $\mathcal{B}^+$ hence equal to $\mathcal{B}^+$. If $f \in \mathcal{B}^+$ and $f \leq g$ then $f \wedge g = f \in \mathcal{F}$. So $\mathcal{F}$ contains all L bounded functions belonging to $\mathcal{B}^+$. Let $f \in \mathcal{B}^+$. By the lemma, choose $g \in U$ such that $f \leq g$, and choose $g_n \in L^+$ with $g_n \nearrow g$. Then $f \wedge g_n \leq g_n$ and so is L bounded, so $f \wedge g_n \in \mathcal{F}$. Since $(f \wedge g_n) \to f$ we see that $f \in \mathcal{F}$. So

$$\mathcal{B}^+ \subset \mathcal{F}.$$

We know that $\mathcal{B}^+$ is a monotone class, in particular an L-monotone class. Hence $\mathcal{F} = \mathcal{B}^+$. □

The space L^1.

Define

$$L^1 := \overline{L}_1 \cap \mathcal{B}.$$

Since $\overline{L}_1$ and $\mathcal{B}$ are both closed under the lattice operations,

$$f \in L^1 \Rightarrow f^{\pm} \in L^1 \Rightarrow |f| \in L^1.$$

Loomis says that the replacement of $\overline{L}_1$ by L^1 is “entirely a matter of convenience”.

Theorem

[12J *If $f \in \mathcal{B}$ then $f \in L^1 \Leftrightarrow \exists g \in L^1$ with $|f| \le g$.*

Proof.

We have proved $\Rightarrow$: simply take $g = |f|$. For the converse we may assume that $f \ge 0$ by applying the result to f^+ and f^-. The family of all $h \in \mathcal{B}^+$ such that $h \wedge g \in L^1$ is monotone and includes L^+ so includes $\mathcal{B}^+$. So $f = f \wedge g \in L^1$. □

Extend I to all of $\mathcal{B}^+$ be setting it $= \infty$ on functions which do not belong to L^1.

Integrable functions

A function $f \in \mathcal{B}^+$ is called **integrable** if either f^+ or f^- is summable. Then

$$I(f) := I(f^+) - I(f^-)$$

is unambiguously defined, but may $= +\infty$ or$-\infty$. So a function f is summable iff it is integrable and $|I(f)| < \infty$.
To summarize:

Theorem

[12K] *If f and g are integrable then $f + g$ is integrable with $I(f + g) = I(f) + I(g)$ provided that $I(f)$ and $I(g)$ are not oppositely infinite. If f_n is integrable, $I(f_1) > -\infty$, and $f_n \nearrow f$ then f is integrable and $I(f_n) \to I(f)$.*

Integrable sets.

Loomis calls a set A **integrable** if $\mathbf{1}_A \in \mathcal{B}$. The monotone class properties of $\mathcal{B}$ imply that the integrable sets form a σ-field. Then define

$$\mu(A) := I(\mathbf{1}_A)$$

and the monotone convergence theorem guarantees that μ is a measure.

An obvious question is to compare $I(f)$ with the integral of f (as we introduced in the last lecture) respect to the measure μ. In other words is

$$I(f) = \int f d\mu?$$

For this we need to

Stone's axiom.

Add **Stone's axiom**

$$f \in L \Rightarrow f \wedge \mathbf{1} \in L.$$

Then the monotone class property implies that this is true with L replaced by $\mathcal{B}$.

Theorem

$f \in \mathcal{B}$ and $a > 0 \Rightarrow$ then

$$A_a := \{p | f(p) > a\}$$

is an integrable set. If $f \in L^1$ then

$$\mu(A_a) < \infty.$$

Proof.

Let

$$f_n := [n(f - f \wedge a)] \wedge \mathbf{1} \in \mathcal{B}.$$

Then

$$f_n(x) = \begin{cases} 1 & \text{if} \quad f(x) \geq a + \frac{1}{n} \\ 0 & \text{if} \quad f(x) \leq a \\ n(f(x) - a) & \text{if} \quad a < f(x) < a + \frac{1}{n} \end{cases}.$$

We have

$$f_n \nearrow \mathbf{1}_{A_a}$$

so $\mathbf{1}_{A_a} \in \mathcal{B}$ and $0 \leq \mathbf{1}_{A_a} \leq \frac{1}{a} f^+$. □

Using an "octave" argument.

Theorem

If $f \geq 0$ and A_a is integrable for all $a > 0$ then $f \in \mathcal{B}$.

Proof.

For $\delta > 1$ define $A_m^\delta := \{x | \delta^m < f(x) \leq \delta^{m+1}\}$ and

$$f_\delta := \sum_m \delta^m \mathbf{1}_{A_m^\delta}.$$

Each $f_\delta \in \mathcal{B}$. Take

$$\delta_n = 2^{2^{-n}}.$$

Then each successive subdivision divides the previous one into "octaves" and $f_{\delta_m} \nearrow f$. □

Using the "octave argument" to show that $I(f) = \int f d\mu$.

Also $\quad f_\delta \leq f \leq \delta f_\delta \quad$ and

$$I(f_\delta) = \sum \delta^n \mu(A_m^\delta) = \int f_\delta d\mu.$$

So we have

$$I(f_\delta) \leq I(f) \leq \delta I(f_\delta)$$

and

$$\int f_\delta d\mu \leq \int f d\mu \leq \delta \int f_\delta d\mu.$$

So if either of $I(f)$ or $\int f d\mu$ is finite they both are and

$$\left| I(f) - \int f d\mu \right| \leq (\delta - 1) I(f_\delta) \leq (\delta - 1) I(f).$$

So passing to the limit $\delta = 1$ shows that

$$\int f d\mu = I(f).$$

$\mathcal{B}^+$ is closed under multiplication.

If $f \in \mathcal{B}^+$ and $a > 0$ then

$$\{x|f(x)^a > b\} = \{x|f(x) > b^{\frac{1}{a}}\}.$$

So $f \in \mathcal{B}^+ \Rightarrow f^a \in \mathcal{B}^+$ and hence the product of two elements of $\mathcal{B}^+$ belongs to $\mathcal{B}^+$ because

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right].$$

Conjugate numbers.

The numbers $p, q > 1$ are called **conjugate** if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

This is the same as

$$pq = p + q$$

or

$$(p-1)(q-1) = 1.$$

This last equation says that if x and y are positive numbers with

$$y = x^{p-1}$$

then

$$x = y^{q-1}.$$

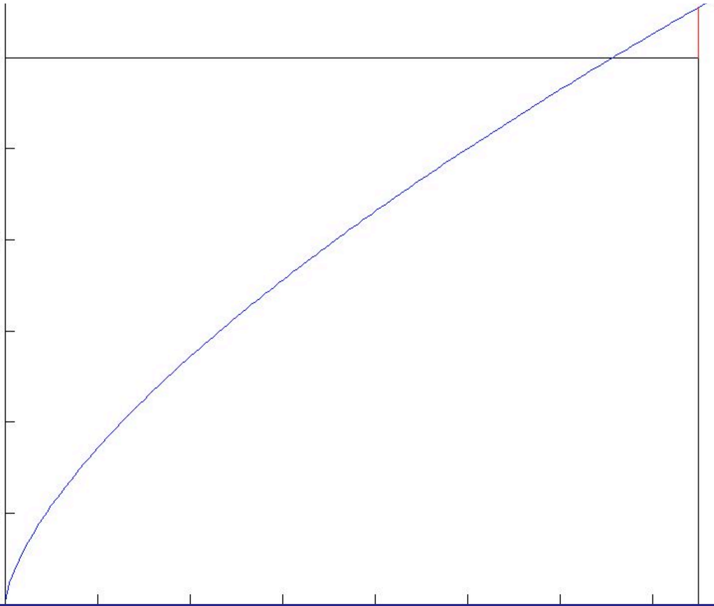
The area under the curve $y = x^{p-1}$ from 0 to a is

$$A = \frac{a^p}{p}$$

while the area between the same curve and the y-axis up to $y = b$

$$B = \frac{b^q}{q}.$$

Suppose $b \leq a^{p-1}$ to fix the ideas. (Otherwise interchange x and y.)Then area ab of the rectangle is less than $A + B$. See the figure in the next slide:



The area under the curve $y = x^{p-1}$ from 0 to a is

$$A = \frac{a^p}{p}$$

while the area between the same curve and the y-axis up to $y = b$

$$B = \frac{b^q}{q}.$$

Suppose $b < a^{p-1}$ to fix the ideas. Then area ab of the rectangle is less than $A + B$, or

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

with equality if and only if $b = a^{p-1}$. Replacing a by $a^{\frac{1}{p}}$ and b by $b^{\frac{1}{q}}$ gives

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

The space L^p.

Let L^p denote the space of functions such that $|f|^p \in L^1$. For $f \in L^p$ define

$$\|f\|_p := \left(\int |f|^p d\mu\right)^{\frac{1}{p}}.$$

We will soon see that if $p \geq 1$ this is a (semi-)norm.

Hölder's inequality.

If $f \in L^p$ and $g \in L^q$ with $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$ take

$$a = \frac{|f|^p}{\|f\|_p^p}, \quad b = \frac{|g|^q}{\|g\|_q^q}$$

as functions. Then using the inequality

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

we see that

$$\int (|f||g|) d\mu$$

$$\leq \|f\|_p \|g\|_q \left(\frac{1}{p} \frac{1}{\|f\|_p^p} \int |f|^p d\mu + \frac{1}{q} \frac{1}{\|g\|_q^q} \int |g|^q d\mu \right) = \|f\|_p \|g\|_q.$$

$$\int (|f||g|)d\mu$$

$$\leq \|f\|_p\|g\|_q \left(\frac{1}{p}\frac{1}{\|f\|_p^p}\int |f|^p d\mu + \frac{1}{q}\frac{1}{\|g\|_q^q}\int |g|^q d\mu\right) = \|f\|_p\|g\|_q.$$

This shows that fg is integrable and that

$$\left|\int fg d\mu\right| \leq \|f\|_p\|g\|_q \tag{1}$$

which is known as **Hölder's inequality**. (If either $||f||_p$ or $\|g\|_q = 0$ then $fg = 0$ a.e. and Hölder's inequality is trivial.)

Minkowski's inequality.

We write

$$(f,g) := \int fg d\mu.$$

Proposition

[Minkowski's inequality] *If* $f, g \in L^p$, $p \geq 1$ *then* $f + g \in L^p$ *and*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

For $p = 1$ this is obvious. If $p > 1$,

$$|f+g|^p \leq [2\max(|f|,|g|)]^p \leq 2^p\left[|f|^p + |g|^p\right]$$

implies that $f + g \in L^p$.

Write

$$\|f+g\|_p^p \leq I(|f+g|^{p-1}|f|) + I(|f+g|^{p-1}|g|).$$

Now $\quad q(p-1) = qp - q = p \quad$ so

$$|f+g|^{p-1} \in L_q$$

and its $\|\cdot\|_q$ norm is

$$I(|f+g|^p)^{\frac{1}{q}} = I(|f+q|^p)^{1-\frac{1}{p}} = I(|f+g|^p)^{\frac{p-1}{p}} = \|f+g\|_p^{p-1}.$$

So we can write the preceding inequality as

$$\|f+g\|_p^p \leq (|f|, |f+g|^{p-1}) + (|g|, |f+g|^{p-1})$$

and apply Hölder's inequality to conclude that

$$\|f+g\|_p^p \leq \|f+g\|_p^{p-1}(\|f\|_p + \|g\|_p).$$

We may divide by $\|f+g\|_p^{p-1}$ to get Minkowski's inequality unless $\|f+g\|_p = 0$ in which case it is obvious. □

Theorem

L^p is complete.

Proof. Suppose $f_n \geq 0$, $f_n \in L^p$, and $\sum \|f_n\|_p < \infty$. Then $f := \sum_1^\infty f_j \in L^p$. Indeed

$$g_n := \sum_1^n f_j \in L^p$$

by Minkowski. Since the $g_n \nearrow f$ we have $|g_n|^p \nearrow f^p$. By the monotone convergence theorem $f :\in L^p$ and $\|f\|_p = \lim \|g\|_p \leq \sum \|f_j\|_p$.

Now let $\{f_n\}$ be any Cauchy sequence in L^p. By passing to a subsequence we may assume that

$$\|f_{n+1} - f_n\|_p < \frac{1}{2^n}.$$

So $\sum_n^\infty |f_{i+1} - f_i| \in L^p$ and hence (new notation)

$$g_n := f_n - \sum_n^\infty |f_{i+1} - f_i| \in L^p \quad \text{and} \quad h_n := f_n + \sum_n^\infty |f_{i+1} - f_i| \in L^p.$$

We have

$$g_{n+1} - g_n = f_{n+1} - f_n + |f_{n+1} - f_n| \geq 0$$

so g_n is increasing and similarly h_n is decreasing. Hence $f := \lim g_n \in L^p$ and $\|f - f_n\|_p \leq \|h_n - g_n\|_p \leq 2^{-n+2} \to 0$. So the subsequence has a limit which then must be the limit of the original sequence. $\square$

Proposition

L is dense in L^p for any $1 \le p < \infty$.

Proof. For $p = 1$ this was a defining property of L^1. More generally, suppose that $f \in L^p$ and that $f \geq 0$. Let

$$A_n := \{x : \frac{1}{n} < f(x) < n\},$$

and let

$$g_n := f \cdot \mathbf{1}_{A_n}.$$

Then $(f - g_n) \searrow 0$ as $n \to \infty$. Choose n sufficiently large so that $\|f - g_n\|_p < \epsilon/2$. Since

$$0 \le g_n \le n\mathbf{1}_{A_n} \quad \text{and} \quad \mu(A_n) < n^p I(|f|^p) < \infty$$

we conclude that

$$g_n \in L^1.$$

Now choose $h \in L^+$ so that

$$\|h - g_n\|_1 < \left(\frac{\epsilon}{2n}\right)^p$$

and also so that $h \le n$. Then

$$\begin{aligned}
\|h - g_n\|_p &= (I(|h - g_n|^p))^{1/p} \\
&= \left(I(|h - g_n|^{p-1}|h - g_n|)\right)^{1/p} \\
&\le \left(I(n^{p-1}|h - g_n|)\right)^{1/p} \\
&= \left(n^{p-1}\|h - g_n\|_1\right)^{1/p} \\
&< \epsilon/2.
\end{aligned}$$

So by the triangle inequality $\|f - h\| < \epsilon$.

Do we pass to the quotient by elements of norm zero?

In the above, we have not bothered to pass to the quotient by the elements of norm zero. In other words, we have not identified two functions which differ on a set of measure zero. We will continue with this ambiguity. But equally well, we could change our notation, and use L^p to denote the quotient space and denote the space before we pass to the quotient by $\mathcal{L}^p$ to conform with our earlier notation. I will continue to be sloppy on this point, in conformity to Loomis' notation.

$\| \cdot \|_\infty$ is the essential sup norm.

Suppose that $f \in \mathcal{B}$ has the property that it is equal almost everywhere to a function which is bounded above. We call such a function **essentially bounded** (from above). We can then define the **essential least upper bound** of f to be the smallest number which is an upper bound for a function which differs from f on a set of measure zero.

Indeed, if b is the greatest lower bound of the set of upper bounds of f, then outside of a set A_n of measure zero, $f \leq b + \frac{1}{n}$. So $f \leq b$ outside of the union of the A_n which is itself a set of measure zero.

Definition

If $|f|$ is essentially bounded, we denote its essential least upper bound by $\|f\|_\infty$. Otherwise we say that $\|f\|_\infty = \infty$.

We let $\mathcal{L}^\infty$ denote the space of $f \in \mathcal{B}$ which have $\|f\|_\infty < \infty$. It is clear that $\|\cdot\|_\infty$ is a semi-norm on this space. The justification for this notation is:

Theorem

[14G] *If $f \in L^p$ for some $p > 0$ then*

$$\|f\|_\infty = \lim_{r\to\infty} \|f\|_r. \tag{2}$$

Remark. In the statement of the theorem, both sides of (2) are allowed to be ∞.

Proof. If $\|f\|_\infty = 0$, then $\|f\|_r = 0$ for all $r > 1$ so the result is trivial in this case. So let us assume that $\|f\|_\infty > 0$ and let a be any positive number smaller that $\|f\|_\infty$. In other words,

$$0 < a < \|f\|_\infty.$$

Let

$$A_a := \{x : |f(x)| > a\}.$$

This set has positive measure by the choice of a, and its measure is finite since $f \in L^p$.

Also

$$\|f\|_r \geq \left(\int_{A_a} |f|^r\right)^{1/r} \geq a\mu(A_a)^{1/r}.$$

$$\|f\|_r \geq \left(\int_{A_a} |f|^r\right)^{1/r} \geq a\mu(A_a)^{1/r}.$$

Letting $r \to \infty$ gives

$$\liminf_{r\to\infty} \|f\|_r \geq a$$

and since a can be any number $< \|f\|_\infty$ we conclude that

$$\liminf_{r\to\infty} \|f\|_r \geq \|f\|_\infty.$$

So we need to prove that

$$\limsup_{r\to\infty} \|f\|_r \leq \|f\|_\infty.$$

This is obvious if $\|f\|_\infty = \infty$.

We need to prove that

$$\limsup_{r\to\infty} \|f\|_r \leq \|f\|_\infty$$

when $\|f\|_\infty$ is finite. For $r > p$ we have

$$|f|^r \leq |f|^p (\|f\|_\infty)^{r-p}$$

almost everywhere. Integrating and taking the r-th root gives

$$\|f\|_r \leq (\|f\|_p)^{\frac{p}{r}} (\|f\|_\infty)^{1-\frac{p}{r}}.$$

Letting $r \to \infty$ gives the desired result. $\square$

Absolute continuity.

Suppose we are given two integrals, I and J on the same space L. That is, both I and J satisfy the three conditions of linearity, positivity, and the monotone limit property that went into our definition of the term "integral". We say that J is **absolutely continuous** with respect to I if every set which is I null (i.e. has measure zero with respect to the measure associated to I) is J null.

Bounded integrals or measures.

The integral I is said to be **bounded** if

$$I(\mathbf{1}) < \infty,$$

or, what amounts to the same thing, that

$$\mu_I(S) < \infty$$

where μ_I is the measure associated to I.

The Radon-Nikodym theorem.

We will first formulate the Radon-Nikodym theorem for the case of bounded integrals, where there is a very clever proof due to von-Neumman which reduces it to the Riesz representation theorem in Hilbert space theory.

Theorem

[Radon-Nikodym] *Let I and J be bounded integrals, and suppose that J is absolutely continuous with respect to I. Then there exists an element $f_0 \in \mathcal{L}^1(I)$ such that*

$$J(f) = I(ff_0) \quad \forall\, f \in \mathcal{L}^1(J). \tag{3}$$

The element f_0 is unique up to equality almost everywhere (with respect to μ_I).

Proof, after von-Neumann.

Consider the linear function

$$K := I + J$$

on L. Then K satisfies all three conditions in our definition of an integral, and in addition is bounded. We know from the case $p = 2$ of the completeness of L^p that $L^2(K)$ is a (real) Hilbert space. (Assume for this argument that we have passed to the quotient space so an element of $L^2(K)$ is an equivalence class of of functions.) The fact that K is bounded, says that $\mathbf{1} := \mathbf{1}_S \in L^2(K)$. If $f \in L^2(K)$ then the Cauchy-Schwartz inequality says that

$$K(|f|) = K(|f| \cdot \mathbf{1}) = (|f|, \mathbf{1})_{2,K} \leq \|f\|_{2,K} \|\mathbf{1}\|_{2,K} < \infty$$

so $|f|$ and hence f are elements of $L^1(K)$.

Furthermore, if $f \in L^2(K)$,

$$|J(f)| \leq J(|f|) \leq K(|f|) \leq \|f\|_{2,K}\|\mathbf{1}\|_{2,K}.$$

So $f \mapsto J(f)$ is a bounded linear function on $L^2(K)$. We conclude from the real version of the Riesz representation theorem, that there exists a unique $g \in L^2(K)$ such that

$$J(f) = (f, g)_{2,K} = K(fg).$$

If A is any K- measurable subset of S , then $0 \leq J(\mathbf{1}_A) = K(\mathbf{1}_A g)$ so g is non-negative. (More precisely, g is equivalent K-almost everywhere to a function which is non-negative.)

We obtain inductively

$$\begin{aligned} J(f) &= K(fg) = \\ I(fg) + J(fg) &= I(fg) + I(fg^2) + J(fg^2) = \\ &\vdots \\ &= I\left(f \cdot \sum_{i=1}^{n} g^i\right) + J(fg^n). \end{aligned}$$

$$\begin{aligned} J(f) &= K(fg) = \\ I(fg) + J(fg) &= I(fg) + I(fg^2) + J(fg^2) = \\ &\vdots \\ &= I\left(f \cdot \sum_{i=1}^{n} g^i\right) + J(fg^n). \end{aligned}$$

Let N be the set of all x where $g(x) \geq 1$. Taking $f = \mathbf{1}_N$ in the preceding string of equalities shows that

$$J(\mathbf{1}_N) \geq nI(\mathbf{1}_N).$$

Since n is arbitrary, we have proved

Lemma

The set where $g \geq 1$ has I measure zero.

$$
\begin{aligned}
J(f) &= K(fg) = \\
I(fg) + J(fg) &= I(fg) + I(fg^2) + J(fg^2) = \\
&\vdots \\
&= I\left(f \cdot \sum_{i=1}^{n} g^i\right) + J(fg^n).
\end{aligned}
$$

We have not yet used the assumption that J is absolutely continuous with respect to I. We now use this assumption to conclude that N is also J-null. This implies that if $f \geq 0$ and $f \in L^1(J)$ then $fg^n \searrow 0$ almost everywhere (J), and hence by the dominated convergence theorem

$$
J(fg^n) \searrow 0.
$$

$$
\begin{aligned}
J(f) &= K(fg) = \\
I(fg) + J(fg) &= I(fg) + I(fg^2) + J(fg^2) = \\
&\vdots \\
&= I\left(f \cdot \sum_{i=1}^{n} g^i\right) + J(fg^n).
\end{aligned}
$$

We have proved that

$$J(fg^n) \searrow 0.$$

Plugging this back into the above string of equalities shows (by the monotone convergence theorem for I) that $f\sum_{i=1}^{\infty} g^n$ converges in the $L^1(I)$ norm to $J(f)$.

$$f \sum_{i=1}^{\infty} g^n$$

converges in the $L^1(I)$ norm to $J(f)$.
In particular, since $J(\mathbf{1}) < \infty$, we may take $f = \mathbf{1}$ and conclude that $\sum_{i=1}^{\infty} g^i$ converges in $L^1(I)$. So set

$$f_0 := \sum_{i=1}^{\infty} g^i \in L^1(I).$$

$$f_0 := \sum_{i=1}^{\infty} g^i \in L^1(I).$$

We have

$$f_0 = \frac{1}{1-g} \quad \text{almost everywhere}$$

so

$$g = \frac{f_0 - 1}{f_0} \quad \text{almost everywhere}$$

and

$$J(f) = I(ff_0)$$

for $f \geq 0, \ f \in L^1(J)$. By breaking any $f \in L^1(J)$ into the difference of its positive and negative parts, we conclude that (3) holds for all $f \in L^1(J)$. The uniqueness of f_0 (almost everywhere (I)) follows from the uniqueness of g in $L^2(K)$. □

Extensions, I.

The Radon Nikodym theorem can be extended in two directions. First of all, let us continue with our assumption that I and J are bounded, but drop the absolute continuity requirement. Let us say that an integral H is **absolutely singular** with respect to I if there is a set N of I-measure zero such that $H(h) = 0$ for any h vanishing on N.

Let us now go back to Lemma 21:

Lemma

The set where $g \geq 1$ has I measure zero.

Define J_{sing} by

$$J_{sing}(f) = J(\mathbf{1}_N f).$$

Then J_{sing} is singular with respect to I, and we can write

$$J = J_{cont} + J_{sing}$$

where

$$J_{cont} = J - J_{sing} = J(\mathbf{1}_{N^c} \cdot).$$

Then we can apply the rest of the proof of the Radon Nikodym theorem to J_{cont} to conclude that

$$J_{cont}(f) = I(ff_0)$$

where $f_0 = \sum_{i=1}^{\infty} (\mathbf{1}_{N^c} g)^i$ is an element of $L^1(I)$ as before. In particular, J_{cont} is absolutely continuous with respect to I.

Extensions, II.

A second extension is to certain situations where S is not of finite measure. We say that a function f is **locally** L^1 if $f\mathbf{1}_A \in L^1$ for every set A with $\mu(A) < \infty$. We say that S is σ-**finite** with respect to μ if S is a countable union of sets of finite μ measure. This is the same as saying that $\mathbf{1} = \mathbf{1}_S \in \mathcal{B}$. If S is σ-finite then it can be written as a disjoint union of sets of finite measure. If S is σ-finite with respect to both I and J it can be written as the disjoint union of countably many sets which are both I and J finite. So if J is absolutely continuous with respect I, we can apply the Radon-Nikodym theorem to each of these sets of finite measure, and conclude that there is an f_0 which is locally L^1 with respect to I, such that $J(f) = I(ff_0)$ for all $f \in L^1(J)$, and f_0 is unique up to almost everywhere equality.

The map from $L^q \to (L^p)^*$

Recall that Hölder's inequality (1) says that

$$\left|\int fg d\mu\right| \leq \|f\|_p \|g\|_q$$

if $f \in L^p$ and $g \in L^q$ where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

For the rest of this lecture we will assume without further mention that this relation between p and q holds. Hölder's inequality implies that we have a map from

$$L^q \to (L^p)^*$$

sending $g \in L^p$ to the continuous linear function on L^p which sends

$$f \mapsto I(fg) = \int fg d\mu.$$

The map from $L^q \to (L^p)^*$ is injective

Furthermore, Hölder's inequality says that the norm of this map from $L^q \to (L^p)^*$ is ≤ 1. In particular, this map is injective.

The theorem we want to prove is that under suitable conditions on S and I (which are more general even that σ-finiteness) this map is surjective for $1 \leq p < \infty$.
We will first prove the theorem in the case where $\mu(S) < \infty$, that is when I is a bounded integral. For this we will will need to discuss the "variations of a of a bounded linear functional":

The "positive part" of a bounded linear functional.

Suppose we start with an arbitrary L and I. For each $1 \leq p \leq \infty$ we have the norm $\|\cdot\|_p$ on L which makes L into a real normed linear space. Let F be a linear function on L which is bounded with respect to this norm, so that

$$|F(f)| \leq C\|f\|_p$$

for all $f \in L$ where C is some non-negative constant. The least upper bound of the set of C which is called $\|F\|_p$ as usual. If $f \geq 0 \in L$, define

$$F^+(f) := \mathrm{lub}\{F(g) : 0 \leq g \leq f, \quad g \in L\}.$$

The "positive part" of a bounded linear functional, continued.

Then $F^+(f) \geq 0$ and

$$F^+(f) \leq \|F\|_p \|f\|_p$$

since $F(g) \leq |F(g)| \leq \|F\|_p \|g\|_p \leq \|F\|_p \|f\|_p$ for all $0 \leq g \leq f$, $g \in L$, since $0 \leq g \leq f$ implies $|g|^p \leq |f|^p$ for $1 \leq p < \infty$ and also implies $\|g\|_\infty \leq \|f\|_\infty$. Also

$$F^+(cf) = cF^+(f) \quad \forall \; c \geq 0$$

as follows directly from the definition.

The "positive part" of a bounded linear functional, continued.

Suppose that f_1 and f_2 are both non-negative elements of L. If $g_1,\ g_2 \in L$ with

$$0 \leq g_1 \leq f_1 \text{ and } 0 \leq g_2 \leq f_2$$

then

$$F^+(f_1+f_2) \geq \operatorname{lub} F(g_1+g_1) = \operatorname{lub} F(g_1)+\operatorname{lub} F(g_2) = F^+(f_1)+F^+(f_2).$$

The "positive part" of a bounded linear functional, continued.

On the other hand, if $g \in L$ satisfies $0 \le g \le (f_1 + f_2)$ then $0 \le g \wedge f_1 \le f_1$, and $g \wedge f_1 \in L$. Also $g - g \wedge f_1 \in L$ and vanishes at points x where $g(x) \le f_1(x)$ while at points where $g(x) > f_1(x)$ we have $g(x) - g \wedge f_1(x) = g(x) - f_{1(x)} \le f_2(x)$. So

$$g - g \wedge f_1 \le f_2$$

and so

$$F^+(f_1+f_2) = \mathrm{lub}\, F(g) \le \mathrm{lub}\, F(g \wedge f_1) + \mathrm{lub}\, F(g - g \wedge f_1) \le F^+(f_1) + F^+(f_2)$$

So

$$F^+(f_1 + f_2) = F^+(f_1) + F^+(f_2)$$

if both f_1 and f_2 are non-negative elements of L.

The "positive part" of a bounded linear functional, continued.

Now write any $f \in L$ as $f = f_1 - g_1$ where f_1 and g_1 are non-negative. (For example we could take $f_1 = f^+$ and $g_1 = f^-$.)
Define: $F^+(f) = F^+(f_1) - F^+(g_1)$.
This is well defined, for if we also had $f = f_2 - g_2$ then $f_1 + g_2 = f_2 + g_1$ so

$$F^+(f_1) + F^+(g_2) = F^+(f_1 + g_2) = F^+(f_2 + g_1) = F^+(f_2) + F^+(g_1)$$

so

$$F^+(f_1) - F^+(g_1) = F^+(f_2) - F^+(g_2).$$

From this it follows that F^+ so extended is linear, and

$$|F^+(f)| \leq F^+(|f|) \leq \|F\|_p \|f_p\|$$

so F^+ is bounded.

The negative part of a bounded linear functional

Define F^- by

$$F^-(f) := F^+(f) - F(f).$$

As F^- is the difference of two linear functions it is linear. Since by its definition, $F^+(f) \geq F(f)$ if $f \geq 0$, we see that $F^-(f) \geq 0$ if $f \geq 0$. Clearly $\|F^-\| \leq \|F^+\|_p + \|F\| \leq 2\|F\|_p$. We have proved:

Proposition

Every linear function on L which is bounded with respect to the $\|\cdot\|_p$ norm can be written as the difference $F = F^+ - F^-$ of two linear functions which are bounded with respect to the $\|\cdot\|_p$ norm and take non-negative values on non-negative functions.

In fact, we could formulate this proposition more abstractly as dealing with a normed vector space which has an order relation consistent with its metric but we shall refrain from this more abstract formulation.

Duality of L^p and L^q when $\mu(S) < \infty$.

Theorem

Suppose that $\mu(S) < \infty$ and that F is a bounded linear function on L^p with $1 \leq p < \infty$. Then there exists a unique $g \in L^q$ such that

$$F(f) = (f, g) = I(fg).$$

Here $q = p/(p-1)$ if $p > 1$ and $q = \infty$ if $p = 1$.

Proof, 1.

Consider the restriction of F to L. We know that $F = F^+ - F^-$ where both F^+ and F^- are linear and non-negative and are bounded with respect to the $\| \cdot \|_p$ norm on L. The monotone convergence theorem implies that if $f_n \searrow 0$ then $\|f_n\|_p \to 0$ and the boundedness of F^+ with respect to the $\| \cdot \|_p$ says that

$$\|f_n\|_p \to 0 \quad \Rightarrow \quad F^+(f_n) \to 0.$$

So F^+ satisfies all the axioms for an integral, and so does F^-. If f vanishes outside a set of I measure zero, then $\|f\|_p = 0$. Applied to a function of the form $f = \mathbf{1}_A$ we conclude that if A has $\mu = \mu_I$ measure zero, then A has measure zero with respect to the measures determined by F^+ or F^-.

Proof, 2- Using Radon-Nikodym.

We can apply the Radon-Nikodym theorem to conclude that there are functions g^+ and g^- which belong to $L^1(I)$ and such that

$$F^{\pm}(f) = I(fg^{\pm})$$

for every f which belongs to $L^1(F^{\pm})$. In particular, if we set $g := g^+ - g^-$ then

$$F(f) = I(fg)$$

for every function f which is integrable with respect to both F^+ and F^-, in particular for any $f \in L^p(I)$. We must show that $g \in L^q$.

Proof, 3.

We first treat the case where $p > 1$. Suppose that $0 \leq f \leq |g|$ and that f is bounded. Then

$$I(f^q) \leq I(f^{q-1} \cdot \operatorname{sgn}(g) g) = F(f^{q-1} \cdot \operatorname{sgn}(g)) \leq \|F\|_p \|f^{q-1}\|_p.$$

So

$$I(f^q) \leq \|F\|_p (I(f^{(q-1)p}))^{\frac{1}{p}}.$$

Now $(q-1)p = q$ so we have

$$I(f^q) \leq \|F\|_p I(f^q)^{\frac{1}{p}} = \|F\|_q I(f^q)^{1-\frac{1}{q}}.$$

This gives

$$\|f\|_q \leq \|F\|_p$$

for all $0 \leq f \leq |g|$ with f bounded. Choose such functions f_n with $f_n \nearrow |g|$. It follows from the monotone convergence theorem that $|g|$ and hence $g \in L^q(I)$ proving the theorem for $p > 1$.

The case $p = 1$

Let us now give the argument for $p = 1$. We want to show that $\|g\|_\infty \leq \|F\|_1$. Suppose that $\|g\|_\infty \geq \|F\|_1 + \epsilon$ where $\epsilon > 0$. Consider the function $\mathbf{1}_A$ where

$$A := \{x : |g(x) \geq \|F\|_1 + \frac{\epsilon}{2}\}.$$

Then

$$(\|F\|_1 + \frac{\epsilon}{2})\mu(A) \leq I(\mathbf{1}_A|g|) = I(\mathbf{1}_A \mathrm{sgn}(g)g) = F(\mathbf{1}_A \mathrm{sgn}(g))$$

$$\leq \|F\|_1 \|\mathbf{1}_A \mathrm{sgn}(g)|_1 = \|F\|_1 \mu(A)$$

which is impossible unless $\mu(A) = 0$, contrary to our assumption.
□