Math212a1412
Constructing outer measures.

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Review: outer measures.

An **outer measure** on a set X is a map m^* to $[0, \infty]$ defined on the collection of *all* subsets of X which satisfies

- $m^*(\emptyset) = 0$,
- **Monotonicity:** If $A \subset B$ then $m^*(A) \leq m^*(B)$, and
- **Countable subadditivity:** $m^* \left(\bigcup_n A_n \right) \leq \sum_n m^*(A_n)$.

Measures from outer measures via Caratheordory.

Given an outer measure, m^*, we defined a set E to be **measurable** (relative to m^*) if

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

for all sets A. Then Caratheodory's theorem that we proved last time asserts that the collection of measurable sets is a σ-field, and the restriction of m^* to the collection of measurable sets is a measure which we shall usually denote by m.

Countable covers.

Let $\mathcal{C}$ be a collection of sets which cover X. For any subset A of X let

$$ccc(A)$$

denote the set of (finite or) **c**ountable **c**overs of A by sets belonging to $\mathcal{C}$. In other words, an element of $ccc(A)$ is a finite or countable collection of elements of $\mathcal{C}$ whose union contains A.

Suppose we are given a function

$$\ell : \mathcal{C} \to [0, \infty].$$

Theorem

There exists a unique outer measure m^* on X such that

- $m^*(A) \leq \ell(A)$ for all $A \in \mathcal{C}$ and
- If n^* is any outer measure satisfying the preceding condition then $n^*(A) \leq m^*(A)$ for all subsets A of X.

This unique outer measure is given by

$$m^*(A) = \inf_{\mathcal{D} \in \mathrm{ccc}(A)} \sum_{D \in \mathcal{D}} \ell(D). \tag{1}$$

In other words, for each countable cover of A by elements of $\mathcal{C}$ we compute the sum above, and then minimize over all such covers of A.

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1.Uniqueness, 2. $m^*(\emptyset) = 0$, 3. monotonicity.

If we had two outer measures satisfying both conditions then each
would have to be $\leq$ the other, so the uniqueness is obvious.

To check that the m^* defined by (1) is an outer measure, observe
that for the empty set we may take the empty cover, and the
convention about an empty sum is that it is zero, so $m^*(\emptyset) = 0$. If
$A \subset B$ then any cover of B is a cover of A, so that
$m^*(A) \leq m^*(B)$.

Subadditivity.

To check countable subadditivity we use the usual $\epsilon/2^n$ trick: If $m^*(A_n) = \infty$ for any A_n the subadditivity condition is obviously satisfied. Otherwise, we can find a $\mathcal{D}_n \in \text{ccc}(A_n)$ with

$$\sum_{D \in \mathcal{D}_n} \ell(D) \leq m^*(A_n) + \frac{\epsilon}{2^n}.$$

Then we can collect all the D together into a countable cover of A so

$$m^*(A) \leq \sum_n m^*(A_n) + \epsilon,$$

and since this is true for all $\epsilon > 0$ we conclude that m^* is countably subadditive.

Checking condition 1: $m^*(A) \leq \ell(A)$ for all $A \in \mathcal{C}$.

We have verified that m^* defined by (1) is an outer measure. We must check that it satisfies the two conditions in the theorem. If $A \in \mathcal{C}$ then the single element collection $\{A\} \in \mathrm{ccc}(A)$, so $m^*(A) \leq \ell(A)$, so the first condition is obvious.

Checking conditions 2: If n^* is any outer measure satisfying $n^*(D) \leq \ell(D)$ then $n^*(A) \leq m^*(A)$ for all subsets A of X.

As to condition 2, suppose n^* is an outer measure with $n^*(D) \leq \ell(D)$ for all $D \in \mathcal{C}$. Then for any set A and any countable cover $\mathcal{D}$ of A by elements of $\mathcal{C}$ we have

$$\sum_{D \in \mathcal{D}} \ell(D) \geq \sum_{D \in \mathcal{D}} n^*(D) \geq n^* \left(\bigcup_{D \in \mathcal{D}} D \right) \geq n^*(A),$$

where in the second inequality we used the countable subadditivity of n^* and in the last inequality we used the monotonicity of n^*. Minimizing over all $\mathcal{D} \in \mathrm{ccc}(A)$ shows that $m^*(A) \geq n^*(A)$. $\square$

This argument is basically a repeat of the construction of Lebesgue measure of the last lecture. However there is some trouble:

A pathological example.

Suppose we take $X = \mathbb{R}$, and let $\mathcal{C}$ consist of all *half open* intervals of the form $[a, b)$. However, instead of taking ℓ to be the length of the interval, we take it to be the square root of the length:

$$\ell([a, b)) := (b - a)^{\frac{1}{2}}.$$

I claim that any half open interval (say $[0, 1)$) of length one has $m^*([a, b)) = 1$. (Since ℓ is translation invariant, it does not matter which interval we choose.)

Claim: $m^*([0,1)) = 1.$

Proof.

$m^*([0,1)) \leq 1$ by the first condition in the theorem, since
$\ell([0,1)) = 1$. On the other hand, if

$$[0,1) \subset \bigcup_i [a_i, b_i)$$

then we know from Heine-Borel that $\sum (b_i - a_i) \geq 1$, so squaring
gives

$$\left(\sum (b_i - a_i)^{\frac{1}{2}} \right)^2 = \sum_i (b_i - a_i) + \sum_{i \neq j} (b_i - a_i)^{\frac{1}{2}} (b_j - a_j)^{\frac{1}{2}} \geq 1.$$

So $m^*([0,1)) = 1$. □

The closed unit interval is not measurable in this example.

On the other hand, consider an interval $[a, b)$ of length 2. Since it covers itself, $m^*([a, b)) \leq \sqrt{2}$.

Consider the closed interval $I = [0, 1]$. Then

$$I \cap [-1, 1) = [0, 1) \quad \text{and} \quad I^c \cap [-1, 1) = [-1, 0)$$

so

$$m^*(I \cap [-1, 1)) + m^*(I^c \cap [-1, 1)) = 2 > \sqrt{2} \geq m^*([-1, 1)).$$

In other words, the closed unit interval is *not measurable* relative to the outer measure m^* determined by the theorem.

A desiratum.

We would like Borel sets to be measurable, and the above computation shows that the measure produced by Method I as above does not have this desirable property. In fact, if we consider two half open intervals I_1 and I_2 of length one separated by a small distance of size ϵ, say, then their union $I_1 \cup I_2$ is covered by an interval of length $2 + \epsilon$, and hence

$$m^*(I_1 \cup I_2) \leq \sqrt{2 + \epsilon} < m^*(I_1) + m^*(I_2).$$

In other words, m^* is not additive even on intervals separated by a finite distance. It turns out that this is the crucial property that is missing:

Outline Constructing outer measures, Method I. Constructing outer measures, Method II. Hausdorff measure. Hausdorff dimensio

Metric outer measures.

Let X be a metric space. An outer measure on X is called a **metric outer measure** if

$$m^*(A \cup B) = m^*(A) + m^*(B) \text{ whenever } d(A, B) > 0. \quad (2)$$

The condition $d(A, B) > 0$ means that there is an $\epsilon > 0$ (depending on A and B) so that $d(x, y) > \epsilon$ for all $x \in A$, $y \in B$. The main result here is due to Caratheodory:

Outline Constructing outer measures, Method I. Constructing outer measures, Method II. Hausdorff measure. Hausdorff dimensio

Metric outer measures.

Caratheodory's theorem on metric outer measures.

Theorem

If m^* is a metric outer measure on a metric space X, then all Borel sets of X are m^* measurable.

Since the σ-field of Borel sets is generated by the closed sets, it is enough to prove that every closed set F is measurable in the sense of Caratheodory, i.e. that for any set A

$$m^*(A) \geq m^*(A \cap F) + m^*(A \setminus F)$$

for any closed set F.

This will require a clever argument due to Caratheodory.

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Metric outer measures.

Let
$$A_j := \{x \in A | d(x, F) \geq \frac{1}{j}\}.$$

We have $d(A_j, A \cap F) \geq 1/j$ so, since m^* is a metric outer measure, we have

$$m^*(A \cap F) + m^*(A_j) = m^*((A \cap F) \cup A_j) \leq m^*(A) \qquad (3)$$

since $(A \cap F) \cup A_j \subset A$. Now
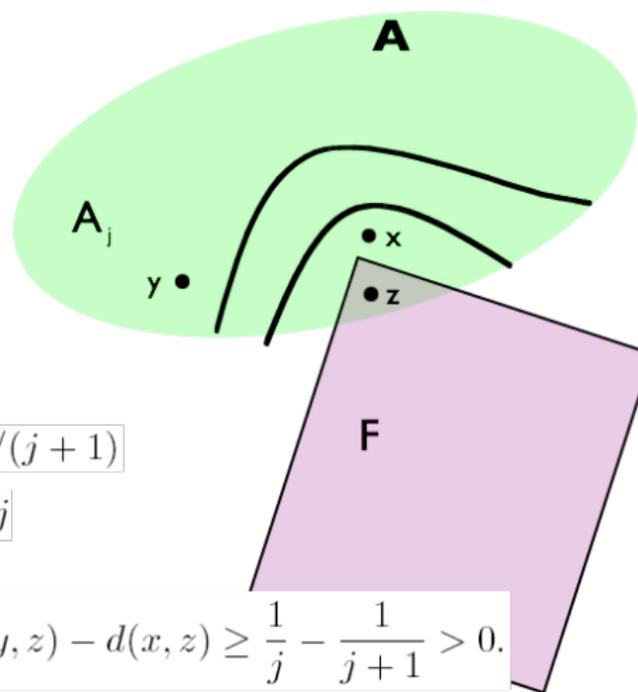
$$A \setminus F = \bigcup A_j$$

since F is closed, and hence every point of A not belonging to F must be at a positive distance from F. We would like to be able to pass to the limit in (3). If the limit on the left is infinite, there is nothing to prove. So we may assume it is finite.

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Metric outer measures.

If $x \in A \setminus (F \cup A_{j+1})$ there is a $z \in F$ with $d(x, z) < 1/(j+1)$
while if $y \in A_j$ we have $d(y, z) \geq 1/j$ so

$$d(x, y) \geq d(y, z) - d(x, z) \geq \frac{1}{j} - \frac{1}{j+1} > 0.$$

See the figure in the next slide, where $x \notin A_{j+1}$ and $y \in A_j$.

Outline Constructing outer measures, Method I. Constructing outer measures, Method II. Hausdorff measure. Hausdorff dimensio
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Metric outer measures.

Let $A_j := \{x \in A | d(x, F) \geq \frac{1}{j}\}.$

$d(x, z) < 1/(j+1)$

$d(y, z) \geq 1/j$

$d(x, y) \geq d(y, z) - d(x, z) \geq \dfrac{1}{j} - \dfrac{1}{j+1} > 0.$

Outline **Constructing outer measures, Method I.** Constructing outer measures, Method II. Hausdorff measure. Hausdorff dimensio
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Metric outer measures.

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while if $y \in A_j$ we have $d(y, z) \geq 1/j$ so

$$d(x, y) \geq d(y, z) - d(x, z) \geq \frac{1}{j} - \frac{1}{j + 1} > 0.$$

Let $B_1 := A_1$ and $B_2 := A_2 \setminus A_1$, $B_3 = A_3 \setminus A_2$ etc. Thus if
$i \geq j + 2$, then $B_j \subset A_j$ and

$$B_i \subset A \setminus (F \cup A_{i-1}) \subset A \setminus (F \cup A_{j+1})$$

and so $d(B_i, B_j) > 0$. So m^* is additive on finite unions of even or
odd B's:

$$m^* \left(\bigcup_{k=1}^{n} B_{2k-1} \right) = \sum_{k=1}^{n} m^*(B_{2k-1}), \quad m^* \left(\bigcup_{k=1}^{n} B_{2k} \right) = \sum_{k=1}^{n} m^*(B_{2k}).$$

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Metric outer measures.

$$m^* \left(\bigcup_{k=1}^n B_{2k-1} \right) = \sum_{k=1}^n m^*(B_{2k-1}), \quad m^* \left(\bigcup_{k=1}^n B_{2k} \right) = \sum_{k=1}^n m^*(B_{2k}).$$

Both of these are $\leq m^*(A_{2n})$ since the union of the sets involved are contained in A_{2n}. Since $m^*(A_{2n})$ is increasing, and assumed bounded, both of the above series converge.

Outline   **Constructing outer measures, Method I.**   Constructing outer measures, Method II.   Hausdorff measure.   Hausdorff dimensio
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Metric outer measures.

Thus

$$
\begin{aligned}
m^*(A/F) &= m^*\left(\bigcup A_i\right) \\
&= m^*\left(A_j \cup \bigcup_{k \geq j+1} B_k\right) \\
&\leq m^*(A_j) + \sum_{k=j+1}^{\infty} m^*(B_k) \\
&\leq \lim_{n\to\infty} m^*(A_n) + \sum_{k=j+1}^{\infty} m^*(B_k).
\end{aligned}
$$

But the sum on the right can be made as small as possible by choosing j large, since the series converges. Hence

$$
m^*(A/F) \leq \lim_{n\to\infty} m^*(A_n). \qquad \square
$$

Comparing the method I construction for two covers.

Let $\mathcal{C} \subset \mathcal{E}$ be two covers, and suppose that ℓ is defined on $\mathcal{E}$, and hence, by restriction, on $\mathcal{C}$. In the definition (1) of the outer measure $m^*_{\ell,\mathcal{C}}$ associated to ℓ and $\mathcal{C}$, we are minimizing over a smaller collection of covers than in computing the metric outer measure $m^*_{\ell,\mathcal{E}}$ using all the sets of $\mathcal{E}$. Hence

$$m^*_{\ell,\mathcal{C}}(A) \geq m^*_{\ell,\mathcal{E}}(A)$$

for any set A.

Applying this remark: the method II construction.

We want to apply this remark to the case where X is a metric space, and we have a cover $\mathcal{C}$ with the property that for every $x \in X$ and every $\epsilon > 0$ there is a $C \in \mathcal{C}$ with $x \in C$ and $\text{diam}(C) < \epsilon$. In other words, we are assuming that the

$$\mathcal{C}_\epsilon := \{C \in \mathcal{C} | \text{ diam } (C) < \epsilon\}$$

are covers of X for every $\epsilon > 0$. Then for every set A the

$$m^*_{\ell, \mathcal{C}_\epsilon}(A)$$

are increasing as ϵ decreases to zero, so we can consider the function on sets given by

$$m^*_{II}(A) := \sup_{\epsilon \to 0} m^*_{\ell, \mathcal{C}_\epsilon}(A).$$

$$m_{II}^*(A) := \sup_{\epsilon \to 0} m_{\ell, \mathcal{C}_\epsilon}^*(A).$$

The axioms for an outer measure are preserved by this limit operation, so m_{II}^* is an outer measure. If A and B are such that $d(A, B) > 2\epsilon$, then any set of $\mathcal{C}_\epsilon$ which intersects A does not intersect B and vice versa, so throwing away extraneous sets in a cover of $A \cup B$ which does not intersect either, we see that $m_{II}^*(A \cup B) = m_{II}^*(A) + m_{II}^*(B)$. The method II construction always yields a metric outer measure.

Binary sequence space.

Let X be the set of all (one sided) infinite sequences of 0's and 1's. So a point of X is an expression of the form

$$a_1 a_2 a_3 \cdots$$

where each a_i is 0 or 1. For any *finite* sequence α of 0's or 1's, let $[\alpha]$ denote the set of all sequences which begin with α. We also let $|\alpha|$ denote the length of α, that is, the number of bits in α.

The metrics d_r on binary sequence space.

For each

$$0 < r < 1$$

we define a metric d_r on X by: If

$$x = \alpha x', \qquad y = \alpha y'$$

where the first bit in x' is different from the first bit in y' then

$$d_r(x, y) := r^{|\alpha|}.$$

In other words, the distance between two sequences is r^k where k is the length of the longest initial segment where they agree.

Proof that d_r is a metric.

Clearly $d_r(x, y) \geq 0$ and $= 0$ if and only if $x = y$, and
$d_r(y, x) = d_r(x, y)$. Also, for three x, y, and z we claim that

$$d_r(x, z) \leq \max\{d_r(x, y), d_r(y, z)\}.$$

Indeed, if two of the three points are equal this is obvious.
Otherwise, let j denote the length of the longest common prefix of
x and y, and let k denote the length of the longest common prefix
of y and z. Let $m = \min(j, k)$. Then the first m bits of x agree
with the first m bits of z and so $d_r(x, z) \leq r^m = \max(r^j, r^k)$.   □

A metric with this property (which is much stronger than the
triangle inequality) is called an **ultrametric**.

The diameter of $[\alpha]$. The spaces (X, d_r) are homeomorphic.

Notice that

$$\text{diam } [\alpha] = r^\alpha. \tag{4}$$

The metrics for different r are different, and we will make use of this fact shortly. But

Proposition.

The spaces (X, d_r) are all homeomorphic under the identity map.

Proof.

It is enough to show that the identity map is a continuous map from (X, d_r) to (X, d_s) since it is one to one and we can interchange the role of r and s. So, given $\epsilon > 0$, we must find a $\delta > 0$ such that if $d_r(x, y) < \delta$ then $d_s(x, y) < \epsilon$. So choose k so that $s^k < \epsilon$. Then letting $\delta = r^k$ will do. $\qquad\square$

So although the metrics are different, the topologies they define are the same.

The case $r = \frac{1}{2}$.

Let $\mathcal{C}$ be the collection of all sets of the form $[\alpha]$ and let ℓ be defined on $\mathcal{C}$ by

$$\ell([\alpha]) = (\frac{1}{2})^{|\alpha|}.$$

We can construct the method II outer measure associated with this function, which will satisfy

$$m_{II}^*([\alpha]) \geq m_I^*([\alpha])$$

where m_I^* denotes the method I outer measure associated with ℓ. What is special about the value $\frac{1}{2}$ is that if $k = |\alpha|$ then

$$\ell([\alpha]) = \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{k+1} + \left(\frac{1}{2}\right)^{k+1} = \ell([\alpha 0]) + \ell([\alpha 1]).$$

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What is special about the value $\frac{1}{2}$ is that if $k = |\alpha|$ then

$$\ell([\alpha]) = \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{k+1} + \left(\frac{1}{2}\right)^{k+1} = \ell([\alpha 0]) + \ell([\alpha 1]).$$

So if we use the metric $d_{\frac{1}{2}}$, we see, by repeating the above, that every $[\alpha]$ can be written as the disjoint union $C_1 \cup \cdots \cup C_n$ of sets in $\mathcal{C}_\epsilon$ with $\ell([\alpha]) = \sum \ell(C_i)$. Thus $m^*_{\ell, \mathcal{C}_\epsilon}([\alpha]) \leq \ell(\alpha)$ and so $m^*_{\ell, \mathcal{C}_\epsilon}([\alpha])(A) \leq m^*_I(A)$ or $m^*_{II} = m^*_I$.

It also follows from the above computation that

$$m([\alpha]) = m^*([\alpha]) = \ell([\alpha]).$$

In particular

$$m(X) = 1.$$

The case $r = \frac{1}{3}$.

There is also something special about the value $r = \frac{1}{3}$: Recall that one of the definitions of the Cantor set **C** is that it consists of all points $x \in [0, 1]$ which have a base 3 expansion involving only the symbols 0 and 2. Let

$$h : X \to \mathbf{C}$$

where h sends the bit 1 into the symbol 2, e.g.

$$h(011001\ldots) = .022002\ldots.$$

In other words, for any sequence z

$$h(0z) = \frac{h(z)}{3}, \qquad h(1z) = \frac{h(z) + 2}{3}. \tag{5}$$

I claim that:

$$\frac{1}{3} d_{\frac{1}{3}}(x, y) \leq |h(x) - h(y)| \leq d_{\frac{1}{3}}(x, y) \tag{6}$$

Proof. If x and y start with different bits, say $x = 0x'$ and
$y = 1y'$ then $d_{\frac{1}{3}}(x, y) = 1$ while $h(x)$ lies in the interval $[0, \frac{1}{3}]$ and
$h(y)$ lies in the interval $[\frac{2}{3}, 1]$ on the real line. So $h(x)$ and $h(y)$
are at least a distance $\frac{1}{3}$ and at most a distance 1 apart, which is
what (6) says. So we proceed by induction.

$$h(0z) = \frac{h(z)}{3}, \qquad h(1z) = \frac{h(z) + 2}{3}. \qquad (5)$$

$$\frac{1}{3} d_{\frac{1}{3}}(x, y) \leq |h(x) - h(y)| \leq d_{\frac{1}{3}}(x, y). \qquad (6)$$

Suppose we know that (6) is true when $x = \alpha x'$ and $y = \alpha y'$ with x', y' starting with different digits, and $|\alpha| \leq n$. (The above case was where $|\alpha| = 0$.) So if $|\alpha| = n + 1$ then either $\alpha = 0\beta$ or $\alpha = 1\beta$ and the argument for either case is similar: We know that (6) holds for $\beta x'$ and $\beta y'$ and

$$d_{\frac{1}{3}}(x, y) = \frac{1}{3} d_{\frac{1}{3}}(\beta x', \beta y')$$

while $|h(x) - h(y)| = \frac{1}{3}|h(\beta x') - h(\beta y')|$ by (5). Hence (6) holds by induction. $\quad \square$

In other words, the map h is a Lipschitz map with Lipschitz inverse from $(X, d_{\frac{1}{3}})$ to the Cantor set **C**.

In a short while, after making the appropriate definitions, these two computations, one with the measure associated to $\ell([\alpha]) = \left(\frac{1}{2}\right)^{|\alpha|}$ and the other associated with $d_{\frac{1}{3}}$ will show that the "Hausdorff dimension" of the Cantor set is $\log 2/\log 3$.

Hausdorff measures on a metric space.

Let X be a metric space. Recall that if A is any subset of X, the **diameter** of A is defined as $\quad \text{diam}(A) = \sup_{x,y \in A} d(x,y)$.

Take $\mathcal{C}$ to be the collection of "all" subsets of X, and for any positive real number s define

$$\ell_s(A) = \text{diam}(A)^s$$

(with $0^s = 0$). The associated method II outer measure is called the s-**dimensional Hausdorff outer measure**, and its restriction to the associated σ-field of (Caratheodory) measurable sets is called the s-**dimensional Hausdorff measure**. We will let $m^*_{s,\epsilon}$ denote the method I outer measure associated to ℓ_s and ϵ, and let $\mathcal{H}^*_s$ denote the Hausdorff outer measure of dimension s, so that

$$\mathcal{H}^*_s(A) = \lim_{\epsilon \to 0} m^*_{s,\epsilon}(A).$$

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For example, we claim that for $X = \mathbb{R}$, $\mathcal{H}_1^*$ is Lebesgue outer measure, which we will denote here by L^*. Indeed, if A has diameter r, then A is contained in a closed interval of length r. Hence $L^*(A) \leq r$. The Method I construction theorem says that $m_{1,\epsilon}^*$ is the largest outer measure satisfying $m^*(A) \leq \text{diam } A$ for sets of diameter less than ϵ. Hence $m_{1,\epsilon}^*(A) \geq L^*(A)$ for all sets A of diameter less than ϵ and this is true for all ϵ. So

$$\mathcal{H}_1^* \geq L^*.$$

On the other hand, any bounded half open interval $[a, b)$ can be broken up into a finite union of half open intervals of length $< \epsilon$, whose sum of diameters is $b - a$. So $m_{1,\epsilon}^*([a, b)) \leq b - a$. But the method I construction theorem says that L^* is the largest outer measure satisfying

$$m^*([a, b)) \leq b - a.$$

Hence $\mathcal{H}_1^* \leq L^*$ So they are equal.

In two or more dimensions, the Hausdorff measure $\mathcal{H}_k$ on $\mathbb{R}^k$ differs from Lebesgue measure by a constant. This is essentially because they assign different values to the ball of diameter one. In two dimensions for example, the Hausdorff measure $\mathcal{H}_2$ assigns the value one to the disk of diameter one, while its Lebesgue measure is $\pi/4$. For this reason, some authors prefer to put this "correction factor" into the definition of the Hausdorff measure, which would involve the Gamma function for non-integral s. I am following the convention that finds it simpler to drop this factor.

Back to the general theory:

The main theorem in the theory.

Theorem

Let $F \subset X$ be a Borel set. Let $0 < s < t$. Then

$$\mathcal{H}_s(F) < \infty \;\Rightarrow \mathcal{H}_t(F) = 0$$

and

$$\mathcal{H}_t(F) > 0 \;\Rightarrow \mathcal{H}_s(F) = \infty.$$

Proof.

If diam $A \leq \epsilon$, then

$$m_{t,\epsilon}^*(A) \leq (\text{diam}\, A)^t \leq \epsilon^{t-s}(\text{diam}\, A)^s$$

so by the method I construction theorem we have

$$m_{t,\epsilon}^*(B) \leq \epsilon^{t-s} m_{s,\epsilon}^*(B)$$

for all B. If we take $B = F$ in this equality, then the assumption $\mathcal{H}_s(F) < \infty$ implies that the limit of the right hand side tends to 0 as $\epsilon \to 0$, so $\mathcal{H}_t(F) = 0$. The second assertion in the theorem is the contrapositive of the first. $\qquad\square$

Definition of the Hausdorff dimension.

This last theorem implies that for any Borel set F, there is a unique value s_0 (which might be 0 or ∞) such that $\mathcal{H}_t(F) = \infty$ for all $t < s_0$ and $\mathcal{H}_s(F) = 0$ for all for all $s > s_0$. This value is called the **Hausdorff dimension** of F.

It is one of many competing (and non-equivalent) definitions of dimension.

Notice that it is a metric invariant, and in fact is the same for two spaces differing by a Lipschitz homeomorphism with Lipschitz inverse.

But it is *not* a topological invariant. In fact, we shall show that the space X of all sequences of zeros and one studied above has Hausdorff dimension 1 relative to the metric $d_{\frac{1}{2}}$ while it has Hausdorff dimension $\log 2/\log 3$ if we use the metric $d_{\frac{1}{3}}$. Since we have shown that $(X, d_{\frac{1}{3}})$ is Lipschitz equivalent to the Cantor set **C**, this will also prove that **C** has Hausdorff dimension $\log 2/\log 3$.

The Hausdorff dimension of $(X, d_{\frac{1}{2}})$.

Lemma

If $\operatorname{diam}(A) > 0$*, then there is an* α *such that* $A \subset [\alpha]$ *and* $\operatorname{diam}([\alpha]) = \operatorname{diam} A$.

Proof.

Given any set A, it has a "longest common prefix". Indeed, consider the set of lengths of common prefixes of elements of A. This is a finite set of non-negative integers since A has at least two distinct elements. Let n be the largest of these, and let α be a common prefix of this length. Then it is clearly the longest common prefix of A. Hence $A \subset [\alpha]$ and $\operatorname{diam}([\alpha]) = \operatorname{diam} A$. $\qquad \square$

Let $\mathcal{C}$ denote the collection of all sets of the form $[\alpha]$ and let ℓ be the function on $\mathcal{C}$ given by

$$\ell([\alpha]) = (\frac{1}{2})^{|\alpha|},$$

and let ℓ^* be the associated method I outer measure, and m the associated measure; all these as we introduced above. We have

$$\ell^*(A) \leq \ell^*([\alpha]) = \text{diam}([\alpha]) = \text{diam}(A).$$

By the method I construction theorem, $m_{1,\epsilon}^*$ is the largest outer measure with the property that $n^*(A) \leq \text{diam}\, A$ for sets of diameter $< \epsilon$. Hence $\ell^* \leq m_{1,\epsilon}^*$, and since this is true for all $\epsilon > 0$, we conclude that

$$\ell^* \leq \mathcal{H}_1^*.$$

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On the other hand, for any α and any $\epsilon > 0$, there is an n such that $2^{-n} < \epsilon$ and $n \geq |\alpha|$. The set $[\alpha]$ is the disjoint union of all sets $[\beta] \subset [\alpha]$ with $|\beta| = n$, and there are $2^{n-|\alpha|}$ of these subsets, each having diameter 2^{-n}. So

$$m_{1,\epsilon}^*([\alpha]) \leq 2^{-|\alpha|}.$$

However ℓ^* is the largest outer measure satisfying this inequality for all $[\alpha]$. Hence $m_{1,\epsilon}^* \leq \ell^*$ for all ϵ so $\mathcal{H}_1^* \leq \ell^*$. In other words

$$\mathcal{H}_1 = m.$$

But since we computed that $m(X) = 1$, we conclude that

The Hausdorff dimension of $(X, d_{\frac{1}{2}})$ is 1.

The Hausdorff dimension of $(X, d_{\frac{1}{3}})$.

The diameter $\text{diam}_{\frac{1}{2}}$ relative to the metric $d_{\frac{1}{2}}$ and the diameter $\text{diam}_{\frac{1}{3}}$ relative to the metric $d_{\frac{1}{3}}$ are given by

$$\text{diam}_{\frac{1}{2}}([\alpha]) = \left(\frac{1}{2}\right)^k, \quad \text{diam}_{\frac{1}{3}}([\alpha]) = \left(\frac{1}{3}\right)^k, \quad k = |\alpha|.$$

If we choose s so that $2^{-k} = (3^{-k})^s$ then

$$\text{diam}_{\frac{1}{2}}([\alpha]) = (\text{diam}_{\frac{1}{3}}([\alpha]))^s.$$
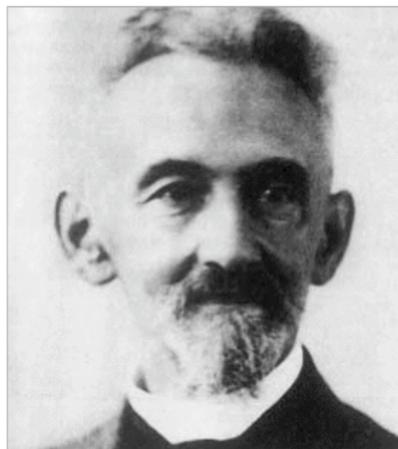
This says that relative to the metric $d_{\frac{1}{3}}$, the previous computation yields

$$\mathcal{H}_s(X) = 1.$$

Hence $\log 2 / \log 3$ is the Hausdorff dimension of the Cantor set.

Felix Hausdorff

Born: 8 Nov 1868 in Breslau, Germany (now Wroclaw, Poland)
Died: 26 Jan 1942 in Bonn, Germany by suicide, to avoid
being sent to an extermination camp.