212a1409
Semigroups, III.
The Spectral Theorem.

Shlomo Sternberg

October 2, 2014

1 The inversion formula.

2 The Hille-Yosida theorem.

3 The spectral theorem, functional calculus form.
- The Dynkin-Hellfer-Sjöstrand formula.
- The existence of almost holomorphic extensions.

4 The key points of the last four lectures.

In today's lecture we cover three big ticket items:

- The Hille-Yosida theorem which gives a necessary and sufficient condition for an operator A to generate an equibounded semi-group.
- The second half of Stone's theorem which asserts that if H is a self-adjoint operator on a Hilbert space then iH generates a one parameter group of unitary transformations $U(t) = e^{iHt}, -\infty < t < \infty$.
- A functional calculus version of the spectral theorem which asserts that we have a map from suitable class of functions on the real line to bounded operators, $f \mapsto f(H)$ where H is a self-adjoint operator. This map is an algebra homomorphism and sends $\overline{f}$ to $f(A)^*$. We will derive this version of the spectral theorem from Stone's theorem.

I will begin, however, with an application of of the inversion formula for the Laplace transform.

In the preceding lecture we started with an equibounded one parameter semigroup T_t, defined its infinitesimal generator, A and found that the resolvent of A was given by the Laplace transform

$$R(z,A) = \int_0^\infty e^{-zt} T_t dt$$

when $\mathrm{Re}\, z > 0$.

In the lecture before that, we obtained, in the case that A is sectorial, an "inversion formula" expressing T_t in terms of the resolvent:

$$T_t = \frac{1}{2\pi i} \int_\Gamma e^{tz} R(z,A) dz.$$

The contour Γ in the integral $\int_\Gamma e^{tz}R(z,A)dz$ could be taken as a large circle centered at the origin in case A was bounded. For A sectorial we "opened up" the circle with rays extending to the left, and had no trouble with the convergence of the contour integral.

But for non-sectorial operators, such as unbounded skew adjoint operators whose spectrum lies on the imaginary axis, the best we can hope for in an inversion formula is to take Γ to be a vertical line to the right of the imaginary axis, i.e. a line of the form $\mathrm{Re}\, z = c > 0$.

We will find that such an inversion formula exists, but that there are some subtleties. First of all, we will only be able to find an "inversion formula" for $T_t x$, $x \in D(A)$. This will be a "strong integral", i.e. an integral in **B** of $R(z, A)x$ rather than a uniform integral (in the space of operators) as in the sectorial case. Also, it won't be an honest improper integral, but rather a "Cauchy principal value".

Suppose that T_t be an equibounded semi-group on a Banach space with generator A. We know that when $\operatorname{Re} z > 0$, the resolvent of A is given by the Laplace transform

$$R(z, A) = \int_0^\infty e^{-zt} T_t dt.$$

We also know that for $x \in D(A)$ the function $t \mapsto T_t x$ is differentiable with bounded derivative $T_t A x$. So we may apply our inversion formula for the Laplace transform (proved in Lecture 6) to conclude that for $t > 0$ we have

$$T_t x = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{zt} R(z, A) x dz$$

where the contour integral is taken as a Cauchy principal value.

Introduction to the Hille-Yosida theorem.

We continue the study of an equibounded semigroup T_t with infinitesimal generator A on a Frechet space $\mathbf{F}$ where we know that the resolvent $R(z, A)$ for Re $z > 0$ is given by

$$R(z, A)x = \int_0^\infty e^{-zt} T_t x dt.$$

This formula shows that $R(z, A)x$ is continuous in z. The resolvent equation

$$R(z, A) - R(w, A) = (w - z)R(z, A)R(w, A)$$

then shows that $R(z, A)x$ is complex differentiable in z with derivative $-R(z, A)^2 x$.

It then follows that $R(z,A)x$ has complex derivatives of all orders given by

$$\frac{d^n R(z,A)x}{dz^n} = (-1)^n n! R(z,A)^{n+1}x.$$

On the other hand, differentiating the integral formula for the resolvent n times gives

$$\frac{d^n R(z,A)x}{dz^n} = \int_0^\infty e^{-zt}(-t)^n T_t x dt$$

where differentiation under the integral sign is justified by the fact that the T_t are equicontinuous in t.
Putting the previous two equations together gives

$$(zR(z,A))^{n+1}x = \frac{z^{n+1}}{n!}\int_0^\infty e^{-zt}t^n T_t x dt.$$

$$(zR(z,A))^{n+1}x = \frac{z^{n+1}}{n!}\int_0^\infty e^{-zt}t^n T_t x dt.$$

This implies that for any semi-norm p we have

$$p((zR(z,A))^{n+1}x) \le \left|\frac{z^{n+1}}{n!}\int_0^\infty e^{-zt}t^n \sup_{t\ge 0} p(T_t x)dt\right| = \sup_{t\ge 0} p(T_t x)$$

since

$$\int_0^\infty e^{-zt}t^n dt = \frac{n!}{z^{n+1}}.$$

Since the T_t are equibounded by hypothesis, we conclude

Proposition

The family of operators $\{(zR(z,A))^n\}$ is equibounded in Re $z > 0$ and $n = 0, 1, 2, \ldots$.

Statement of the Hille-Yosida theorem.

Theorem

[Hille -Yosida.] *Let A be an operator with dense domain $D(A)$, and such that the resolvents*

$$R(n, A) = (nI - A)^{-1}$$

exist and are bounded operators for $n = 1, 2, \ldots$. Then A is the infinitesimal generator of a uniquely determined equibounded semigroup if and only if the operators

$$\{(I - n^{-1}A)^{-m}\}$$

are equibounded in $m = 0, 1, 2 \ldots$ and $n = 1, 2, \ldots$.

Idea of the proof

If A is the infinitesimal generator of an equibounded semi-group then we know that the $\{(I - n^{-1}A)^{-m}\}$ are equibounded by virtue of the preceding proposition. So we must prove the converse. Our proof of the converse will be in several stages.

The idea of the proof is to construct bounded operators J_n so that we can form the semigroup $s \mapsto \exp sJn$ via the exponential series and use these semi-groups to construct approximations to the desired semi-group generated by A.

So we begin by constructing the J_n.

The definition of J_n.

Set

$$J_n = (I - n^{-1}A)^{-1}$$

so $J_n = n(nI - A)^{-1}$ and so for $x \in D(A)$ we have

$$J_n(nI - A)x = nx$$

or

$$J_nAx = n(J_n - I)x.$$

Similarly $(nI - A)J_n = nI$ so $AJ_n = n(J_n - I)$. Thus we have

$$AJ_nx = J_nAx = n(J_n - I)x \quad \forall\ x \in D(A). \tag{1}$$

Since the J_n are bounded, we can construct the one parameter semi-group $s \mapsto \exp(sJ_n)$ via the exponential series. Set $s = nt$. We can then form $e^{-nt} \exp(ntJ_n)$ which we can write as $\exp(tn(J_n - I)) = \exp(tAJ_n)$ by virtue of (1). We expect from

$$\lim_{s \to \infty} sR(s)x = x \quad \forall\ x \in \mathbf{F}$$

that

$$\lim_{n \to \infty} J_n x = x \quad \forall \quad x \in \mathbf{F}. \tag{2}$$

This then suggests that the limit of the $\exp(tAJ_n)$ be the desired semi-group.

Proof that $\lim_{n\to\infty} J_n x = x \quad \forall \quad x \in \mathbf{F}. \quad (2).$

We first prove it for $x \in D(A)$. For such x we have $(J_n - I)x = n^{-1}J_nAx$ by (1) and this approaches zero since the J_n are equibounded. But since $D(A)$ is dense in $\mathbf{F}$ and the J_n are equibounded we conclude that (2) holds for all $x \in \mathbf{F}$. $\square$

Defining the approximating semi-groups.

Now define

$$T_t^{(n)} = \exp(tAJ_n) := \exp(nt(J_n - I)) = e^{-nt}\exp(ntJ_n).$$

We know from our study of the exponential series that

$$p(\exp(ntJ_n)x) \leq \sum \frac{(nt)^k}{k!} p(J_n^k x) \leq e^{nt} K q(x)$$

which implies that

$$p(T_t^{(n)} x) \leq K q(x). \tag{3}$$

Thus the family of operators $\{T_t^{(n)}\}$ is equibounded for all $t \geq 0$ and $n = 1, 2, \ldots$.

The $\{T_t^{(n)}\}$ converge as $n \to \infty$ uniformly on each compact interval of t.

The J_n commute with one another by their definition, and hence J_n commutes with $T_t^{(m)}$. By the semi-group property we have

$$\frac{d}{dt}T_t^{(m)}x = AJ_mT_t^{(m)}x = T_t^{(m)}AJ_mx$$

so

$$T_t^{(n)}x - T_t^{(m)}x = \int_0^t \frac{d}{ds}(T_{t-s}^{(m)}T_s^{(n)})xds = \int_0^t T_{t-s}^{(m)}(AJ_n - AJ_m)T_s^{(n)}xds.$$

Applying the semi-norm p and using the equiboundedness we see that

$$p(T_t^{(n)}x - T_t^{(m)}x) \leq Ktq((J_n - J_m)Ax).$$

$$p(T_t^{(n)}x - T_t^{(m)}x) \leq Ktq((J_n - J_m)Ax).$$

From (2) which tells us that the $J_nAx \to Ax$ this implies that the $T_t^{(n)}x$ converge (uniformly in every compact interval of t) for $x \in D(A)$, and hence since $D(A)$ is dense and the $T_t^{(n)}$ are equicontinuous for all $x \in \mathbf{F}$. The limiting family of operators T_t are equicontinuous and form a semi-group because the $T_t^{(n)}$ have this property.

We still need to prove that the infinitesimal generator of this semi-group is A. Let us temporarily denote the infinitesimal generator of T_t by B. So we want to prove that $A = B$.

Let $x \in D(A)$. We know that

$$p(T_t^{(n)}x) \leq Kq(x). \quad (3).$$

We claim that

$$\lim_{n\to\infty} T_t^{(n)} A J_n x = T_t A x \tag{4}$$

uniformly in in any compact interval of t. Indeed, for any semi-norm p we have

$$\begin{aligned} p(T_t Ax - T_t^{(n)} AJ_n x) &\leq p(T_t Ax - T_t^{(n)} Ax) + p(T_t^{(n)} Ax - T_t^{(n)} AJ_n x) \\ &\leq p((T_t - T_t^{(n)})Ax) + Kq(Ax - J_n Ax) \end{aligned}$$

where we have used (3) to get from the second line to the third. The second term on the right tends to zero as $n \to \infty$ and we have already proved that the first term converges to zero uniformly on every compact interval of t. This establishes (4)

Now

$$\begin{aligned} T_t x - x &= \lim_{n\to\infty} (T_t^{(n)} x - x) \\ &= \lim_{n\to\infty} \int_0^t T_s^{(n)} A J_n x ds \\ &= \int_0^t (\lim_{n\to\infty} T_s^{(n)} A J_n x) ds \\ &= \int_0^t T_s A x ds \end{aligned}$$

where the passage of the limit under the integral sign is justified by the uniform convergence in t on compact sets. It follows from $T_t x - x = \int_0^t T_s A x ds$ that x is in the domain of the infinitesimal operator B of T_t and that $Bx = Ax$. So B is an extension of A in the sense that $D(B) \supset D(A)$ and $Bx = Ax$ on $D(A)$.

B is an extension of A in the sense that $D(B) \supset D(A)$ and $Bx = Ax$ on $D(A)$.

But since B is the infinitesimal generator of an equibounded semi-group, we know that $(I - B)$ maps $D(B)$ onto $\mathbf{F}$ bijectively, and we are assuming that $(I - A)$ maps $D(A)$ onto $\mathbf{F}$ bijectively. Hence $D(A) = D(B)$.

This concludes the proof of the Hille-Yosida theorem.

The case of a Banach space.

In case **F** is a Banach space, so there is a single norm $p = \| \ \|$, the hypotheses of the theorem read: $D(A)$ is dense in **F**, the resolvents $R(n, A)$ exist for all integers $n = 1, 2, \ldots$ and there is a constant K independent of n and m such that

$$\|(I - n^{-1}A)^{-m}\| \leq K \quad \forall \ n = 1, 2, \ldots, \ m = 1, 2, \ldots. \tag{5}$$

Contraction semigroups.

In particular, if A satisfies

$$\|(I - n^{-1}A)^{-1}\| \leq 1 \tag{6}$$

condition (5) is satisfied, and such an A then generates a semi-group. Under this stronger hypothesis we can draw a stronger conclusion: In (3) we now have $p = q = \|\cdot\|$ and $K = 1$. Since $\lim_{n\to\infty} T_t^n x = T_t x$ we see that under the hypothesis (6) we can conclude that

$$\|T_t\| \leq 1 \quad \forall\ t \geq 0.$$

A semi-group T_t satisfying this condition is called a **contraction semi-group**.

The other half of Stone's theorem.

We have already given a direct proof that if S is a self-adjoint operator on a Hilbert space then the resolvent exists for all non-real z and satisfies

$$\|R(z,S)\| \leq \frac{1}{|\mathrm{Im}\ (z)|}.$$

This implies (6) for $A = iS$ and $-iS$ giving us a proof of the existence of $U(t) = \exp(iSt)$ for any self-adjoint operator S, a proof which is independent of the spectral theorem.

We have

$$\frac{d}{dt}(U(t)U(t)^*) = U(t)(A + A^*)U(t)^* = 0$$

and $U(0) = I$, so

$$U(t)U(t)^* \equiv I.$$

Thus the operators $U(t)$ are unitary for all t

A similar argument show that $U(t)U(-t) \equiv I$. We conclude that the $U(t)$ form a one parameter group of unitary operators.

The functional calculus for functions in $\mathcal{S}$.

Recall that the Fourier inversion formula for functions f whose Fourier transform $\hat{f}$ belongs to L_1 (say for $f \in \mathcal{S}$, for example) says that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t) e^{itx} dt.$$

If we replace x by H and write $U(t)$ instead of e^{itH} this suggests that we define

$$f(H) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t) U(t) dt. \tag{7}$$

We want to check that this assignment $f \mapsto f(H)$ has the properties that we would expect from a functional calculus.

Checking that $(fg)(H) = f(H)g(H)$.

To check this we use fact that the Fourier transform takes multiplication into convolution, i.e. that $(fg)\hat{} = \hat{f} \star \hat{g}$ so

$$(fg)(H) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(t-s)\hat{g}(s)U(t)dsdt$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(r)\hat{g}(s)U(r+s)drds$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(r)\hat{g}(s)U(r)U(s)drds$$

$$= f(H)g(H).$$

Checking that the map $f \mapsto f(H)$ sends $\overline{f} \mapsto (f(H))^*$.

For the standard Fourier transform we know that the Fourier transform of $\overline{f}$ is given by

$$\hat{\overline{f}}(\xi) = \overline{\hat{f}(-\xi)}.$$

Substituting this into the right hand side of (7) gives

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \overline{\hat{f}(-t)}U(t)dt = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \overline{\hat{f}(-t)}U^*(-t)dt$$

$$= \left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(-t)U(-t)dt\right)^*$$

$$= (f(H))^*$$

by making the change of variables $s = -t$.

Checking that $\|f(H)\| \leq \|f\|_\infty$.

Let $\|f\|_\infty$ denote the sup norm of f, and let $c > \|f\|_\infty$. Define g by

$$g(s) := c - \sqrt{c^2 - |f(s)|^2}.$$

So g is a real element of $\mathcal{S}$ and

$$\begin{aligned} g^2 &= c^2 - 2c\sqrt{c^2 - |f|^2} + c^2 - |f|^2 \\ &= 2cg - f\overline{f} \end{aligned}$$

so

$$f\overline{f} - 2cg + g^2 = 0.$$

So by our previous results,

$$f(H)^*f(H) - cg(H) - cg(H)^* + g(H)^*g(H) = 0.$$

$$f(H)^* f(H) - cg(H) - cg(H)^* + g(H)^* g(H) = 0$$

i.e.

$$f(H)^* f(H) + (c - g(H))^*(c - g(H)) = c^2.$$

So for any $v \in \mathfrak{H}$ we have

$$\|f(H)v\|^2 \leq \|f(H)v\|^2 + \|(c - g(H))v\|^2 = c^2\|v\|^2$$

proving that

$$\|f(H)\| \leq \|f\|_\infty. \tag{8}$$

Enlarging the functional calculus to continuous functions vanishing at infinity.

The inequality

$$\|f(H)\| \leq \|f\|_\infty \tag{8}$$

allows us to extend the functional calculus to all continuous functions vanishing at infinity. Indeed if $\hat{f}$ is an element of L_1 so that its inverse Fourier transform f is continuous and vanishes at infinity (by Riemann-Lebesgue) we can approximate f in the $\|\cdot\|_\infty$ norm by elements of $\mathcal{S}$ and so the formula (7) applies to f.

We will denote the space of continuous functions vanishing at infinity by $C_0(\mathbb{R})$.

Checking that (7) is non-trivial and unique.

I claim that we know from the preceding lectures that for z not real the function r_z given by

$$r_z(x) = \frac{1}{z-x}$$

has the property that

$$r_z(H) = R(z,H) = (zI - H)^{-1}$$

is given by an integral of the type (7). Indeed, suppose, for example, that $z = a - ib$, $b > 0$ so that $w = iz$ has positive real part. We know that for $A = iH$,

$$R(w,A) = \int_0^\infty e^{-wt} U(t) dt.$$

Now

$$R(z,H) = iR(w,iH) \;\text{ and }\; \frac{1}{z-x} = \frac{i}{iz-ix}$$

so

$$R(z,H) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} i\sqrt{2\pi} e^{-wt} \mathbf{1}_{[0,\infty)} U(t) dt = r_z(H)$$

where $\mathbf{1}_{[0,\infty)}$ is the indicator function of $[0,\infty)$, i.e. $\mathbf{1}_{[0,\infty)}(x) = 0$ for $x < 0$ and $= 1$ for $x \geq 0$. A similar argument works for z with positive imaginary part.

The proof of our formula for the resolvent of the infinitesimal generator of an equibounded semigroup involved some heavy lifting but not the spectral theorem. This shows that (7), is not trivial. Once we know that $r_z(H) = R(z,H)$ the Stone-Weierstrass theorem gives uniqueness.

There is a formula due to Dynkin, Hellfer and Sjöstrand for $f(H)$ when f is a C^∞ function of compact support which is very useful for applications. It depends on the concept of an almost holomorphic extension which I will now explain.

A variant of the Cauchy integral formula.

I begin with a variant of the Cauchy integral formula for a C^∞ function g of compact support on $\mathbb{C}$:

$$\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial g}{\partial \overline{z}} \cdot \frac{1}{z-w} dxdy = -g(w). \tag{9}$$

As with the usual Cauchy formula, (9) is a consequence of Stokes' theorem. Before proving it, let me explain (remind you of?) some notation:

In the (x, y)-plane we define the ($\mathbb{C}$-valued) linear differential forms $dz := dx + idy, \quad d\overline{z} := dx - idy$ so that

$$d\overline{z} \wedge dz = 2idx \wedge dy$$

and the left hand side of (9) can be written as

$$\frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\partial g}{\partial \overline{z}}\cdot\frac{1}{z-w}d\overline{z}\wedge dz.$$

In this integral $\frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$. I will also use the notation $\overline{\partial}$ for $\frac{\partial}{\partial \overline{z}}$.

Any holomorphic function h of z (in particular any rational function) satisfies $\frac{\partial h}{\partial \overline{z}} \equiv 0$. We define $\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$. Then you can check that for any function F we have

$$dF = \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial \overline{z}}d\overline{z}.$$

Since $\frac{\partial}{\partial \overline{z}}\left(\frac{1}{z-w}\right) = 0$ the the left hand side of (9) is

$$\frac{1}{2\pi i}\int_{\mathbb{C}} d\left(\frac{g}{z-w}dz\right).$$

We can now prove (9):

Proof.

The integral on the left is the limit of the integral over $\mathbb{C}\setminus D_\delta$ where D_δ is a disk of radius δ centered at w. Since g has compact support, we can apply Stokes' theorem to the the integral over $\mathbb{C}\setminus D_\delta$ to get

$$-\frac{1}{2\pi i}\int_{\partial D_\delta}\frac{g(z)}{z-w}dz = -\frac{1}{2\pi}\int_0^{2\pi}\frac{g(w+\delta e^{i\theta})}{\delta}\delta d\theta \to -g(w).$$

$\square$

Almost holomorphic extensions.

Given $f \in C_0^\infty(\mathbb{R})$, an **almost holomorphic extension** of f is a function $\tilde{f} \in C_0^\infty(\mathbb{C})$ such that $\tilde{f}(x + i0) = f(x)$ with the property that

$$\left| \frac{\partial \tilde{f}}{\partial \overline{z}}(x + iy) \right| \leq C_N |y|^N$$

for all $N \in \mathbb{N}$. It is easy to show that almost holomorphic extensions exist. In order not to interrupt the flow of ideas, I will postpone the construction of almost holomorphic extensions to the end of this lecture.

If w is a real number we can apply our Cauchy style formula to $\tilde{f}$ to conclude that

$$f(w) = -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\tilde{f}\frac{1}{z-w}dxdy.$$

Since $\overline{\partial}\tilde{f}$ vanishes to infinite order in y along the real axis, there is no trouble in the convergence of this integral. Indeed, we can differentiate under the integral sign with respect to w and get

$$f^{(n)}(w) = -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\tilde{f}\frac{n!}{(z-w)^{n+1}}dxdy.$$

From our functional calculus we know that the operator corresponding to $1/(z-w)$ is $R(z,H)$. So we conclude that

$$f(H) = -\frac{1}{\pi}\int \overline{\partial}\tilde{f} R(z,H)dxdy. \tag{10}$$

This is the **Dynkin-Hellfer-Sjöstrand formula.**

We also have the formula

$$\frac{1}{n!}f^{(n)}(H) = -\frac{1}{\pi}\int \overline{\partial}\tilde{f} R(z,H)^{n+1}dxdy. \tag{11}$$

As a consequence of the Dynkin-Hellfer-Sjöstrand formula we have the following result:

Proposition

If $\mathrm{supp}(f) \cap \mathrm{Spec}(H) = \emptyset$ *then* $f(H) = 0$.

Proof.

It is enough to prove this result for $f \in C_0^\infty(\mathbb{R})$. In particular, we may assume that $R(z, H)$ exists (and is holomorphic as a function of z) on $\mathrm{supp}(\tilde{f})$. So we may apply Stokes' theorem to conclude that

$$f(H) = -\frac{1}{\pi} \int_\Gamma \tilde{f} R(z, H) dz = 0$$

if we choose Γ so that $\tilde{f}$ vanishes on Γ. $\square$

To summarize:

Let $C_0(\mathbb{R})$ denote the space of continuous functions on $\mathbb{R}$ which vanish at $\pm\infty$ and with the sup norm $\|\cdot\|_\infty$.

Theorem

If H is a self-adjoint operator on a Hilbert space $\mathfrak{H}$ then there exists a unique linear map

$$f \mapsto f(H)$$

from $C_0(\mathbb{R})$ to bounded operators on $\mathfrak{H}$ such that:

1. The map $f \mapsto f(H)$ is an algebra homomorphism,
2. $\overline{f}(H) = f(H)^*$,
3. $\|f(H)\| \leq \|f\|_\infty$,
4. If w is a complex number with non-zero imaginary part and $r_w(x) = (w - x)^{-1}$ then
$$r_w(H) = R(w, H)$$
5. If the support of f is disjoint from the spectrum of H then $f(H) = 0$.

In order to get the full spectral theorem we will have to extend this functional calculus from $C_0(\mathbb{R})$ to a larger class of functions, for example to the class of bounded continuous functions or even to the class of bounded Borel measurable functions once we learn what these are.

For example, for each real number t we might want to consider the function $x \mapsto e^{itx}$ and so use our functional calculus to construct e^{itH}. We have already constructed this one parameter group of unitary transformations directly.

I will postpone the discussion of this extension until after we have studied measure theory, and a version of the Riesz representation theorem will come to our rescue.

But a key step will be the following elementary monotonicity fact:

Proposition

If $f \in C_0(\mathbb{R})$ is a non-negative real valued function, then $f(H) \geq 0$.

Since f is real valued, so that $f = \overline{f}$ we know that $f(H)$ is self-adjoint. The assertion $f(H) \geq 0$ means that for all $v \in \mathfrak{H}, \quad (f(H)v, v) \geq 0$.

Proof.

Since $f \geq 0$, we can write $f = \overline{g}g$ with $g \in C_0(\mathbb{R})$. Then

$$(f(H)v, v) = (g(H)^* g(H)v, v) = (g(H)v, g(H)v) \geq 0.$$

$\square$

We now turn to the existence of almost holomorphic extensions, an idea due to Hörmander. We follow the discussion in Dimassi-Sjöstrand. The key idea (as is frequent in this course) will be integration by parts:

Before proceeding to the full proof of the existence of almost holomorphic extensions, let me prove a baby version which is enough to get the Dynkin-Hellfer-Sjöstrand formula. Recall in the proof of this formula we use the fact that $R(z, H)$ has a singularity of order $\mathrm{Im} z^{-1}$ as we approach the x-axis. So if $\tilde{f}$ is an extension of f such that $\overline{\partial}\tilde{f}$ vanishes to third order (or higher) there is no trouble with the integral in the Dynkin-Hellfer-Sjöstrand formula.

A baby version of the Dynkin-Hellfer-Sjöstrand formula.

Let ϕ be a smooth function of support in [-2, 2] such that $\phi \equiv 1$ on [-1,1] and let $\sigma(x, y) = \phi(y/(1 + x^2))$. Thus σ_x and σ_y vanish identically in some neighborhood of the x-axis. Let

$$\tilde{f}(z) := \left(f(x) + f'(x)(iy) + \frac{1}{2} f''(x)(iy)^2 + \frac{1}{6} f'''(x)(iy)^3 \right) \sigma(x, y).$$

Clearly the restriction of $\tilde{f}$ to the x-axis is f. In computing $\overline{\partial}\tilde{f}$ the terms involving the expression in parentheses will be $1/2\times$

$$f'(x) + f''(x)(iy) + \frac{1}{2} f'''(x)(iy)^2 + \frac{1}{6} f^{iv}(x)(iy)^3$$

$$-f'(x) - f''(x)(iy) - \frac{1}{2} f'''(x)(iy)^2.$$

Since $\overline{\partial}\sigma = 0$ in a neighborhood of the x-axis we see that near the x-axis we have

$$\overline{\partial}\tilde{f} = \frac{1}{2} f^{iv}(x) \frac{(iy)^3}{3!}$$

giving us our desired result.

The full almost holomorphic extension theorem

Let $f \in C_0^\infty(\mathbb{R})$, $\quad \psi \in C_0^\infty(\mathbb{R})$, with $\psi \equiv 1$ on $\mathrm{Supp}(f)$, and $\chi \in C_0^\infty(\mathbb{R})$ with $\chi \equiv 1$ near 0. Define

$$\tilde{f}(x+iy) := \frac{\psi(x)}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(x+iy)\xi} \chi(y\xi) \hat{f}(\xi) d\xi,$$

where $\hat{f}$ is the Fourier transform of f. By the Fourier inversion formula

$$\tilde{f}_{|\mathbb{R}} = f. \tag{12}$$

$$\tilde{f}(x+iy) := \frac{\psi(x)}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(x+iy)\xi} \chi(y\xi)\hat{f}(\xi)d\xi, \quad \tilde{f}_{|\mathbb{R}} = f.$$

With $\overline{\partial} := \frac{1}{2}(\partial_x + i\partial_y)$ we have

$$\overline{\partial}\tilde{f} = \frac{i}{2}\frac{\psi(x)}{\sqrt{2\pi}} \int e^{i(x+iy)\xi)}(-\chi(y\xi) + \chi'(y\xi))\xi\hat{f}(\xi)d\xi$$

$$+\frac{1}{2}\frac{\psi(x)}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(x+iy)\xi} i\xi\chi(y\xi)\hat{f}(\xi)d\xi + \frac{1}{2}\frac{\psi'(x)}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(x+iy)\xi}\chi(y\xi)\hat{f}(\xi)d\xi$$

$$= \frac{i}{2}\frac{\psi(x)}{\sqrt{2\pi}} \int e^{i(x+iy)\xi)}\chi'(y\xi)\xi\hat{f}(\xi)d\xi + \frac{1}{2}\frac{\psi'(x)}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(x+iy)\xi}\chi(y\xi)\hat{f}(\xi)d\xi.$$

$$\overline{\partial}\tilde{f} = \frac{i}{2}\frac{\psi(x)}{\sqrt{2\pi}}\int e^{i(x+iy)\xi)}\chi'(y\xi)\xi\hat{f}(\xi)d\xi + \frac{1}{2}\frac{\psi'(x)}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i(x+iy)\xi}\chi(y\xi)\hat{f}(\xi)d\xi$$

Define

$$\chi_N(t) := t^{-N}\chi'(t).$$

We can insert and extract a factor of y^N in the first integral above and write this first integral as

$$y^N\frac{i}{2}\frac{\psi(x)}{\sqrt{2\pi}}\int e^{i(x+iy)\xi)}\chi_N(y\xi)\xi^{N+1}\hat{f}(\xi)d\xi$$

and so get a bound on this first integral of the form

$$C_N|y|^N\|\xi^{N+1}\hat{f}(\xi)\|_{L^1}.$$

$$\overline{\partial}\tilde{f} = \frac{i}{2}\frac{\psi(x)}{\sqrt{2\pi}}\int e^{i(x+iy)\xi)}\chi'(y\xi)\xi\hat{f}(\xi)d\xi + \frac{1}{2}\frac{\psi'(x)}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i(x+iy)\xi}\chi(y\xi)\hat{f}(\xi)d\xi$$

For the second integral we put in the expression of $\hat{f}$ as the Fourier transform of f to get

$$\frac{1}{2}\frac{\psi'(x)}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-r+iy)\xi}\chi(y\xi)f(r)drd\xi.$$

Now $\psi' = 0$ on $\mathrm{Supp}(f)$ so $x - r \neq 0$ on $\mathrm{Supp}(\psi'(x)f(r))$ so this becomes

$$\frac{1}{4\pi}\psi'(x)\int\int D_\xi\left(e^{i(x-r+iy)\xi}\right)\frac{\chi(y\xi)}{x-r+iy}f(r)drd\xi. \qquad (*)$$

Here I am using the notation $D_s := \frac{1}{i}\frac{\partial}{\partial s}$ for any variable s.

Integration by parts turns $(*)$ into

$$\frac{1}{4\pi}\psi'(x)\int\int e^{i(x-r+iy)\xi}\frac{\chi'(y\xi)y}{x-r+iy}f(r)drd\xi.$$

Since $\chi'(y\xi)=y^N\xi^N\chi_N(y\xi)$ the double integral becomes

$$y^N\int\int e^{i(x-r+iy)\xi}\xi^N\frac{\chi_N(y\xi)y}{(x-r+iy)}f(r)drd\xi.$$

Inserting and extracting a factor of $(\xi+i)^2$ this becomes

$$=y^N\int\int(i-D_r)^2(-D_r)^N\left(e^{i(x-r+iy)\xi}\right)\frac{\chi_N(y\xi)y\xi^N}{(x-r+iy)}f(r)\frac{1}{(\xi+i)^2}drd\xi.$$

Integration by parts again brings the derivatives over to the the term $\frac{f(r)}{x-r+iy}$ and shows that the second integral is also $O(|y|^N)$. So we have proved that

$$|\overline{\partial}\tilde{f}(z)| \leq C_N |\mathrm{Im} z|^N. \tag{13}$$

Thus for any $f \in C_0^\infty(\mathbb{R})$ we have produced an "almost holomorphic" extension $\tilde{f}$ of f. $\square$

The key ideas and logic of the last four lectures

- Facts about the Fourier transform. In particular the Fourier inversion formula.
- The infinitesimal generator of a(n equicontinuous equibounded) semigroup.
- The resolvent of the infinitesimal generator is the Laplace transform of the semigroup.
- The Hille-Yosida theorem.
- Stone's theorem about one parameter groups of unitary transformations on a Hilbert space, a special case of the Hille-Yosida theorem.
- The spectrum of a self-adjoint operator is real.
- The spectral theorem, functional calculus form, for continuous functions vanishing at infinity, via Stone's theorem and the Fourier inversion formula.