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Semigroups, II.
Self-adjoint operators.

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In this lecture we discuss one parameter semi-groups T_t whose "infinitesimal generator" A is not necessarily sectorial. We want to understand the meaning of the equation

$$T_t = e^{tA}$$

for such operators (where we impose certain conditions on the semi-group T_t or on the operator A).

The reason that sectorial operators are not enough is quantum mechanics. In quantum mechanics the fundamental object of study is a unitary one parameter group $U(t)$ and Stone's theorem asserts that all such $U(t)$ are of the form $U(t) = e^{tA}$ where A is skew adjoint, i.e. $A = iH$ where H is a self-adjoint operator. Now the spectrum of a self-adjoint operator lies on the real axis (as we shall prove in a moment), so the spectrum of a skew adjoint operator lies on the imaginary axis. If A is unbounded (as is usually the case in quantum mechanics) then it will not be sectorial.

1. Recall about unbounded operators and their spectra.

2. The spectrum of a self-adjoint operator is real.

3. Equibounded continuous semi-groups.

Recall from last time that if $D(H)$ is a dense subspace of a Banach space $\mathbf{B}$ and $H : D(H) \to \mathbf{C}$ (another Banach space) then it has a well defined adjoint H^* whose graph $\Gamma(H)^*$ is given by

$$\{\ell, m\} \in \Gamma(H)^* \quad \Leftrightarrow \quad \langle \ell, Hx \rangle = \langle m, x \rangle \quad \forall\, x \in D(H).$$

Furthermore, H^* is always closed.

We now (temporarily) restrict to the case that $\mathbf{B} = \mathbf{C} = \mathfrak{H}$ is a Hilbert space. We may identify $\mathbf{B}^* = \mathbf{C}^*$ with $\mathfrak{H}$ via the Riesz representation theorem which says that the most general continuous linear function on $\mathfrak{H}$ is given by scalar product with an element of $\mathfrak{H}$.

The adjoint of an operator on a Hilbert space.

If $H : D(H) \to \mathfrak{H}$ is an operator with $D(H)$ dense in $\mathfrak{H}$ we may identify the graph of H^* as consisting of all $\{g, h\} \in \mathfrak{H} \oplus \mathfrak{H}$ such that

$$(Hx, g) = (x, h) \quad \forall x \in D(H)$$

and then write

$$(Hx, g) = (x, H^*g) \quad \forall\ x \in D(H), \quad g \in D(H^*).$$

Notice that we can describe $\Gamma(H^*)$, the graph of H^*, as being the orthogonal complement in $\mathfrak{H} \oplus \mathfrak{H}$ of the subspace

$$M := \{Hx, -x\} \quad x \in D(H)\}.$$

Indeed, the condition that $\{g, h\}$ be orthogonal to M is

$$(Hx, g) - (x, h) = 0,$$

i.e. $(Hx, g) = (x, h)$ for all x in $D(H)$ which is precisely the condition for $\{g, h\}$ to belong to the graph of H^*.

The domain of the adjoint.

The domain $\mathcal{D}$ of H^* consists of those g such that there is an h with $(Hx, g) = (x, h)$ for all x in the domain of H. We claim that $\mathcal{D}$ is dense in $\mathfrak{H}$. Suppose not. Then there would be some $z \in \mathfrak{H}$ with $(z, g) = 0$ for all $g \in \mathcal{D}$. Thus $\{z, 0\} \perp M^\perp = \Gamma(H^*)$. But $(M^\perp)^\perp$ is the closure $\overline{M}$ of M. This means that there is a sequence $x_n \in D(H)$ such that $Hx_n \to z$ and $x_n \to 0$. So if we assume that H is closed, we conclude that $z = 0$. In short, if H is a closed densely defined operator so is H^*.

The definition of a self-adjoint operator on a Hilbert space.

We now come to a crucial definition: An operator H defined on a domain $D(H) \subset \mathfrak{H}$ is called **self-adjoint** if

- $D(H)$ is dense in $\mathfrak{H}$,
- $D(H) = D(H^*)$, and
- $Hx = H^*x \; \forall x \in D(H)$.

The conditions about the domain $D(H)$ are rather subtle. For the moment we record one immediate consequence of the theorem asserting that the adjoint of a densely defined operator is closed:

Proposition

Any self adjoint operator is closed.

Symmetric operators.

A densely defined operator S on a Hilbert space is called **symmetric** if

- $D(S) \subset D(S^*)$ and
- $Sx = S^*x \ \ \forall\, x \in D(S)$.

Another way of saying the same thing is: S is symmetric if $D(S)$ is dense and

$$(Sx, y) = (x, Sy) \quad \forall \ \ x, y \in D(S).$$

Every self-adjoint operator is symmetric but not every symmetric operator is self adjoint. This subtle difference will only become clear as we go along.

A sufficient condition for a symmetric operator to be self-adjoint.

Let A be a symmetric operator on a Hilbert space $\mathfrak{H}$. The following theorem will be very useful:

Theorem

If there is a complex number z such that $A + zI$ and $A + \overline{z}I$ both map $D(A)$ surjectively onto $\mathfrak{H}$ then A is self-adjoint.

Proof.

We must show that if ψ and f are such that

$$(f,\phi) = (\psi, A\phi) \quad \forall\ \phi \in D(A)$$

$$\text{then} \qquad \psi \in D(A) \ \text{ and } \ A\psi = f.$$

Choose $w \in D(A)$ such that $(A + \bar{z}I)w = f + \bar{z}\psi$. Then for any $\phi \in D(A)$

$$(\psi, (A + zI)\phi) = (f + \bar{z}\psi, \phi) = (Aw + \bar{z}w, \phi) = (w, A\phi + z\phi).$$

Then choose $\phi \in D(A)$ such that $(A + zI)\phi = \psi - w$. So $(\psi, \psi - w) = (w, \psi - w)$ and hence $\|\psi - w\|^2 = 0$, i.e $\psi = w$, so

$$\psi \in D(A) \ \text{ and } \ A\psi = f.$$

□

Multiplication operators.

Here is an important application of the theorem we just proved: Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\mathfrak{H} := L_2(X, \mu)$. Since we have not yet done measure theory, take $\mathfrak{H} := L_2(\mathbb{R}^n)$.
Let a be a real valued $\mathcal{F}$ measurable function (say continuous) on X (on $\mathbb{R}^n$) with the property that a is bounded on any measurable subset of X of finite measure (on bounded subsets of $\mathbb{R}^n$). Let

$$\mathcal{D} := \{u \in \mathfrak{H} | \int_X (1 + a^2)|u|^2 d\mu < \infty.$$

Notice that $\mathcal{D}$ is dense in $\mathfrak{H}$. Let S be the linear operator

$$u \mapsto au$$

defined on the domain $\mathcal{D}$. Notice that S is symmetric.

Proposition

The operator S with domain $\mathcal{D}$ is self-adjoint.

Proof.

The operator consisting of multiplication by

$$\frac{1}{i+a}$$

is bounded since $\left|\frac{1}{i+a}\right| \leq 1$ and clearly maps $\mathfrak{H}$ to $\mathcal{D}$. Its inverse is multiplication by $i+a$. Similarly multiplication by $-i+a$ maps $\mathcal{D}$ onto $\mathfrak{H}$. So we may take $z=i$ in the previous Theorem.

□

Using the Fourier transform.

The Fourier transform is a unitary operator on $L_2(\mathbb{R}^n)$ (Plancherel's theorem), and carries constant coefficient partial differential operators into multiplication by a polynomial. A consequence of a theorem about multiplication operators that we just proved is:

Proposition

If D is a constant coefficient differential operator which is carried by the Fourier transform into a real *polynomial, then D is self-adjoint.*

An example is the Laplacian, which goes over into multiplication by $-\|k\|^2$ under the Fourier transform. The domain of the Laplacian consists of those $f \in L_2$ whose Fourier transform $\hat{f}$ have the property that $\|k\|^2 \hat{f}(k) \in L_2$.

An even simpler example is the operator $\frac{1}{i}\frac{d}{dx}$: We know that the Fourier transform of f' is $i\xi\hat{f}(\xi)$ (for $f \in \mathcal{S}$.) So we know that the operator $H := \frac{1}{i}\frac{d}{dx}$ is self adjoint with domain consisting of all $f \in L_2(\mathbb{R})$ such that the function $\xi \mapsto \xi\hat{f}(\xi)$ belongs to L_2. (We will find that the spectrum of H is the entire real line.)

In quantum mechanics (and elsewhere) one is interested in the "one parameter group" $U(t) = e^{-iHt}$. For the H as above, this sends $\hat{f}(\xi)$ to $e^{-i\xi t}\hat{f}(\xi)$ which is the Fourier transform of the function $x \mapsto f(x-t)$. Now the operator $U(t)$ sending $f(x)$ into $f(x-t)$ is well defined on all of L_2 and is unitary. It acts on f by "shifting it" t units to the right. So we can hope that for any self-adjoint operator we can construct $U(t) = e^{-iHt}$ even though $-iH$ is not sectorial, and that the operators $U(t)$ are unitary. This assertion is one half of Stone's theorem. I plan to prove this in the next lecture.

But then the operator $\frac{1}{i}\frac{d}{dx}$ on $L_2([0,1])$ (with a similar domain) can *not* be self adjoint for the following reason: If it were, it would generate generate a one parameter group, call it $V = V(t)$. Suppose that f is a (say continuous) function supported on the subinterval $[a,b]$. f can not tell if it is to be thought of as an element of $L_2([0,1])$ or of $L_2(\mathbb{R})$. So for small values of $|t|$, we must have $(V(t)f(x)) = f(x-t)$. What happens when the support of $f(x-t)$ crosses the point 1? It can not simply disappear for then $V(t)$ would not be unitary. So it must reappear at 0. But should it reappear as itself or with a minus sign or by being multiplied by $e^{i\theta}$ for some θ? Any such choice would be ok, but this is not determined by H alone. Some "boundary constraints" must be specified.

I hope that this convinces you that the condition of being self-adjoint is rather subtle.

The following theorem will be central for us. Once we will have stated and proved the spectral theorem, the following theorem will be an immediate consequence. But we will proceed in the opposite direction, first proving the theorem and then using it to prove the spectral theorem:

Theorem

Let H be a self-adjoint operator on a Hilbert space $\mathfrak{H}$ with domain $D = D(H)$. Let

$$c = \lambda + i\mu, \quad \mu \neq 0$$

be a complex number with non-zero imaginary part. Then

$$(cI - H) : D(H) \to \mathfrak{H}$$

is bijective. Furthermore the inverse transformation

$$(cI - H)^{-1} : \mathfrak{H} \to D(H)$$

is bounded and in fact

$$\|(cI - H)^{-1}\| \leq \frac{1}{|\mu|}. \tag{1}$$

We will prove this theorem in stages: Let $g \in D(H)$ and set $f := (cI - H)g = [\lambda I - H]g + i\mu g$.
We begin by showing that

$$\|f\|^2 = \|(\lambda I - H)g\|^2 + \mu^2\|g\|^2.$$

We have $\|f\|^2 = (f, f) =$

$$\|[\lambda I - H]g\|^2 + \mu^2\|g\|^2 + ([\lambda I - H]g, i\mu g) + (i\mu g, [\lambda I - H]g).$$

The last two terms cancel: Indeed, since $g \in D(H)$ and H is self adjoint we have

$$(\mu g, [\lambda I - H]g) = (\mu[\lambda I - H]g, g) = ([\lambda I - H]g, \mu g)$$

since μ is real. Hence

$$([\lambda I - H]g, i\mu g) = -i(\mu g, [\lambda I - H]g).$$

We have thus proved that

$$\|f\|^2 = \|(\lambda I - H)g\|^2 + \mu^2\|g\|^2. \tag{2}$$

We next show that $\|(cI - H)^{-1}\| \leq \frac{1}{|\mu|}$. Indeed, it follows from (2) that

$$\|f\|^2 \geq \mu^2\|g\|^2$$

for all $g \in D(H)$. Since $|\mu| > 0$, we see that $f = 0 \Rightarrow g = 0$ so $(cI - H)$ is injective on $D(H)$, and furthermore that $(cI - H)^{-1}$ (which is defined on im $(cI - H)$) satisfies

$$\|(cI - H)^{-1}\| \leq \frac{1}{|\mu|}.$$

We must show that the image of $(cI - H)$ is all of $\mathfrak{H}$.

We show the image of $(cI - H)$ is dense in $\mathfrak{H}$.

For this it is enough to show that there is no $h \neq 0 \in \mathfrak{H}$ which is orthogonal to im $(cI - H)$. So suppose that

$$([cI - H]g, h) = 0 \quad \forall g \in D(H).$$

Then

$$(g, \overline{c}h) = (cg, h) = (Hg, h) \ \ \forall g \in D(H)$$

which says that $h \in D(H^*)$ and $H^* h = \overline{c}h$. But H is self adjoint so $h \in D(H)$ and $Hh = \overline{c}h$. (Here is where we use the fact that H is self-adjoint.)Thus

$$\overline{c}(h, h) = (\overline{c}h, h) = (Hh, h) = (h, Hh) = (h, \overline{c}h) = c(h, h).$$

Since $c \neq \overline{c}$ this is impossible unless $h = 0$.

We show that image of $(cI - H)$ is all of $\mathfrak{H}$, completing the proof of the theorem.

Let $f \in \mathfrak{H}$. We know that we can find

$$f_n = (cI - H)g_n, \quad g_n \in D(H) \quad \text{with } f_n \to f.$$

The sequence f_n is convergent, hence Cauchy, and from

$$\|(cI - H)^{-1}\| \leq \frac{1}{|\mu|} \quad (1)$$

applied to elements of im $D(H)$ we know that

$$\|g_m - g_n\| \leq |\mu|^{-1}\|f_n - f_m\|.$$

Hence the sequence $\{g_n\}$ is Cauchy, so $g_n \to g$ for some $g \in \mathfrak{H}$. But we know that H is a closed operator. Hence $g \in D(H)$ and $(cI - H)g = f$. □

Equibounded continuous semi-groups.

After this excursion into self-adjoint operators, I now want to turn to the study of semi-groups and their infinitesimal operators. If T_t is a family of bounded operators on a Banach space $\mathbf{B}$ defined for all $t \geq 0$ satisfying

$$T_{s+t} = T_t \circ T_s$$

and

$$T_0 = I$$

then we will call T_t a **semi-group**.

Comment.

If T_t satisfies $T_{s+t} = T_t \circ T_s$ and $T_0 = I$ and is strongly continuous at the origin then it follows from the uniform boundedness theorem that T_t is uniformly continuous at all $t \geq 0$ and that there are constants C and M such that

$$\|T_t\| \leq Ce^{Mt}$$

for all $t \geq 0$. For the elementary proofs of these facts see the recent book on semi-groups by Kantorovitz. I will not use these facts but simply take a more restricted definition as we will give below.

But if we start with the more general definition and multiply T_t by e^{-Mt} we obtain a semi-group all of whose operators are uniformly bounded. So without much loss of generality we can restrict ourselves to this case.

For certain applications (especially to partial differential equations) it is useful to work in Frechet spaces (which are a bit more general than Banach spaces) so the next few slides will be devoted to the setting of Frechet spaces. A Frechet space **F** is a vector space with a topology defined by a sequence of semi-norms and which is complete. An important example is the Schwartz space $\mathcal{S}$. Let **F** be such a space.

Equibounded continuous semi-groups.

We want to consider a one parameter family of operators T_t on $\mathbf{F}$ defined for all $t \geq 0$ and which satisfy the following conditions:

- $T_0 = I$
- $T_t \circ T_s = T_{t+s}$
- $\lim_{t \to t_0} T_t x = T_{t_0} x \ \forall t_0 \geq 0$ and $x \in \mathbf{F}$.
- For any defining seminorm p there is a defining seminorm q and a constant K such that $p(T_t x) \leq K q(x)$ for all $t \geq 0$ and all $x \in \mathbf{F}$.

We call such a family an **equibounded continuous semigroup**. We will usually drop the adjective "continuous" and even "equibounded" since we will not be considering any other kind of semigroup.

The treatment here will essentially follow that of Yosida, *Functional Analysis* especially Chapter IX.

The infinitesimal generator.

We are going to begin by showing that every such semigroup has an "infinitesimal generator", i.e. can be written in some sense as $T_t = e^{At}$. We define the operator A as

$$Ax = \lim_{t \searrow 0} \frac{1}{t}(T_t - I)x.$$

That is, A is the operator so defined on the domain $D(A)$ consisting of those x for which the limit exists.

Our first task is to show that $D(A)$ is dense in $\mathbf{F}$.

The resolvent

For this we begin with a "putative resolvent"

$$R(z) := \int_0^\infty e^{-zt} T_t dt \tag{3}$$

which is defined (by the boundedness and continuity properties of T_t) for all z with $\mathrm{Re}\, z > 0$. Our experience with bounded or sectorial operators shows that this should be a good candidate for the resolvent of A.

One of our tasks will be to show that $R(z)$ as defined in (3) is in fact the resolvent of A.

We begin by checking that every element of $\mathrm{im}\, R(z)$ belongs to $D(A)$: We have

$$\frac{1}{h}(T_h - I)R(z)x = \frac{1}{h}\int_0^\infty e^{-zt}T_{t+h}xdt - \frac{1}{h}\int_0^\infty e^{-zt}T_txdt =$$

$$\frac{1}{h}\int_h^\infty e^{-z(r-h)}T_rxdr - \frac{1}{h}\int_0^\infty e^{-zt}T_txdt$$

$$= \frac{e^{zh}-1}{h}\int_h^\infty e^{-zt}T_txdt - \frac{1}{h}\int_0^h e^{-zt}T_txdt$$

$$= \frac{e^{zh}-1}{h}\left[R(z)x - \int_0^h e^{-zt}T_txdt\right] - \frac{1}{h}\int_0^h e^{-zt}T_txdt.$$

$R(z)$ is a right inverse of $zI - A$.

$$\frac{1}{h}(T_h - I)R(z)x = \frac{e^{zh}-1}{h}\left[R(z)x - \int_0^h e^{-zt}T_t x dt\right] - \frac{1}{h}\int_0^h e^{-zt}T_t x dt.$$

If we now let $h \to 0$, the integral inside the bracket tends to zero, and the expression on the right tends to x since $T_0 = I$. We thus see that

$$R(z)x \in D(A)$$

and

$$AR(z) = zR(z) - I,$$

or, rewriting this in a more familiar form,

$$(zI - A)R(z) = I. \tag{4}$$

$$(zI - A)R(z) = I. \qquad (4)$$

This equation says that $R(z)$ is a right inverse for $zI - A$. It will require a lot more work to show that it is also a left inverse.

We show that $D(A)$ is dense in $\mathbf{F}$.

We will prove that $D(A)$ is dense in $\mathbf{F}$ by showing that, taking s to be real, that

$$\lim_{s\to\infty} sR(s)x = x \quad \forall\, x \in \mathbf{F}. \tag{5}$$

Indeed,

$$\int_0^\infty se^{-st}dt = 1$$

for any $s > 0$. So we can write

$$sR(s)x - x = s\int_0^\infty e^{-st}[T_t x - x]dt.$$

Applying any seminorm p we obtain

$$p(sR(s)x - x) \le s\int_0^\infty e^{-st}p(T_t x - x)dt.$$

We know that

$$p(sR(s)x - x) \leq s\int_0^\infty e^{-st}p(T_tx - x)dt.$$

For any $\epsilon > 0$ we can, by the continuity of T_t, find a $\delta > 0$ such that

$$p(T_tx - x) < \epsilon \quad \forall \quad 0 \leq t \leq \delta.$$

Now let us write

$$s\int_0^\infty e^{-st}p(T_tx - x)dt = s\int_0^\delta e^{-st}p(T_tx - x)dt + s\int_\delta^\infty e^{-st}p(T_tx - x)dt$$

The first integral is bounded by

$$\epsilon s\int_0^\delta e^{-st}dt \leq \epsilon s\int_0^\infty e^{-st}dt = \epsilon.$$

$$s\int_0^\infty e^{-st}p(T_tx-x)dt = s\int_0^\delta e^{-st}p(T_tx-x)dt + s\int_\delta^\infty e^{-st}p(T_tx-x)dt$$

The first integral is bounded by ϵ. As to the second integral, let M be a bound for $p(T_tx)+p(x)$ which exists by the uniform boundedness of T_t. The triangle inequality says that $p(T_tx-x)\leq p(T_tx)+p(x)$ so the second integral is bounded by

$$M\int_\delta^\infty se^{-st}dt = Me^{-s\delta}.$$

This tends to 0 as $s\to\infty$, completing the proof that $sR(s)x\to x$ and hence that $D(A)$ is dense in $\mathbf{F}$.

The differential equation.

Theorem

If $x \in D(A)$ then for any $t > 0$

$$\lim_{h \to 0} \frac{1}{h}[T_{t+h} - T_t]x = AT_t x = T_t A x.$$

In colloquial terms, we can formulate the theorem as saying that

$$\frac{d}{dt} T_t = AT_t = T_t A$$

in the sense that the appropriate limits exist when applied to $x \in D(A)$.

Since T_t is continuous in t, we have

$$T_tAx = T_t \lim_{h\searrow 0} \frac{1}{h}[T_h - I]x = \lim_{h\searrow 0} \frac{1}{h}[T_tT_h - T_t]x =$$

$$\lim_{h\searrow 0} \frac{1}{h}[T_{t+h} - T_t]x = \lim_{h\searrow 0} \frac{1}{h}[T_h - I]T_tx$$

for $x \in D(A)$. This shows that $T_tx \in D(A)$ and

$$\lim_{h\searrow 0} \frac{1}{h}[T_{t+h} - T_t]x = AT_tx = T_tAx.$$

To prove the theorem we must show that we can replace $h \searrow 0$ by $h \to 0$. Our strategy is to show that with the information that we already have about the existence of right handed derivatives, we can conclude that

$$T_tx - x = \int_0^t T_sAxds.$$

As $t \mapsto T_t$ is continuous, this is enough to give the desired result.

Using Hahn-Banach.

So we wish to prove that

$$T_t x - x = \int_0^t T_s A x ds.$$

In order to establish the above equality, it is enough, by the Hahn-Banach theorem to prove that for any $\ell \in \mathbf{F}^*$ we have

$$\ell(T_t x) - \ell(x) = \int_0^t \ell(T_s A x) ds.$$

In turn, it is enough to prove this equality for the real and imaginary parts of ℓ.

So it all boils down to the following lemma in the theory of functions of a real variable:

Lemma

Suppose that f *is a continuous real valued function of* t *with the property that the right hand derivative*

$$\frac{d^+}{dt}f := \lim_{h \searrow 0} \frac{f(t+h) - f(t)}{h} = g(t)$$

exists for all t *and* $g(t)$ *is continuous. Then* f *is differentiable with* $f' = g$.

Proof of the lemma, 1.

We first prove that $\frac{d^+}{dt}f \geq 0$ on an interval $[a, b]$ implies that $f(b) \geq f(a)$. Suppose not. Then there exists an $\epsilon > 0$ such that

$$f(b) - f(a) < -\epsilon(b - a).$$

Set

$$F(t) := f(t) - f(a) + \epsilon(t - a).$$

Then $F(a) = 0$ and

$$\frac{d^+}{dt}F > 0.$$

At a this implies that there is some $c > a$ near a with $F(c) > 0$. On the other hand, since $F(b) < 0$, and F is continuous, there will be some point $s < b$ with $F(s) = 0$ and $F(t) < 0$ for $s < t \leq b$. This contradicts the fact that $[\frac{d^+}{dt}F](s) > 0$.

Proof of the lemma, 2.

Thus if $\frac{d^+}{dt}f \geq m$ on an interval $[t_1, t_2]$ we may apply the above result to $f(t) - mt$ to conclude that

$$f(t_2) - f(t_1) \geq m(t_2 - t_1),$$

and if $\frac{d^+}{dt}f(t) \leq M$ we can apply the above result to $Mt - f(t)$ to conclude that $f(t_2) - f(t_1) \leq M(t_2 - t_1)$. So if $m = \min g(t) = \min \frac{d^+}{dt}f$ on the interval $[t_1, t_2]$ and M is the maximum, we have

$$m \leq \frac{f(t_2) - f(t_1)}{t_2 - t_1} \leq M.$$

Since we are assuming that g is continuous, this is enough to prove that f is indeed differentiable with derivative g. $\square$.

We will now conclude that we have indeed constructed the resolvent of A, part 1:

We have already verified that

$$R(z) = \int_0^\infty e^{-zt} T_t dt$$

maps **F** into $D(A)$ and satisfies

$$(zI - A)R(z) = I$$

for all z with Re $z > 0$.
We shall show that for this range of z

$$(zI - A)x = 0 \quad \Rightarrow \quad x = 0 \quad \forall\, x \in D(A),$$

that $(zI - A)^{-1}$ exists, and that it is given by $R(z)$.

Suppose that

$$Ax = zx \quad x \in D(A), \quad x \neq 0$$

and choose $\ell \in \mathbf{F}^*$ with $\ell(x) = 1$. Consider

$$\phi(t) := \ell(T_t x).$$

By the result of the preceding section we know that ϕ is a differentiable function of t and satisfies the differential equation

$$\phi'(t) = \ell(T_t A x) = \ell(T_t z x) = z\ell(T_t x) = z\phi(t), \quad \phi(0) = 1.$$

So

$$\phi(t) = e^{zt}$$

which is impossible since $\phi(t)$ is a bounded function of t and the right hand side of the above equation is not bounded for $t \geq 0$ since the real part of z is positive. Thus

$$(zI - A)x = 0 \ \Rightarrow \ x = 0.$$

We have from (4) that

$$(zI - A)R(z)(zI - A)x = (zI - A)x$$

and we know that $R(z)(zI - A)x \in D(A)$. From the injectivity of $zI - A$ we conclude that $R(z)(zI - A)x = x$.
From $(zI - A)R(z) = I$ we see that $zI - A$ maps $\mathrm{im}\, R(z) \subset D(A)$ onto $\mathbf{F}$ so certainly $zI - A$ maps $D(A)$ onto $\mathbf{F}$ bijectively. Hence

$$\mathrm{im}(R(z)) = D(A), \quad \mathrm{im}(zI - A) = \mathbf{F}$$

and

$$R(z) = (zI - A)^{-1}.$$

Summary of where we are.

The resolvent $R(z) = R(z, A) := \int_0^\infty e^{-zt} T_t dt$ is defined as a strong limit for Re $z > 0$ and, for this range of z:

$$\begin{aligned} D(A) &= \mathrm{im}(R(z, A)) && (6) \\ AR(z, A)x &= \\ R(z, A)Ax &= (zR(z, A) - I)x, \;\; x \in D(A) && (7) \\ AR(z, A)x &= (zR(z, A) - I)x, \;\forall\; x \in \mathbf{F} && (8) \\ \lim_{z \nearrow \infty} zR(z, A)x &= x \text{ for } z \text{ real } \;\forall x \in \mathbf{F}. && (9) \end{aligned}$$

The operator A is closed.

We claim that

Theorem

The operator A is closed.

Suppose that $x_n \in D(A),\ x_n \to x$ and $y_n \to y$ where $y_n = Ax_n$. We must show that $x \in D(A)$ and $Ax = y$.

Proof.

Set

$$z_n := (I - A)x_n \quad \text{so} \quad z_n \to x - y.$$

Since $R(1, A) = (I - A)^{-1}$ is a bounded operator, we conclude that

$$x = \lim x_n = \lim (I - A)^{-1} z_n = (I - A)^{-1}(x - y).$$

From (6) we see that $x \in D(A)$ and from the preceding equation that $(I - A)x = x - y$ so $Ax = y$. □

Application to Stone's theorem.

We now have enough information to prove one half of Stone's theorem, namely that any continuous one parameter group of unitary transformations on a Hilbert space has an infinitesimal generator which is skew adjoint:

Suppose that $U(t)$ is a one-parameter group of unitary transformations on a Hilbert space $\mathfrak{H}$. We have $(U(t)x, y) = (x, U(t)^{-1}y) = (x, U(-t)y)$ and so differentiating at the origin shows that the infinitesimal generator A, which we know to be closed, is skew-symmetric:

$$(Ax, y) = -(x, Ay) \quad \forall\, x, y \in D(A).$$

The resolvents $(zI - A)^{-1}$ exist for all z which are not purely imaginary, and $(zI - A)$ maps $D(A)$ onto all of the Hilbert space $\mathfrak{H}$.

Writing $A = iH$ we see that H is symmetric and that $\pm iI + H$ is surjective. Hence H is self-adjoint. This proves that every one parameter group of unitary transformations has an infinitesimal generator form iH with H self-adjoint.

We now want to turn to the other half of Stone's theorem: We want to start with a self-adjoint operator H, and construct a (unique) one parameter group of unitary operators $U(t)$ whose infinitesimal generator is iH. This fact is an immediate consequence of the spectral theorem. But we want to derive the spectral theorem from Stone's theorem, so we need to provide a proof of this half of Stone's theorem which is independent of the spectral theorem. We will state and prove the Hille-Yosida theorem and find that this other half of Stone's theorem is a special case.

I plan to do this next time.