2121407
Semigroups.

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Reminder:

No class this Thursday.

The semi-group generated by an operator

In today's lecture I want to discuss the semi-group generated by an operator A, that is the semi-group

$$t \mapsto e^{tA}.$$

I will start with the simplest case of a bounded operator, and then generalize.

Over the course of this and the next two lectures I hope to cover the Hille-Yosida theorem on semigroups, derive Stone's theorem on unitary one parameter groups on a Hilbert space (a theorem which lies at the foundations of quantum mechanics) and the spectral theorem for (possibly) unbounded self-adjoint operators on a Hilbert space.

e^{tA} when A is bounded

Suppose that A is a bounded operator on a Banach space. For example, any linear operator on a finite dimensional space. Then the series

$$e^{tA} = \sum_0^\infty \frac{t^k}{k!} A^k$$

converges for any t. (We will concentrate on t real, and eventually on $t \geq 0$ when we get to more general cases.) Convergence is guaranteed as a result of the convergence of the usual exponential series in one variable. (There are serious problems with this definition from the point of view of numerical implementation which we will not discuss here.)

Identities of e^{tA} when A is bounded

The standard proof using the binomial formula shows that

$$e^{(s+t)A} = e^{sA} \cdot e^{tA}.$$

Also, the standard proof for the usual exponential series shows that the operator valued function $t \mapsto e^{tA}$ is differentiable (in the uniform topology) and that

$$\frac{d}{dt}\left(e^{tA}\right) = A \cdot e^{tA} = e^{tA} \cdot A.$$

The resolvent set and the resolvent.

A point $z \in \mathbb{C}$ is said to belong to the **resolvent set** of A if the operator $zI - A$ has a bounded (two sided) inverse. Then this inverse is called the **resolvent** of A and is denoted by $R(z, A)$. So

The series for the resolvent when $|z| > \|A\|$

$$R(z,A) := (zI - A)^{-1}.$$

For example, if $z \neq 0$ we have $zI - A = z(I - z^{-1}A)$. If $|z| > \|A\|$ the geometric series

$$I + z^{-1}A + z^{-2}A^2 + z^{-3}A^3 + \cdots$$

converges to $(I - z^{-1}A)^{-1}$. So all z satisfying $|z| > \|A\|$ belong to the resolvent set of A and for such z we have the convergent series expansion

$$R(z,A) = z^{-1}I + z^{-2}A + z^{-3}A^2 + z^{-4}A^3 + \cdots .$$

The spectrum

Of course, in finite dimensions, $\lambda \in \mathbb{C}$ is by definition an eigenvalue of A if and only if the operator $\lambda I - A$ is not invertible. So in finite dimensions the resolvent set of A consists of all complex numbers which are **not** eigenvalues.

In general, we will *define* the **spectrum** to be the complement of the resolvent set.

The resolvent from the semi-group.

Suppose that $\operatorname{Re} z > \|A\|$ so that z belongs to the resolvent set of A and also the function

$$t \mapsto e^{-zt}e^{tA}$$

is integrable over $(0, \infty)$. We have

$$(zI - A)\int_0^\infty e^{-zt}e^{tA}dt = \int_0^\infty (zI - A)e^{-t(zI-A)}dt$$

$$= -\int_0^\infty \frac{d}{dt}\left(e^{-t(zI-A)}\right)dt = I.$$

So

$$R(z,A) = \int_0^\infty e^{-zt} e^{tA} dt.$$

In words: for $\operatorname{Re} z > \|A\|$ the resolvent is the Laplace transform of the one parameter group.

If we differentiate both sides of the above equation n times with respect to z we find that

$$R(z,A)^{n+1} = \frac{1}{n!} \int_0^\infty t^n e^{-zt} e^{tA} dt.$$

The semigroup from the resolvent.

Let Γ be a circle of radius $> \|A\|$ centered at the origin. The claim is that

$$e^{tA} = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}R(\lambda, A)d\lambda.$$

Proof.

From the power series expansion of the resolvent, the contour integral is

$$\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{\infty}\int_\Gamma \lambda^n\frac{A^k}{\lambda^{k+1}}d\lambda=\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{\infty}A^k\frac{1}{2\pi i}\int_\Gamma \lambda^{n-k-1}d\lambda.$$

By the Cauchy integral formula, $\frac{1}{2\pi i}\int_\Gamma \lambda^{n-k-1}d\lambda=0$ unless $n=k$ in which case the integral is 1. So the above expression becomes

$$\sum_{n=0}^{\infty}\frac{t^nA^n}{n!}=e^{At}.$$

□

If we integrate the equation

$$e^{tA} = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}R(\lambda, A)d\lambda$$

by parts we obtain

$$te^{tA} = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}R(\lambda, A)^2 d\lambda.$$

More generally, integrating by parts n times gives

$$\frac{t^n}{n!}e^{tA} = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}R(\lambda, A)^{n+1} d\lambda.$$

The first resolvent identity.

If z and w both belong to the resolvent set of A, then we can multiply the equation

$$zI - A = wI - A + (z - w)I$$

on the left by $R(z, A)$ and on the right by $R(w, A)$ to obtain

$$R(w, A) = R(z, A) + (z - w)R(z, A)R(w, A)$$

which is known as the **first resolvent identity** and dates back to the 19th century. We will make much use of this identity.

The second resolvent identity.

The first resolvent identity relates the resolvents of a single operator A at two different points in its resolvent set. The second resolvent identity, which also dates back to the 19th century relates the resolvents of two different operators at a point which belongs to the resolvent set of both:

Let ϕ and ψ be invertible operators. Clearly

$$\psi = \phi + \psi(\phi^{-1} - \psi^{-1})\phi.$$

Let A and B be two operators (at the moment both bounded) and let z belong to the resolvent set of both A and B. Apply the above equation to $\phi = R(z, A)$ and $\psi = R(z, B)$ so as to get the **second resolvent identity**

$$R(z, B) = R(z, A) + R(z, B)(B - A)R(z, A).$$

$e^A \cdot e^B \neq e^{A+B}$ in general.

If A and B are bounded operators, then it is not true in general that $e^A \cdot e^B = e^{A+B}$. Indeed, the two sides agree up to terms which are linear in A and B, but the quadratic terms on the left are $\frac{1}{2}\left[A^2 + 2AB + B^2\right]$ while the quadratic terms on the right are $\frac{1}{2}\left[A^2 + AB + BA + B^2\right]$. These do not agree unless $AB = BA$. So it is not true in general that

$$e^B = e^{(B-A)} \cdot e^A.$$

Indeed, the Campbell-Baker-Hausdorff formula gives a rather complicated formula for the operator C such that $e^B = e^C \cdot e^A$. However consider the following idea of Kantorovitz:

Define

$$X_0 := I, \; X_1 = B - A, \; X_2 := B^2 - 2BA + A^2,$$

and, in general,

$$X_n := B^n - nB^{n-1}A + \binom{n}{2} B^{n-2}A^2 + \cdots \pm A^n. \tag{1}$$

In other words, X_n looks like the binomial expansion of $(B-A)^n$ with all the B's moved to the left and all the A's to the right. Then, claim:

$$e^{tB} = \left(I + tX_1 + \frac{1}{2}t^2 X_2 + \cdots \right) e^{tA}. \tag{2}$$

To prove:

$$e^{tB} = \left(I + tX_1 + \frac{1}{2}t^2X_2 + \cdots \right) e^{tA}. \qquad (2)$$

Proof.

If A and B commute, this is simply the assertion that $e^{tB} = e^{t(B-A)}e^{tA}$. But in trying to verify (2) all the A's lie to the right of all the B's, and we never move an A past a B, so (2) is true in general. □

Unbounded operators

Up until now we have dealt with bounded operators. But we are interested in partial differential operators such as the heat operator which (at least when acting on a fixed Banach space) are unbounded. So we must discuss unbounded operators.

We will find that for certain types of operators (sectorial operators - see later for the definition) the above discussion about the semi-group generated by an operator goes through with minor modification in the statements and a good bit of work in the proofs. But for more general unbounded operators (as in the Hille-Yosida theorem, which I plan to discuss later) we will have to do major reworking.

The direct sum of two Banach spaces

Let B and C be Banach spaces. We make $B \oplus C$ into a Banach space via

$$\|\{x, y\}\| = \|x\| + \|y\|.$$

Here we are using $\{x, y\}$ to denote the ordered pair of elements $x \in B$ and $y \in C$ so as to avoid any conflict with our notation for scalar product in a Hilbert space. So $\{x, y\}$ is just another way of writing $x \oplus y$.

Linear operators and their graphs.

A subspace

$$\Gamma \subset B \oplus C$$

will be called a **graph** (more precisely a graph of a linear transformation) if

$$\{0, y\} \in \Gamma \quad \Rightarrow \quad y = 0.$$

Another way of saying the same thing is

$$\{x, y_1\} \in \Gamma \text{ and } \{x, y_2\} \in \Gamma \quad \Rightarrow \quad y_1 = y_2.$$

In other words, if $\{x, y\} \in \Gamma$ then y is determined by x.

The domain and the map of a graph.

Let $D(\Gamma)$ denote the set of all

$$x \in B \text{ such that there is a } y \in C \text{ with } \{x, y\} \in \Gamma.$$

Then $D(\Gamma)$ is a linear subspace of B, but, and this is very important, $D(\Gamma)$ is *not* necessarily a closed subspace. We have a linear map

$$T(\Gamma): \; D(\Gamma) \to C, \quad Tx = y \;\; \text{where} \;\; \{x, y\} \in \Gamma.$$

The graph of a linear transformation.

Equally well, we could start with the linear transformation: Suppose we are given a (not necessarily closed) subspace $D(T) \subset B$ and a linear transformation

$$T : D(T) \to C.$$

We can then consider its graph $\Gamma(T) \subset B \oplus C$ which consists of all

$$\{x, Tx\}, \quad x \in D(T).$$

Thus the notion of a graph, and the notion of a linear transformation defined only on a subspace of B are logically equivalent. When we start with T (as usually will be the case) we will write $D(T)$ for the domain of T and $\Gamma(T)$ for the corresponding graph.

There is a certain amount of abuse of language here, in that when we write T, we mean to include $D(T)$ and hence $\Gamma(T)$ as part of the definition.

Closed linear transformations.

A linear transformation is said to be **closed** if its graph is a closed subspace of $B \oplus C$.

Let us disentangle what this says for the operator T. It says that if $f_n \in D(T)$ then

$$f_n \to f \quad \text{and} \quad Tf_n \to g \quad \Rightarrow \quad f \in D(T) \quad \text{and} \quad Tf = g.$$

This is a much weaker requirement than continuity. Continuity of T would say that $f_n \to f$ alone would imply that Tf_n converges to Tf. Closedness says that if we know that **both**

$$f_n \;\; \text{converges } \textbf{and} \;\; g_n = Tf_n \text{ converges to } \;\; g$$

then $f = \lim f_n$ lies in $D(T)$ and $Tf = g$.

Avoiding the closed graph theorem

It is here that we run up against a famous theorem - the closed graph theorem - which says that if T is defined on all of a Banach space B and has a closed graph, then T must be bounded.

So if we are considering operators which are not bounded, we have to deal with operators whose domain is not all of B.

The resolvent, the resolvent set, and the specturm.

Let $T : B \to B$ be an operator with domain $D = D(T)$. A complex number z is said to belong to the **resolvent set** of T if the operator

$$zI - T$$

maps D onto all of B and has a two sided bounded inverse. As before, we denote this bounded inverse by $R(z, T)$ or $R_z(T)$ or simply by R_z if T is understood. So

$$R(z, T) := (zI - T)^{-1} \text{ maps } B \to D(T)$$

and is bounded. $R(z, T)$ is called the **resolvent** of T at the complex number z. The complement of the resolvent set is called the **spectrum** of T and is denoted by $\mathrm{Spec}(T)$.

The spectrum is a closed subset of $\mathbb{C}$.

Theorem

The set $\mathrm{Spec}(T)$ *is a closed subset of* $\mathbb{C}$. *In fact, if* $z \notin \mathrm{Spec}(T)$ *and* $c := \|R(z, T)\|$ *then the spectrum does not intersect the disk*

$$\{w \in \mathbb{C} |\ |(w - z)| < c^{-1}\}.$$

For w *in this disk* $R(w, T) = \sum_0^\infty (-(w - z))^n R(z, T)^{n+1}$ *and so is an analytic operator valued function of* w. *Differentiating this series term by term shows that*

$$\frac{d}{dz} R(z, T) = -R(z, T)^2.$$

Proof, part 1.

The series given in the theorem certainly converges in operator norm to a bounded operator for w in the disk. For a fixed w in the disk, let C denote the operator which is the sum of the series. Then

$$C = R(z, T) - (w - z)R(z, T)C.$$

This shows that C maps B to $D(T)$ and has kernel equal to the kernel of $R(z, T)$ which is $\{0\}$. So C is a bounded injective operator mapping B into D.

Also

$$C = R(z, T) - (w - z)CR(z, T)$$

which shows that the image of $R(z, T)$ is contained in the image of C and so the image of C is all of D.

Proof, part 2.

$$C := \sum_0^\infty (-(w-z))^n R(z,T)^{n+1}.$$

If $f \in D$ and $g = (zI - T)f$ then $f = R(z,T)g$ and so $Cg = f - (w-z)Cf$ and hence

$$C(zf - Tf) = f - (w-z)Cf$$

or

$$C(-Tf) = f - wCf \quad \text{so} \quad C(wI - T)f = f$$

showing that C is a left inverse for $wI - T$.

A similar argument shows that it is a right inverse. So we have proved that the series converges to the resolvent proving that the resolvent set is open and hence that the spectrum is closed. The rest of the theorem is immediate. □

The two resolvent identities follow just as in the bounded case.

Lemma

If $T : B \to B$ is an operator on a Banach space whose spectrum is not the entire plane then T is closed.

Proof.

Assume that $R = R(z, T)$ exists for some z. Suppose that f_n is a sequence of elements in the domain of T with $f_n \to f$ and $Tf_n \to g$. Set $h_n := (zI - T)f_n$ so

$$h_n \to zf - g.$$

Then $R(zf - g) = \lim Rh_n = \lim f_n = f$. Since R maps B to the domain of T this shows that f lies in this domain. Multiplying $R(zf - g) = f$ by $zI - T$ gives

$$zf - g = zf - Tf$$

showing that $Tf = g$. □

The adjoint of a densely defined linear operator.

Suppose that we have a linear operator $T : D(T) \to C$ and let us make the hypothesis that

$$D(T) \textit{ is dense in } B.$$

Any element of B^* is then completely determined by its restriction to $D(T)$. Now consider

$$\Gamma(T)^* \subset C^* \oplus B^*$$

defined by

$$\{\ell, m\} \in \Gamma(T)^* \quad \Leftrightarrow \quad \langle \ell, Tx \rangle = \langle m, x \rangle \quad \forall\, x \in D(T). \qquad (3)$$

The adjoint of a densely defined linear operator.

$$\langle \ell, Tx\rangle = \langle m, x\rangle \quad \forall\, x \in D(T). \qquad (3)$$

Since m is determined by its restriction to $D(T)$, we see that $\Gamma^* = \Gamma(T^*)$ is indeed a graph. (It is easy to check that it is a linear subspace of $C^* \oplus B^*$.) In other words we have defined a linear transformation

$$T^* := T(\Gamma(T)^*)$$

whose domain consists of all $\ell \in C^*$ such that there exists an $m \in B^*$ for which $\langle \ell, Tx\rangle = \langle m, x\rangle \quad \forall\, x \in D(T)$.

The adjoint of a linear transformation is closed.

If $\ell_n \to \ell$ and $m_n \to m$ then the definition of convergence in these spaces implies that for any $x \in D(T)$ we have

$$\langle \ell, Tx \rangle = \lim \langle \ell_n, Tx \rangle = \lim \langle m_n, x \rangle = \langle m, x \rangle.$$

If we let x range over all of $D(T)$ we conclude that Γ^* is a closed subspace of $C^* \oplus B^*$. In other words we have proved

Theorem

If $T : D(T) \to C$ is a linear transformation whose domain $D(T)$ is dense in B, it has a well defined adjoint T^* whose graph is given by (3). Furthermore T^* is a closed operator.

Most of the material for the rest of today's lecture (including the figures) is taken from:

Analytic Semigroups and Reaction-Diffusion Problems
Internet Seminar 2004 - 2005
by
Luca Lorenzi, Alessandra Lunardi, Giorgio Metafune, Diego Pallara

and from

Introduzione ai problemi parabolici nonlineari
by
Alessandra Lunardi

via the internet.

Sectorial operators.

A closed operator A is called **sectorial** if its resolvent set contains a sector S of the form

$$S = \{\lambda \in \mathbb{C} | \lambda \neq 0, |\arg \lambda| < \theta, \theta > \pi/2\}.$$

More precisely, we will assume that there is a positive constant M such that

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda|}, \quad \lambda \in S.$$

For example, we have seen that the spectrum of the Laplacian $A = -\Delta$ on a compact manifold is discrete, lies along the negative real axis and tends to $-\infty$. Hence A is sectorial.

Definition of e^{tA} when A is sectorial.

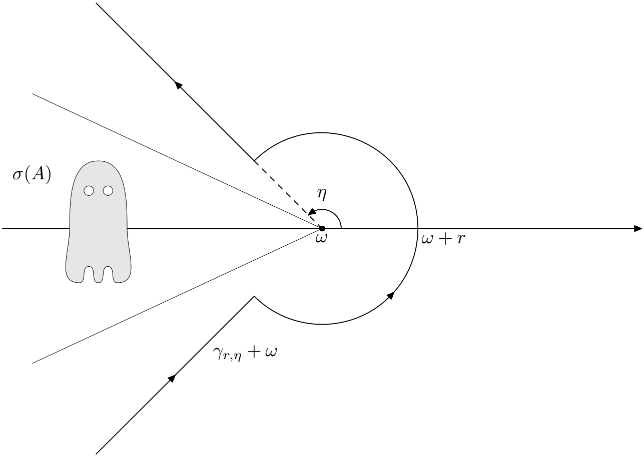
Recall that in the case of a bounded operator A, the exponential series for e^{tA} converges, and if Γ is a circle of radius $> \|A\|$ centered at the origin, we proved that

$$e^{tA} = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}R(\lambda, A)d\lambda.$$

For sectorial operators we will **define** e^{tA} by a contour integral of $e^{t\lambda}R(\lambda, A)$ over a modified contour. I think that the idea of doing this goes back to Dunford.

Here is the picture of the proposed contour, where we have "opened up" the circle by two branches going off diagonally to the left:

Definition of e^{tA} when A is sectorial.



In the figure, we have slightly generalized the notion of a sectorial operator to allow for some positive spectrum, so we assume that the resolvent set contains the set

$$S_{\theta,\omega} := \{\lambda \in \mathbb{C} | \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$$

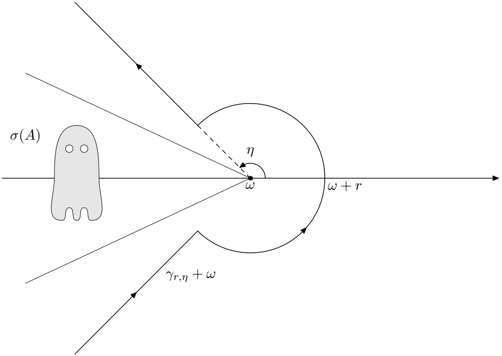
where $\omega \in \mathbb{R}$ and $\theta \in (\pi/2, \pi)$. We also assume that

$$\|R(\lambda, A\| \leq \frac{M}{|\lambda - \omega|} \quad \lambda \in S_{\theta,\omega}$$

for some positive constant M. We let $\gamma_{r,\eta}$ be the curve

$$\{\lambda \in \mathbb{C} | \ |\arg(\lambda)| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} | \ |\arg \lambda| \leq \eta, \ |\lambda| = r\}.$$

The curve in the figure is the curve $\gamma_{r,\eta}$ shifted by ω.



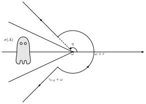
For each $t > 0$ we define

$$e^{tA} := \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}R(\lambda, A)d\lambda.$$

We must show that:

- This is well defined, i.e. that the integral converges and defines a bounded operator.
- This has the properties we expect of the exponential, e.g. $e^{sA}e^{tA} = e^{(s+t)A}$, and $A^k e^{tA} = e^{tA}A^k$ (when defined).
- The definition is independent of the choice of r and η.
- This coincides with the old definition when A is bounded.

The integral converges.



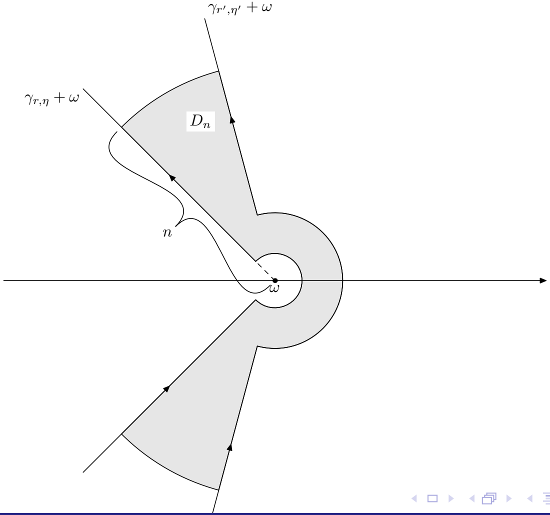
The integral breaks up into three pieces given by

$$\frac{e^{\omega t}}{2\pi i}\left(-\int_r^\infty e^{(\rho\cos\eta - i\rho\sin\eta)t}R(\omega+\rho e^{-i\eta},A)e^{-i\eta}d\rho\right.$$
$$+\int_{-\eta}^{\eta} e^{(r\cos\alpha + ir\sin\alpha)t}R(\omega+re^{i\alpha},A)ire^{i\alpha}d\alpha$$
$$\left.+\int_r^\infty e^{(\rho\cos\eta + i\rho\sin\eta)t}R(\omega+\rho e^{i\eta},A)e^{i\eta}d\rho\right).$$

The middle integral causes no convergence problems. For the first and third terms, we know that $\|R(\omega + \rho e^{\pm i\eta}, A)\|$ is bounded by M/r and $\cos \eta < 0$ so there is no trouble with these integrals.

We now show that the integral is independent of the choice of r and η: For this choose a different r', η' and consider the region D lying between the two curves $\gamma_{r,\eta} + \omega$ and $\gamma_{r'\eta'} + \omega$ lying between them and the cutoff regions $D_n := D \cap \{|z - \omega| < n\}$ for $n \in \mathbb{N}$. See the figure on the next slide.

Definition of e^{tA} when A is sectorial.



The function $\lambda \mapsto e^{t\lambda}R(\lambda, A)$ is holomorphic on the domain D_n and hence its integral over ∂D_n vanishes. The integral over the pieces of the circle $|\lambda - \omega| = n$ lying on the boundary tend to zero by the estimate we gave above, namely that over these integrals the term $\cos\alpha$ is negative. This proves that the definition of e^{tA} does not depend on the choice of r and η.

If A is a bounded operator, we choose $r > \|A\|$ and let $\eta \to \pi$. The same argument shows that the integral tends to the integral over the circle, as the two radial integrals will tend to terms which cancel one another. So the new definition coincides with the old one for bounded operators.

We define $e^{0A} = I$.

We now prove:

Proposition

For all $t > 0$, $\quad e^{tA} : X \to \mathrm{Dom}(A)$ *and for* $x \in \mathrm{Dom}(A)$

$$Ae^{tA}x = e^{tA}Ax.$$

For this purpose, we first prove a lemma about closed operators A:

Lemma

Let $I = (a, b)$ with $-\infty \leq a < b \leq +\infty$ and $f : I \to \mathrm{Dom}(A)$ be a function which is Riemann integrable, and such that the function $\lambda \mapsto Af(\lambda)$ is also Riemann integrable. Then

$$\int_I f(\lambda)d\lambda \in \mathrm{Dom}(A) \quad \textit{and} \quad A\int_I f(\lambda)d\lambda = \int_I Af(\lambda)d\lambda.$$

Proof of the lemma.

First suppose that I is bounded. Then the assertion $S \in \mathrm{Dom}(A)$ is true by linearity where S is a Riemann approximating sum $S = \sum f(m_i)(\lambda_i - \lambda_{i-1})$ as is the assertion

$$AS = \sum Af(m_i)(\lambda_i - \lambda_{i-1}).$$

Proof of the lemma, 2.

By assumption $S \to \int_I f(\lambda)d\lambda$ and $AS \to \int_I Af(\lambda)d\lambda$. The fact that A is closed then implies that

$$\int_I f(\lambda)d\lambda \in \mathrm{Dom}(A) \quad \text{and} \quad A\int_I f(\lambda)d\lambda = \int_I Af(\lambda)d\lambda.$$

The non-bounded case follows by the same argument since the convergence of the integrals means that
$\int_I = \lim \int_{\text{bounded intervals}}$. $\square$

Proof of the proposition.

Replacing A by $A - \omega I$ we may assume that $\omega = 0$ to simplify the notation. Take $f(\lambda) = e^{t\lambda}R(\lambda, A)x$ in the lemma: the resolvent maps the entire Banach space X into $\mathrm{Dom}(A)$ and hence the lemma implies that $e^{tA}x \in \mathrm{Dom}(A)$ for any $x \in X$ and that

$$Ae^{tA}x = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{t\lambda}AR(\lambda, A)x d\lambda.$$

Now by the definition of the resolvent, $(\lambda I - A)R(\lambda, A) = I$ so $AR(\lambda, A) = \lambda R(\lambda, A) - I$. By Cauchy, $\int_{\gamma_{r,\eta}} e^{t\lambda} d\lambda = 0$, so

$$Ae^{tA}x = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} \lambda e^{t\lambda}R(\lambda, A)x d\lambda.$$

We have shown that

$$Ae^{tA}x = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{t\lambda}AR(\lambda,A)xd\lambda$$

$$= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} \lambda e^{t\lambda}R(\lambda,A)xd\lambda.$$

For $x \in \mathrm{Dom}(A)$ we have $R(\lambda,A)Ax = AR(\lambda,A)x$ so the first equation above shows that for $x \in \mathrm{Dom}(A)$

$$Ae^{tA}x = e^{tA}Ax. \qquad \square$$

We continue with the assumption that $t > 0$.

We know that $e^{tA}x \in \mathrm{Dom}(A)$ for any $x \in X$ and if $x \in \mathrm{Dom}(A)$ then $Ae^{tA}x = e^{tA}Ax = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} \lambda e^{t\lambda}R(\lambda, A)xd\lambda$.

The lemma applied to the function $f(\lambda) := \lambda e^{t\lambda}R(\lambda, A)x$ tells us that $Ae^{tA}x$ belongs to $\mathrm{Dom}(A)$. In other words, $e^{tA}x$ belongs to $\mathrm{Dom}(A^2)$ and furthermore

$$A^2e^{tA}x = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} A\lambda e^{\lambda t}R(\lambda, A)xd\lambda.$$

If $x \in \mathrm{Dom}(A)$ we can move the A past the $R(\lambda, A)$ to conclude that

$$A^2e^{tA}x = Ae^{tA}Ax.$$

If $x \in \mathrm{Dom}(A)$ we can move the A past the $R(\lambda, A)$ to conclude that

$$A^2 e^{tA} x = A e^{tA} A x.$$

If we assume in addition that $x \in \mathrm{Dom}(A^2)$ so that (by definition) $Ax \in \mathrm{Dom}(A)$ we can apply the proposition to Ax and conclude that

$$A^2 e^{tA} x = e^{tA} A^2 x.$$

Continuing in this way we conclude

Theorem

If $t > 0$ *then* $e^{tA}x \in \mathrm{Dom}(A^k)$ *for all positive integers* k*, and all* $x \in X$*. If* $x \in \mathrm{Dom}(A^k)$ *then*

$$A^k e^{tA}x = e^{tA}A^k x.$$

Furthermore

$$A^k e^{tA} = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} \lambda^k e^{t\lambda} R(\lambda, A)d\lambda.$$

The first resolvent identity.

Recall that this says that for λ and μ in the resolvent set of A we have:

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A).$$

Proof.

Since $(\mu I - A)R(\mu, A) = I$ and $(\lambda I - A)R(\lambda, A) = I$ we have

$$\begin{aligned} R(\lambda, A) &= [\mu R(\mu, A) - AR(\mu, A)]R(\lambda, A) \\ R(\mu, A) &= [\lambda R(\lambda, A) - AR(\lambda, A)]R(\mu, A). \end{aligned}$$

Subtract the second equation from the first and use the fact that the resolvents commute to give the first resolvent identity. □

We are going to use the first resolvent identity in the form

$$R(\lambda, A)R(\mu, A) = \frac{R(\lambda, A) - R(\mu, A)}{\mu - \lambda} \quad \text{if} \quad \mu \neq \lambda$$

together with the Cauchy integral formula to prove that

$$e^{tA}e^{sA} = e^{(t+s)A}.$$

For this purpose we write

$$\begin{aligned} e^{sA} &= \frac{1}{2\pi i}\int_{\gamma_{2r,\eta'}} e^{\mu s}R(\mu, A)d\mu \\ e^{tA} &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{\lambda t}R(\lambda, A)d\lambda \end{aligned}$$

where $\eta' \in (\eta, \pi/2)$.

So $e^{sA}e^{tA} =$

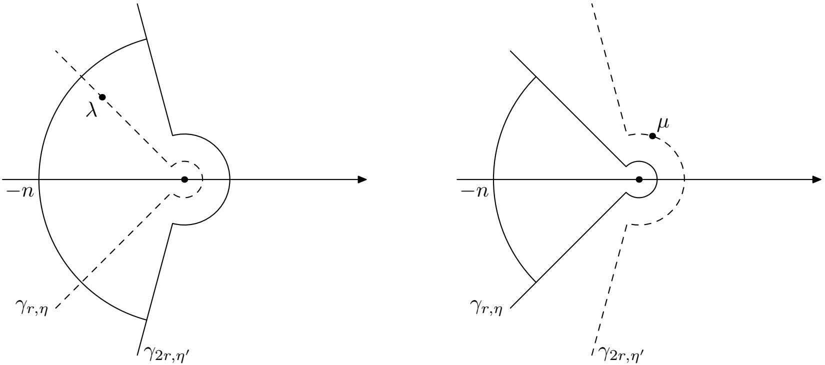
$$\left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r,\eta}} \int_{\gamma_{2r,\eta'}} \frac{1}{\mu - \lambda} e^{\mu s} e^{\lambda t} R(\lambda, A) d\mu d\lambda$$

$$- \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{2r,\eta'}} \int_{\gamma_{r,\eta}} \frac{1}{\mu - \lambda} e^{\mu s} e^{\lambda t} R(\mu, A) d\lambda d\mu.$$

(The order of integration is irrelevant due to the exponential convergence.) The 2nd integral, where we first integrate over the interior curve, vanishes by Cauchy's theorem, as the integrand is holomorphic. See the right hand figure on the next slide. Cauchy's integral formula applied to the inner integrand of the first integral where we integrate over the outer curve first, gives,

$$\frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} e^{\lambda s} R(\lambda, A) d\lambda = e^{(s+t)A}. \quad \square$$

The semi-group property.



Proposition

There is a constant M_0 such that

$$\|e^{tA}\| \leq M_0 e^{\omega t}.$$

Once again, for the proof we may assume that $\omega = 0$. Also remember that we may replace r by r' in the contour over which we integrate. In particular, we may replace r by r/t in the contour for e^{sA}, in particular for $s = t$. We make the change of variables $\xi = \lambda t$ and obtain:

Proof.

$$
\begin{aligned}
e^{tA} &= \frac{1}{2\pi i}\int_{\gamma_{rt,\eta}} e^{\xi} R\left(\frac{\xi}{t}, A\right)\frac{d\xi}{t} = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{\xi} R\left(\frac{\xi}{t}, A\right)\frac{d\xi}{t} \\
&= \frac{1}{2\pi i}\left(\int_r^{+\infty} e^{\rho e^{i\eta}} R\left(\frac{\rho e^{i\eta}}{t}, A\right)\frac{e^{i\eta}}{t}d\rho - \int_r^{+\infty} e^{\rho e^{-i\eta}} R\left(\frac{\rho e^{-i\eta}}{t}, A\right)\frac{e^{-i\eta}}{t}d\rho \right. \\
&\qquad \left. + \int_{-\eta}^{\eta} e^{re^{i\alpha}} R\left(\frac{re^{i\alpha}}{t}, A\right) ire^{i\alpha}\frac{d\alpha}{t}\right).
\end{aligned}
$$

It follows that

$$
\|e^{tA}\| \le \frac{1}{\pi}\left\{\int_r^{+\infty} Me^{\rho\cos\eta}\frac{d\rho}{\rho} + \frac{1}{2}\int_{-\eta}^{\eta} Me^{r\cos\alpha}d\alpha\right\}.
$$

Estimating Ae^{tA}.

Recall that $\|R(\lambda-\omega,A)\| \leq \frac{M}{|\lambda-\omega|}$ is part of our assumption, and that $AR(\lambda,A) = \lambda R(\lambda,A) - I$ so on the path $\gamma_{r,\eta}$ we have

$$\|AR(\lambda,A)\| \leq M+1.$$

Hence

$$\|Ae^{tA}\| \leq \frac{M+1}{2\pi}\left[2\int_r^\infty e^{\rho t\cos\eta}d\rho + r\int_{-\eta}^{\eta} e^{rt\cos\alpha}d\alpha\right].$$

$$\|Ae^{tA}\| \le \frac{M+1}{2\pi}\left[2\int_r^\infty e^{\rho t\cos\eta}d\rho + r\int_{-\eta}^{\eta} e^{rt\cos\alpha}d\alpha\right].$$

Let $r \to 0$. The second integral disappears. In the first integral, make the change of variable $\rho \mapsto s = t|\cos\eta|\rho$. We get $\|Ae^{tA}\| \le \frac{M+1}{\pi t|\cos\eta|}\int_0^\infty e^{-s}ds$ so

$$\|Ae^{tA}\| \le \frac{M+1}{\pi t|\cos\eta|} =: \frac{M_1}{t}.$$

$$\|Ae^{tA}\| \leq \frac{M+1}{\pi t|\cos\eta|} =: \frac{M_1}{t}.$$

Now e^{tA} maps $X \to \mathrm{Dom}(A)$ and for every $x \in \mathrm{Dom}(A)$ we have $Ae^{tA}x = e^{tA}Ax$ hence

$$(Ae^{\frac{t}{k}A})^k = A^k e^{tA} \quad \forall \quad k \in \mathbb{N}.$$

So applying the above inequality we obtain:

$$\|A^k e^{tA}\| \leq \frac{M_k}{t^k} \quad \text{where} \quad M_k := (kM_1)^k.$$

The derivatives of e^{tA} for $t > 0$.

I will continue with the harmless assumption that $\omega = 0$. Differentiating the definition

$$e^{tA} := \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{t\lambda}R(\lambda, A)d\lambda.$$

under the integral sign we see that for $t > 0$ we have

$$\frac{d}{dt}e^{tA} := \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} \lambda e^{t\lambda}R(\lambda, A)d\lambda,$$

and this is equal to Ae^{tA} as we have already seen. So for $t > 0$ we have

$$\frac{d}{dt}e^{tA} = Ae^{tA}.$$

Iterating the formula

$$\frac{d}{dt}e^{tA} = Ae^{tA}$$

gives

$$\frac{d^k}{dt^k}e^{tA} = A^k e^{tA}$$

for $t > 0$. So $t \mapsto e^{tA}$ is C^∞ as a map from $\mathbb{R}^+$ to bounded operators on X and its derivatives are given as above.

In fact, it extends to a holomorphic function in a wedge about the positive x-axis as we shall now see.

The holomorphic character of e^{tA}.

The exponent in the integral along the rays in

$$\frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{z\lambda}R(\lambda,A)d\lambda.$$

where $\lambda = \rho e^{i\eta}$ and $z = |z|e^{i\tau}$ is

$$\rho|z|e^{i(\eta+\tau)} = \rho|z|(\cos(\eta+\tau) + i\sin(\eta+\tau)).$$

So as long as $\cos(\eta+\tau) < 0$, which is the same as $|\tau| < \eta - \frac{\pi}{2}$ this integral converges, and we can differentiate under the integral sign. So if θ is the angle that enters into the sectorial character of A, we see that the function $z \mapsto e^{zA}$ defined by the above integral for some $|\eta| < \theta$ is holomorphic as a function of z for $\arg(z) < \theta - \frac{\pi}{2}$.

The limit of $e^{tA}x$ as $t \to 0$ for $x \in \mathrm{Dom}(A)$.

Let $x \in \mathrm{Dom}(A)$, choose $\mathbb{R} \ni \xi > \omega$ so that ξ is in the resolvent set of A, and choose $0 < r < \xi - \omega$. Let $y := \xi x - Ax$ so that $x = R(\xi, A)y$. Then by the first resolvent identity

$$\begin{aligned} e^{tA}x &= e^{tA}R(\xi,A)y = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}R(\lambda,A)R(\xi,A)y\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}\frac{R(\lambda,A)}{\xi-\lambda}y\,d\lambda - \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}\frac{R(\xi,A)}{\xi-\lambda}y\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}\frac{R(\lambda,A)}{\xi-\lambda}y\,d\lambda, \end{aligned}$$

as the second integral vanishes by the Cauchy integral formula as ξ is outside the region to the left of the curve of integration. The term $R(\lambda,\xi)/(\xi-\lambda)$ in the first integral is of order $|\lambda|^{-2}$ for large λ so we can pass to the limit $t \to 0^+$ under the integral sign to obtain

$$\lim_{t\to 0^+} e^{tA}x = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} \frac{1}{\xi-\lambda}R(\lambda,A)yd\lambda.$$

We have shown that

$$\lim_{t \to 0^+} e^{tA}x = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}+\omega} \frac{1}{\xi - \lambda} R(\lambda, A) y d\lambda.$$

Now we can cut off the radial pieces of the integral at $|\lambda - \omega| = n$ and close up the curve via the circular arc $|\lambda - \omega| = n$ going from $\arg(\lambda - \omega) = \eta$ to $\arg(\lambda - \omega) = -\eta$. On this circular arc the integrand is of order $|\lambda|^{-2}$ so this portion of the integral goes to zero as $n \to \infty$. The point ξ is in the interior of the closed curve. Taking into account that the curve is oriented clockwise rather that counter clockwise, the Cauchy integral formula tells us that the integral equals $R(\xi, A)y = x$. So we have proved that

$$\lim_{t \to 0^+} e^{tA}x = x$$

for $x \in \mathrm{Dom}(A)$.

We have proved that

$$\lim_{t \to 0^+} e^{tA}x = x$$

for $x \in \mathrm{Dom}(A)$.

We also know that $\|e^{tA}\|$ is bounded for $0 < t < 1$ by a constant independent of t, so it follows from the above that

Proposition

If $x \in \overline{\mathrm{Dom}(A)}$ *then*

$$\lim_{t \to 0^+} e^{tA}x = x$$

Conversely, suppose that $\lim_{t\to 0^+} e^{tA}x = y$. We know that $e^{tA}x \in \mathrm{Dom}(A)$ so $y \in \overline{\mathrm{Dom}(A)}$. For any ξ in the resolvent set of A we know that $R(\xi, A)$ is a bounded operator which maps all of X into $\mathrm{Dom}(A)$. So

$$R(\xi, A)y = R(\xi, A) \lim_{t\to 0^+} e^{tA}x = \lim_{t\to 0^+} e^{tA}R(\xi, A)x = R(\xi, A)x.$$

So $y = x$.

Proposition

For every $x \in X$ and $t \geq 0$

1. $\int_0^t e^{sA} x ds \in \mathrm{Dom}(A)$ *and*
2. $A \int_0^t e^{sA} x ds = e^{tA} x - x.$
3. *If, in addition, the function $s \mapsto A e^{sA} x$ is integrable on $(0, \epsilon)$ for some $\epsilon > 0$ then*

$$e^{tA} x - x = \int_0^t A e^{sA} x ds.$$

for any $t \geq 0$.

Remark. The highlighted step 2 will be very important for us.

Proof of 1.

Choose ξ in the resolvent set of A. For any $\epsilon \in (0, t)$

$$\begin{aligned}
\int_\epsilon^t e^{sA} x ds &= \int_\epsilon^t (\xi I - A) R(\xi, A) e^{sA} x ds \\
&= \xi \int_\epsilon^t R(\xi, A) e^{sA} x ds - \int_\epsilon^t \frac{d}{ds}[R(\xi, A) e^{sA} x] ds \\
&= \xi R(\xi, A) \int_\epsilon^t e^{sA} x ds - e^{tA} R(\xi, A) x + e^{\epsilon A} R(\xi, A) x.
\end{aligned}$$

Since $R(\xi, A)x \in \mathrm{Dom}(A)$, the limit of the the last term as $\epsilon \to 0$ is $R(\xi, A)x$. So

$$\int_0^t e^{sA} x ds = \xi R(\xi, A) \int_0^t e^{sA} x ds - e^{tA} R(\xi, A) x + R(\xi, A) x.$$

Proof of 1, continued, proof of 2 and 3.

We have proved that

$$\int_0^t e^{sA} x ds = \xi R(\xi, A) \int_0^t e^{sA} x ds - e^{tA} R(\xi, A) x + R(\xi, A) x.$$

Since $R(\xi, A)$ maps X into $\mathrm{Dom}(A)$, we have proved 1.
If we apply $\xi I - A$ to both sides of the above equations we get

$$(\xi I - A) \int_0^t e^{sA} x ds = \xi \int_0^t e^{sA} x ds - (e^{tA} x - x)$$

which after rearranging the terms says that
$A \int_0^t e^{sA} x ds = e^{tA} x - x$ which is 2. Under the integrability hypothesis we may move the A inside the integral which is 3. □

Proposition

If $x \in \mathrm{Dom}(A)$ *and* $Ax \in \overline{Dom(A)}$ *then*

$$\lim_{t \to 0^+} \frac{1}{t}[e^{tA}x - x] = Ax.$$

Conversely, if the limit of the left hand side exists, call it z*, then* $x \in \mathrm{Dom}(A)$ *and* $Ax = z$*,*

Proof. By statement 3 of the preceding Proposition, we may write the left hand side of the equation in our proposition as $\frac{1}{t}\int_0^t e^{sA}Axdx$. Since the function $s \mapsto e^{sA}Ax$ is continuous, we can apply the fundamental theorem of the calculus to conclude that the limit is indeed Ax.

Conversely: If the limit exists, then $e^{tA}x \to x$ so both x and z belong to $\overline{\mathrm{Dom}(A)}$. For any ξ in the resolvent set of A we have

$$R(\xi, A)z = \lim_{t\to 0^+} R(\xi, A)\frac{1}{t}(e^{tA}x - x) = \lim_{t\to 0^+} \frac{1}{t}R(\xi, A)A\int_0^t e^{tA}xdx$$

$$= (\xi R(\xi, A) - I)\lim_{t\to 0^+} \frac{1}{t}\int_0^t e^{tA}xdx = \xi R(\xi, A)x - x.$$

This shows that $x \in \mathrm{Dom}(A)$. If we apply $\xi I - A$ to both sides of this equation we get

$$z = \xi x - (\xi I - A)x$$

so

$$z = Ax. \quad \square$$

Recall the formula $A\int_0^t e^{sA}xds = e^{tA}x - x$. Replace A by $A-\lambda I$ for $\lambda > 0$ to obtain

$$e^{-\lambda t}e^{tA}x - x = (A-\lambda I)\int_0^t e^{-\lambda s}e^{sA}xds.$$

Letting $t\to\infty$ gives (since e^{tA} is uniformly bounded in $t>0$)

$$x = (\lambda I - A)\int_0^\infty e^{-\lambda t}e^{tA}xdt.$$

Applying $R(\lambda, A)$ to this equation gives

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t}e^{tA}dt.$$

This was proved for real positive λ. But by analytic continuation it is true for all λ in the sector S.

Summary:

1. The power series expansion of e^{tA} when A is bounded.
   - The resolvent from the semi-group.
   - The semigroup from the resolvent.
   - The two resolvent identities.
2. Unbounded operators, their resolvents and their spectra.
3. Sectorial operators.
   - Definition of a sectorial operator.
   - Definition of e^{tA} when A is sectorial.
   - The semi-group property.
   - Bounds on e^{tA}.
   - The derivatives of e^{tA}, its holomorphic charcter.
   - The limit of e^{tA} as $t \to 0^+$.
   - The resolvent as the Laplace transform of the semigroup.