Math212a Lecture 5
Applications of the spectral theorem for compact self-adjoint operators, 2.
Gårding's inequality and its consequences.

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1. Review of Sobolev spaces.
   - Distributions and Schwartz's theorem.
2. Gårding's inequality.
   - Differential operators.
   - Rellich's lemma
   - Some numerical inequalities.
   - Elliptic operators.
   - Statement and proof of Gårding's inequality.
3. Consequences of Gårding's inequality.
4. Extension of the basic lemmas to manifolds.
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The space $P(\mathbb{T})$ and its scalar products.

Recall that $\mathbb{T}$ now stands for the n-dimensional torus. Let $\mathbf{P} = \mathbf{P}(\mathbb{T})$ denote the space of trigonometric polynomials. These are functions on the torus of the form

$$u(x) = \sum a_\ell e^{i\ell\cdot x}$$

where $\ell = (\ell_1, \ldots, \ell_n)$ is an n-tuplet of integers and the sum is finite. For each integer t (positive, zero or negative) we introduced the scalar product

$$(u, v)_t := \sum_\ell (1 + \ell \cdot \ell)^t a_\ell \overline{b}_\ell. \qquad (1)$$

$$(u, v)_t := \sum_{\ell} (1 + \ell \cdot \ell)^t a_\ell \overline{b}_\ell.$$

For $t = 0$ this is the scalar product

$$(u, v)_0 = \frac{1}{(2\pi)^n} \int_{\mathbb{T}} u(x)\overline{v(x)} dx.$$

We denote the norm corresponding to the scalar product $(\, , \,)_s$ by $\| \; \|_s$.

Relations between the norms.

If

$$\Delta := -\left(\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^n)^2}\right)$$

the operator $(1+\Delta)$ satisfies

$$(1+\Delta)u = \sum (1+\ell\cdot\ell)a_\ell e^{i\ell\cdot x}$$

and so

$$((1+\Delta)^t u, v)_s = (u, (1+\Delta)^t v)_s = (u, v)_{s+t}$$

and

$$\|(1+\Delta)^t u\|_s = \|u\|_{s+2t}. \tag{2}$$

The generalized Cauchy Schwarz inequality.

We then get the generalized Cauchy-Schwarz inequality

$$|(u,v)_s| \le \|u\|_{s+t}\|v\|_{s-t} \tag{3}$$

for any t, as a consequence of the usual Cauchy-Schwarz inequality. Indeed,

$$\begin{aligned}
\sum_\ell (1+\ell\cdot\ell)^s a_\ell \overline{b}_\ell &= \sum_\ell (1+\ell\cdot\ell)^{\frac{s+t}{2}} a_\ell (1+\ell\cdot\ell)^{\frac{s-t}{2}} \overline{b}_\ell \\
&= ((1+\Delta)^{\frac{s+t}{2}} u, (1+\Delta)^{\frac{s-t}{2}} v)_0 \\
&\le \|(1+\Delta)^{\frac{s+t}{2}} u\|_0 \|(1+\Delta)^{\frac{s-t}{2}} v\|_0 \\
&= \|u\|_{s+t}\|v\|_{s-t}.
\end{aligned}$$

The generalized Cauchy-Schwarz inequality reduces to the usual Cauchy-Schwartz inequality when $t = 0$.

Clearly we have

$$\|u\|_s \leq \|u\|_t \quad \text{if } s \leq t.$$

If D^p denotes a partial derivative,

$$D^p = \frac{\partial^{|p|}}{\partial(x^1)^{p_1} \cdots \partial(x^n)^{p_m}}$$

then

$$D^p u = \sum (i\ell)^p a_\ell e^{i\ell \cdot x}.$$

Notation.

In these equations we are using the following notations:

- If $p = (p_1, \ldots, p_n)$ is a vector with non-negative integer entries we set
$$|p| := p_1 + \cdots + p_n.$$
- If $\xi = (\xi_1, \ldots, \xi_n)$ is a (row) vector we set
$$\xi^p := \xi_1^{p_1} \cdot \xi_2^{p_2} \cdots \xi_n^{p_n}$$

It is then clear that

$$\|D^p u\|_t \leq \|u\|_{t+|p|} \tag{4}$$

and similarly

$$\|u\|_t \leq (\text{constant depending on } t) \sum_{|p| \leq t} \|D^p u\|_0 \quad \text{if } t \geq 0. \tag{5}$$

In particular,

Theorem

The norms

$$u \mapsto \|u\|_t$$

$t \geq 0$ *and*

$$u \mapsto \sum_{|p| \leq t} \|D^p u\|_0$$

are equivalent.

The Sobolev spaces $\mathbf{H}_t$.

We let $\mathbf{H}_t$ denote the completion of the space $\mathbf{P}$ with respect to the norm $\| \ \|_t$. Each $\mathbf{H}_t$ is a Hilbert space, and we have natural embeddings

$$\mathbf{H}_t \hookrightarrow \mathbf{H}_s \ \text{ if } s < t.$$

The equation

$$\|(1+\Delta)^t u\|_s = \|u\|_{s+2t}$$

says that $(1+\Delta)^t$,initially defined on $\mathbf{P}$, extends to a map

$$(1+\Delta)^t : \mathbf{H}_{s+2t} \to \mathbf{H}_s$$

and is an isometry.

The duality between $\mathbf{H}_t$ and $\mathbf{H}_{-t}$.

From the generalized Cauchy-Schwartz inequality we also have a natural pairing of $\mathbf{H}_t$ with $\mathbf{H}_{-t}$ given by the extension of $(\ ,\ )_0$, so

$$|(u,v)_0| \leq \|u\|_t \|v\|_{-t}. \tag{6}$$

In fact, this pairing allows us to identify $\mathbf{H}_{-t}$ with the space of continuous linear functions on $\mathbf{H}_t$. Let us state this in more detail:

Theorem

Let $v \in \mathbf{H}_{-t}$*. Then* v *defines a continuous linear function* ϕ_v *on* $\mathbf{H}_t$ *by*

$$\phi_v(u) = (u, v)_0$$

and $\|\phi_v\| = \|v\|_{-t}$*, i.e.*

$$\|v\|_{-t} = \sup |(u, v)_0|, \quad \|u\|_t = 1.$$

Conversely, every continuous linear function ϕ *on* $\mathbf{H}_t$ *is of the form* ϕ_v *for a unique* $v \in \mathbf{H}_{-t}$*.*

Proof.

If ϕ is a continuous linear function on $\mathbf{H}_t$ the Riesz representation thm says that there is a unique $w \in \mathbf{H}_t$ with $\phi(u) = (u, w)_t$ and

$$\|\phi\| = \sup_{\|u\|_t = 1} |(u, w)_t| = \|w\|_t.$$

Set

$$v := (1 + \Delta)^t w.$$

Then $v \in \mathbf{H}_{-t}$ and

$$(u, v)_0 = (u, (1 + \Delta)^t w)_0 = (u, w)_t = \phi(u).$$

Starting with $v \in \mathbf{H}_{-t}$ we get the continuous linear function $\phi : u \mapsto (u, v)_0$ with $\phi = \phi_v$ (and $w = (1 + \Delta)^{-t} v$). □

We record the theorem as

$$\mathbf{H}_{-t} = (\mathbf{H}_t)^* . \tag{7}$$

As an illustration of (7), observe that the series

$$\sum_{\ell} (1 + \ell \cdot \ell)^s$$

converges for $s < -\frac{n}{2}$. This means that if define v by taking

$$b_\ell \equiv 1$$

then $v \in \mathbf{H}_s$ for $s < -\frac{n}{2}$.

The Dirac delta function.

This means that if define v by taking

$$b_\ell \equiv 1$$

then $v \in \mathbf{H}_s$ for $s < -\frac{n}{2}$.
If u is given by $u(x) = \sum_\ell a_\ell e^{i\ell\cdot x}$ is any trigonometric polynomial, then

$$(u, v)_0 = \sum a_\ell = u(0).$$

So the natural pairing (6) allows us to extend the linear function sending $u \mapsto u(0)$, initially defined only on $\mathbf{P}(\mathbb{T})$, to all of $\mathbf{H}_t$ if $t > \frac{n}{2}$. We can now give v its "true name": it is the Dirac "delta function" δ (on the torus) where

$$(u, \delta)_0 = u(0).$$

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$$(u, \delta)_0 = u(0).$$

So $\delta \in H_{-t}$ for $t > \frac{n}{2}$, and the preceding equation is usually written symbolically as

$$\frac{1}{(2\pi)^n} \int_{\mathbb{T}} u(x)\delta(x)dx = u(0);$$

but the true mathematical interpretation is as given above.

We set

$$\mathbf{H}_\infty := \bigcap \mathbf{H}_t, \quad \mathbf{H}_{-\infty} := \bigcup \mathbf{H}_t.$$

Sobolev's Lemma.

The space $\mathbf{H}_0$ is just $L_2(\mathbb{T})$, and we can think of the space $\mathbf{H}_t,\quad t > 0$ as consisting of those functions having "generalized L_2 derivatives up to order t". Certainly a function of class C^t belongs to $\mathbf{H}_t$. With a loss of degree of differentiability the converse is true:

Lemma

[Sobolev.] *If $u \in \mathbf{H}_t$ and*

$$t \geq \left[\frac{n}{2}\right] + k + 1$$

then $u \in C^k(\mathbb{T})$ and

$$\sup_{x\in\mathbb{T}} |D^p u(x)| \leq \text{const.}\|u\|_t \quad \textit{for } |p| \leq k. \tag{8}$$

Distributions aka generalized functions.

A **distribution** on $\mathbb{T}^n$ is a linear function T on $C^\infty(\mathbb{T}^n)$ with the continuity condition that

$$\langle T, \phi_k \rangle \to 0$$

whenever

$$D^p \phi_k \to 0$$

uniformly for each fixed p. If $u \in \mathbf{H}_{-t}$ we may define

$$\langle u, \phi \rangle := (\phi, \overline{u})_0$$

and since $C^\infty(\mathbb{T})$ is dense in $\mathbf{H}_t$ we may conclude

Schwartz's theorem.

Lemma

$\mathbf{H}_{-t}$ is the space of those distributions T which are continuous in the $\| \ \|_t$ norm, i.e. which satisfy

$$\|\phi_k\|_t \to 0 \quad \Rightarrow \ \langle T, \phi_k \rangle \to 0.$$

We then obtain

Theorem

[Laurent Schwartz.] *$\mathbf{H}_{-\infty}$ is the space of all distributions. In other words, any distribution belongs to $\mathbf{H}_{-t}$ for some t.*

Multiplication by a smooth function.

Suppose that f is a C^∞ function on $\mathbb{T}$. Multiplication by f is clearly a bounded operator on $C^\infty(\mathbb{T})$ in the L_2 norm and so extends to a bounded operator on $\mathbf{H}_0 = L_2(\mathbb{T})$. Similarly, it extends to a bounded operator on $\mathbf{H}_t, \quad t > 0$ since we can expand $D^p(fu)$ by applications of Leibnitz's rule for $u \in C^\infty(\mathbb{T})$. For $t = -s < 0$ we know by our theorem that $\mathbf{H}_{-s}$ is the dual space of $\mathbf{H}_s$ (and using the norm on $\mathbf{H}_{-s}$) that

$$\|fu\|_t = \sup |(v, fu)_0|/\|v\|_s = \sup |(u, \overline{f}v)_0|/\|v\|_s \leq \|u\|_t \sup \|\overline{f}v\|_s/\|v\|_s.$$

So in all cases we have

$$\|fu\|_t \leq (\text{const. depending on } f \text{and } t)\|u\|_t. \tag{9}$$

Differential operators with smooth coefficients.

Let

$$L = \sum_{|p| \leq m} \alpha_p(x) D^p$$

be a differential operator of degree m with C^∞ coefficients defined on $\mathbb{T}$. Then it follows from the above that

$$\|Lu\|_{t-m} \leq \text{constant}\|u\|_t \tag{10}$$

where the constant depends on L and t.

Rellich's lemma

Lemma

[Rellich's lemma.] *If $s < t$ the embedding $\mathbf{H}_t \hookrightarrow \mathbf{H}_s$ is compact.*

We must show that the image of the unit ball B of $\mathbf{H}_t$ in $\mathbf{H}_s$ can be covered by finitely many balls of radius ϵ.

To start the proof, Choose N so large that $(1 + \ell \cdot \ell)^{(s-t)/2} < \frac{\epsilon}{2}$ when $\ell \cdot \ell > N$.

Proof.

Let Z_t be the subspace of $\mathbf{H}_t$ consisting of all u such that $a_\ell = 0$ when $\ell \cdot \ell \leq N$.This is a space of finite codimension, and hence the unit ball of $Z_t^\perp \subset \mathbf{H}_t$ can be covered by finitely many balls of radius $\frac{\epsilon}{2}$. The space $Z_t^\perp$ consists of all u such that $a_\ell = 0$ when $\ell \cdot \ell > N$. The image of $Z_t^\perp$ in $\mathbf{H_s}$ is the orthogonal complement of the image of Z_t. On the other hand, for $u \in B \cap Z_t$ we have

$$\|u\|_s^2 \leq (1+N)^{s-t}\|u\|_t^2 \leq \left(\frac{\epsilon}{2}\right)^2.$$

So the image of $B \cap Z_t$ is contained in a ball of radius $\frac{\epsilon}{2}$. Every element of the image of B can be written as a(n orthogonal) sum of an element in the image of $B \cap Z_t^\perp$ and an element of $B \cap Z_t$ and so the image of B is covered by finitely many balls of radius ϵ. □

A useful numerical inequality.

Let $x > 0$ be a positive number, and a and b be non-negative numbers. Then

$$x^a + x^{-b} \geq 1$$

because if $x \geq 1$ the first summand is ≥ 1 and if $x \leq 1$ the second summand is ≥ 1. Setting $x = \epsilon^{1/a} A$ gives

$$1 \leq \epsilon A^a + \epsilon^{-b/a} A^{-b}$$

if ϵ and A are positive. Indeed, $x^a = \epsilon A^a$ and $x^{-b} = \epsilon^{-b/a} A^{-b}$.

A consequence.

$$1 \leq \epsilon A^a + \epsilon^{-b/a} A^{-b}$$

if ϵ and A are positive. Suppose that $t_1 > s > t_2$ and we set $a = t_1 - s,\ b = s - t_2$ and $A = 1 + \ell \cdot \ell$. Then we get

$$(1 + \ell \cdot \ell)^s \leq \epsilon (1 + \ell \cdot \ell)^{t_1} + \epsilon^{-(s-t_2)/(t_1-s)} (1 + \ell \cdot \ell)^{t_2}$$

and therefore

$$\|u\|_s \leq \epsilon \|u\|_{t_1} + \epsilon^{-(s-t_2)/(t_1-s)} \|u\|_{t_2} \text{ if } t_1 > s > t_2, \quad \epsilon > 0 \quad (11)$$

for all $u \in \mathbf{H}_{t_1}$. This elementary inequality will be the key to several arguments in today's lecture where we will combine (11) with integration by parts.

$\|u\|_s \leq \epsilon\|u\|_{t_1} + \epsilon^{-(s-t_2)/(t_1-s)}\|u\|_{t_2}$ if $t_1 > s > t_2, \quad \epsilon > 0.$ (11).

We may sometimes refer to (11) as the "little constant - big constant" inequality. It says that we can estimate $\|u\|_s$ in terms of a small constant times $\|u\|_{t_1}$ for $t_1 > s$ provided we add a large constant times $\|u\|_{t_2}$ for $t_2 < s$.

Elliptic differential operators.

A differential operator $L = \sum_{|p| \leq m} a_p(x) D^p$ with real coefficients and m even is called **elliptic** if there is a constant $c > 0$ such that

$$(-1)^{m/2} \sum_{|p|=m} a_p(x)\xi^p \geq c(\xi \cdot \xi)^{m/2} \quad \forall\ x,\ \xi. \tag{12}$$

For example, the operator $\Delta := -\left(\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^n)^2}\right)$ is elliptic. The vector ξ in (12) is a "dummy variable". (Its true significance is that it is a covector, i.e. an element of the cotangent space at x.) The expression on the left of (12) is called the **symbol** of the operator L. It is a homogeneous polynomial of degree m in the variable ξ whose coefficients are functions of x.

The symbol of L is sometimes written as $\sigma(L)$ or $\sigma(L)(x,\xi)$. Another way of expressing the ellipticity condition (12) is:

There is a positive constant c such that

$$\sigma(L)(x,\xi) \geq c \;\text{ for all } x \;\text{ and } \xi \text{ such that } \;\xi\cdot\xi = 1.$$

Statement of Gårding's inequality.

We will assume until further notice that the operator L is elliptic and that m is a positive even integer.

Theorem

[Gårding's inequality.] *For every $u \in C^\infty(\mathbb{T})$ we have*

$$(u, Lu)_0 \geq c_1 \|u\|_{m/2}^2 - c_2 \|u\|_0^2 \tag{13}$$

where $c_1 > 0$ and c_2 are constants depending on L.

Remark. If $u \in \mathbf{H}_{m/2}$, both sides of (13) make sense, and we can approximate u in the $\| \ \|_{m/2}$ norm by C^∞ functions. So once we prove the theorem, we conclude that it is true for all of $\mathbf{H}_{m/2}$.

Outline of steps in the proof of Gårding's inequality.

We will prove the theorem in stages:

1. When L is constant coefficient and homogeneous.
2. When L is homogeneous and approximately constant.
3. When the L can have lower order terms but the (top order) homogeneous part of L is approximately constant.
4. The general case.

Stage 1: L is constant coefficient and homogeneous.

$L = \sum_{|p|=m} \alpha_p D^p$ where the α_p are constants. Then

$$(u, Lu)_0 = \left(\sum a_\ell e^{i\ell\cdot x}, \sum_\ell \left(\sum_{|p|=m} \alpha_p (i\ell)^p \right) a_\ell e^{i\ell\cdot x} \right)_0$$

$$\geq c \sum_\ell (\ell \cdot \ell)^{m/2} |a_\ell|^2 \quad \text{by (12), the ellipticity condition,}$$

$$= c \sum [1 + (\ell \cdot \ell)^{m/2}] |a_\ell|^2 - c\|u\|_0^2 \geq cC\|u\|_{m/2}^2 - c\|u\|_0$$

where

$$C = \inf_{r \geq 0} \frac{1 + r^{m/2}}{(1+r)^{m/2}} > 0.$$

Stage 1: L is constant coefficient and homogeneous, continued.

Indeed, we have proved the estimate which involves the coefficient $[1+(\ell\cdot\ell)^{m/2}]$ and we want to have $(1+\ell\cdot\ell)^{m/2}$. But the quotient

$$\frac{1+r^{m/2}}{(1+r)^{m/2}}$$

is positive everywhere, equals 1 at $r=0$ and tends to one as $r\to\infty$. So it has a positive minimum.
This takes care of stage 1.

Stage 2.

Here $L = L_0 + L_1$ where L_0 is as in stage 1 and $L_1 = \sum_{|p|=m} \beta_p(x) D^p$ and

$$\max_{p,x} |\beta_p(x)| < \eta,$$

where η sufficiently small. (How small will be determined very soon in the course of the discussion.) We have

$$(u, L_0 u)_0 \geq c' \|u\|_{m/2}^2 - c\|u\|_0^2$$

from stage 1.

Stage 2, continued.

We integrate $(u, L_1 u)_0$ by parts $m/2$ times. There are no boundary terms since we are on the torus. In integrating by parts some of the derivatives will hit the coefficients. Let us collect all the these terms as I_2. The other terms we collect as I_1, so

$$I_1 = \sum \int b_{p'+p''} D^{p'} u \overline{D^{p''} u} dx$$

where $|p'| = |p''| = m/2$ and $b_r = \pm\beta_r$. We can estimate this sum by

$$|I_1| \leq \eta \cdot \text{const.} \|u\|^2_{m/2}$$

and so will require that $\eta \cdot (\text{const.}) < c'$.
The constant depends on m but not, otherwise, on L.

Stage 2, continued.

The remaining terms give a sum of the form

$$I_2 = \sum \int b_{p'q} D^{p'} u \overline{D^q u} dx$$

where $p' \leq m/2, q' < m/2$ so we have

$$|I_2| \leq \text{const.} \|u\|_{\frac{m}{2}} \|u\|_{\frac{m}{2}-1}.$$

Recall the "little constant big constant inequality":

$$\|u\|_s \leq \epsilon \|u\|_{t_1} + \epsilon^{-(s-t_2)/(t_1-s)} \|u\|_{t_2} \quad \text{if } t_1 > s > t_2, \quad \epsilon > 0 \quad (11).$$

Stage 2, continued, using

$$\|u\|_s \leq \epsilon\|u\|_{t_1}+\epsilon^{-(s-t_2)/(t_1-s)}\|u\|_{t_2} \text{ if } t_1 > s > t_2, \ \ \epsilon > 0 \quad (11).$$

.

Take $s = \frac{m}{2} - 1, t_1 = \frac{m}{2}, t_2 = 0$, so $s - t_2 = \frac{m}{2} - 1, \ t - s = 1$ so we get, for any $\epsilon > 0$,

$$\|u\|_{\frac{m}{2}-1} \leq \epsilon\|u\|_{\frac{m}{2}} + \epsilon^{1-\frac{m}{2}}\|u\|_0.$$

Stage 2, continued.

We have

$$|I_2| \leq \text{const.} \|u\|_{\frac{m}{2}} \|u\|_{\frac{m}{2}-1}$$

and

$$\|u\|_{\frac{m}{2}-1} \leq \epsilon \|u\|_{\frac{m}{2}} + \epsilon^{1-\frac{m}{2}} \|u\|_0.$$

Substituting this inequality into the above estimate for I_2 gives

$$|I_2| \leq \epsilon \cdot \text{const.} \|u\|^2_{m/2} + \epsilon^{1-\frac{m}{2}} \text{const.} \|u\|_{m/2} \|u\|_0.$$

We now will use another numerical inequality to control the second term:

Stage 2, continued.

For any positive numbers a, b and ζ the inequality $(\zeta a - \zeta^{-1} b)^2 \geq 0$ implies that $2ab \leq \zeta^2 a^2 + \zeta^{-2} b^2$. Taking $\zeta^2 = \epsilon^{\frac{m}{2}+1}$ we can replace $\epsilon^{1-\frac{m}{2}} \|u\|_{m/2} \|u\|_0$ by

$$\frac{1}{2}\left[\epsilon^2 \|u\|^2_{\frac{m}{2}} + \epsilon^{-m} \|u\|^2_0\right].$$

We can absorb $\frac{1}{2}\epsilon \times$ const. into the constant in the first term in

$$|I_2| \leq \epsilon \cdot \text{const.} \|u\|^2_{m/2} + \epsilon^{1-\frac{m}{2}} \text{const.} \|u\|_{m/2} \|u\|_0$$

if ϵ is small enough.

Stage 2, concluded.

We have thus established that

$$|I_1| \leq \eta \cdot (\text{const.})_1 \|u\|_{m/2}^2$$

where the constant depends only on m, and

$$|I_2| \leq \epsilon(\text{const.})_2 \|u\|_{m/2}^2 + \epsilon^{-m}\text{const.}\|u\|_0^2$$

where the constants depend on L_1 but ϵ is at our disposal. So if $\eta(\text{const.})_1 < c'$ and we then choose ϵ so that $\epsilon(\text{const.})_2 < c' - \eta \cdot (\text{const.})_1$ we obtain Gårding's inequality for Stage 2.

Stage 3.

Here $L = L_0 + L_1 + L_2$ where L_0 and L_1 are as in stage 2, and L_2 is a lower order operator. Here we integrate by parts and argue as in stage 2.

Stage 4, the general case.

Choose an open covering of $\mathbb{T}$ such that the variation of each of the highest order coefficients in each open set is less than the η of stage 2. (Recall that this choice of η depended only on m and the c that entered into the definition of ellipticity.) Thus, if v is a smooth function supported in one of the sets of our cover, the action of L on v is the same as the action of an operator as in case 3) on v, and so we may apply Gårding's inequality. Choose a finite subcover and a partition of unity $\{\phi_i\}$ subordinate to this cover. Write $\phi_i = \psi_i^2$ (where we choose the ϕ so that the ψ are smooth). So $\sum \psi_i^2 \equiv 1$.

Stage 4, continued.

So $\sum \psi_i^2 \equiv 1$. Now $\quad (\psi_i u, L(\psi_i u))_0 \geq c'' \|\psi_i u\|^2_{m/2} - \text{const.} \|\psi_i u\|_0^2$ where c'' is a positive constant depending only on c, η, and on the lower order terms in L. We have

$$(u, Lu)_0 = \int (\sum \psi_i^2 u) \overline{Lu} dx = \sum (\psi_i u, L\psi_i u)_0 + R$$

where R involves derivatives of the ψ_i and hence lower order derivatives of u. These can be estimated as in case 2) above, and so we get

$$(u, Lu)_0 \geq c''' \sum \|\psi_i u\|^2_{m/2} - \text{const.} \|u\|_0^2 \qquad (14)$$

since $\|\psi_i u\|_0 \leq \|u\|_0$.

Stage 4, continued.

$$(u, Lu)_0 \geq c''' \sum \|\psi_i u\|^2_{m/2} - \text{const.}\|u\|_0^2 \quad (14)$$

since $\|\psi_i u\|_0 \leq \|u\|_0$. Now $\|u\|_{m/2}$ is equivalent, as a norm, to $\sum_{p \leq m/2} \|D^p u\|_0$ as we verified in the last lecture. Also

$$\sum \|D^p(\psi_i u)\|_0^2 = \sum \|\psi_i D^p u\|_0^2 + R'$$

where R' involves terms differentiating the ψ and so lower order derivatives of u. Hence

$$\sum \|\psi_i u\|^2_{m/2} \geq \text{pos. const.}\|u\|^2_{m/2} - \text{const.}\|u\|_0^2$$

by the integration by parts argument again.

Stage 4, concluded.

Hence by (14)

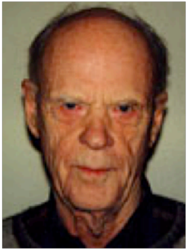
$$(u, Lu)_0 \geq c''' \sum \|\psi_i u\|^2_{m/2} - \text{const.}\|u\|^2_0$$

$$\geq \text{pos. const.}\|u\|^2_{m/2} - \text{const.}\|u\|^2_0$$

which is Gårding's inequality. □

For the time being we will continue to study the case of the torus. But a look ahead is in order. In this last step of the argument, where we applied the partition of unity argument, we have really freed ourselves of the restriction of being on the torus. Once we make the appropriate definitions, we will then get Gårding's inequality for elliptic operators on manifolds. Furthermore, the consequences we are about to draw from Gårding's inequality will be equally valid in the more general setting.

Lars Gårding



Born in 1919

Died July 7, 2014.

Lemma

For every integer t there is a constant $c(t) = c(t, L)$ and a positive number $\Lambda = \Lambda(t, L)$ such that

$$\|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m} \qquad (15)$$

when

$$\lambda > \Lambda$$

for all smooth u, and hence for all $u \in \mathbf{H}_t$.

Proof. Let s be some non-negative integer. We will first prove (15) for $t = s + \frac{m}{2}$.

To prove: $\|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m} \quad (15).$

Case $t = s + \frac{m}{2}$.

We have

$$\|u\|_t\|Lu + \lambda u\|_{t-m} = \|u\|_t\|Lu + \lambda u\|_{s-\frac{m}{2}}$$

$$= \|u\|_t\|(1+\Delta)^s Lu + \lambda(1+\Delta)^s u\|_{-s-\frac{m}{2}}$$

$$\geq |(u, (1+\Delta)^s Lu + \lambda(1+\Delta)^s u)_0|$$

by the generalized Cauchy - Schwarz inequality.

To prove: $\|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m} \quad (15)$.

Case $t = s + \frac{m}{2}$, continued.

The operator $(1+\Delta)^s L$ is elliptic of order $m + 2s$ so

$$\|(1+\Delta)^t u\|_s = \|u\|_{s+2t}$$

and Gårding's inequality gives

$$(u, (1+\Delta)^s Lu + \lambda(1+\Delta)^s u)_0 \geq c_1\|u\|^2_{s+\frac{m}{2}} - c_2\|u\|_0^2 + \lambda\|u\|_s^2.$$

Since $\|u\|_s \geq \|u\|_0$ we can combine the two previous inequalities to get

$$\|u\|_t\|Lu + \lambda u\|_{t-m} \geq c_1\|u\|_t^2 + (\lambda - c_2)\|u\|_0^2.$$

If $\lambda > c_2$ we can drop the second term and divide by $\|u\|_t$ to obtain (15).

To prove: $\|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m} \quad (15)$.

Case $t = \frac{m}{2} - s$.

We now prove the lemma for the case $t = \frac{m}{2} - s$ by the same sort of argument: We have

$$\begin{aligned}\|u\|_t\|Lu + \lambda u\|_{-s-\frac{m}{2}} &= \|(1+\Delta)^{-s}u\|_{s+\frac{m}{2}}\|Lu + \lambda u\|_{-s-\frac{m}{2}} \\ &\geq ((1+\Delta)^{-s}u, L(1+\Delta)^s(1+\Delta)^{-s}u + \lambda u)_0.\end{aligned}$$

Now use the fact that $L(1+\Delta)^s$ is elliptic of order $m+2s$ and Gårding's inequality to continue the above inequalities as

$$\begin{aligned}&\geq c_1\|(1+\Delta)^{-s}u\|^2_{s+\frac{m}{2}} - c_2\|(1+\Delta)^{-s}u\|^2_0 + \lambda\|u\|^2_{-s} \\ &= c_1\|u\|^2_t - c_2\|u\|^2_{-2s} + \lambda\|u\|^2_{-s} \geq c_1\|u\|^2_t\end{aligned}$$

if $\lambda > c_2$. Again we may then divide by $\|u\|_t$ to get the result. □

Using (15).

$$\|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m}. \qquad (15)$$

The operator $L + \lambda I$ is a bounded operator from $\mathbf{H}_t$ to $\mathbf{H}_{t-m}$ (for any t). Suppose we fix t and choose λ so large that (15) holds. Then (15) says that $(L + \lambda I)$ is invertible on its image, and this inverse is bounded there with a bound independent of $\lambda > \Lambda$.

The boundedness of $(L + \lambda I)^{-1}$ on its image, implies that this image is a closed subspace of $\mathbf{H}_{t-m}$. Indeed, if $v_n = (L + \lambda I)u_n$ and $v_n \to v$ then the v_n form a Cauchy sequence and hence so do the u_n. So $u_n \to u$ and we conclude that $v = (L + \lambda I)u$.

$L + \lambda I$ is surjective for large λ.

Let us show that this image is all of $\mathbf{H}_{t-m}$ for λ large enough. Suppose not, which means that there is some $w \in \mathbf{H}_{t-m}$ with

$$(w, Lu + \lambda u)_{t-m} = 0$$

for all $u \in \mathbf{H}_t$. We can write this last equation as

$$((1 + \Delta)^{t-m} w, Lu + \lambda u)_0 = 0.$$

Introducing the formal adjoint.

We are assuming that

$$((1+\Delta)^{t-m}w, Lu+\lambda u)_0 = 0$$

and want to prove that $w = 0$. Integration by parts gives the adjoint differential operator L^* characterized by

$$(\phi, L\psi)_0 = (L^*\phi, \psi)_0$$

for all smooth functions ϕ and ψ, and by passing to the limit this holds for all $\phi \in \mathbf{H}_s$, $\psi \in \mathbf{H}_t$ with $s+t \geq m$. The operator L^* has the same leading term as L and hence is elliptic.

Using $\|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m}$ (15), applied to L^*.

So let us choose λ sufficiently large that (15) holds for L^* as well as for L. Now

$$0 = \left((1+\Delta)^{t-m}w, Lu + \lambda u\right)_0$$

$$= \left(L^*(1+\Delta)^{t-m}w + \lambda(1+\Delta)^{t-m}w, u\right)_0$$

for all $u \in \mathbf{H}_t$. Write this as

$$(L^*v + \lambda v, u)_0 = 0 \quad v := (1+\Delta)^{t-m}w.$$

Hence (by (15)) applied to L^* and v, we get that $(1+\Delta)^{t-m}w = 0$ so $w = 0$. We have proved

Theorem

For every t and for λ large enough (depending on t) the operator $L + \lambda I$ maps $\mathbf{H}_t$ bijectively onto $\mathbf{H}_{t-m}$ and $(L + \lambda I)^{-1}$ is bounded independently of λ.

As an immediate application we get the important

Theorem

If u is a distribution and $Lu \in \mathbf{H}_s$ then $u \in \mathbf{H}_{s+m}$.

Proof.

Write $f = Lu$. By Schwartz's theorem, we know that $u \in \mathbf{H}_k$ for some k. So $f + \lambda u \in \mathbf{H}_{\min(k,s)}$ for any λ. Choosing λ large enough, we conclude that $u = (L + \lambda I)^{-1}(f + \lambda u) \in \mathbf{H}_{\min(k+m,s+m)}$. If $k + m < s + m$ we can repeat the argument to conclude that $u \in \mathbf{H}_{\min(k+2m,s+m)}$. we can keep going until we conclude that $u \in \mathbf{H}_{s+m}$. □

The key idea in this argument goes back to a 1940 paper by Hermann Weyl in the *Annals of Mathematics* entitled "The method of orthogonal projection in potential theory."

Hermann Klaus Hugo Weyl (1885-1955)

Notice as an important corollary that any solution of the homogeneous equation $Lu = 0$ is C^∞.
Replacing the operator L by $L - \lambda I$ we conclude that any solution of $Lu = \lambda u$ is C^∞.

So if we have found an eigenvector of L, we know automatically that it is C^∞.

We have proved

Theorem

For every t and for λ large enough (depending on t) the operator $L + \lambda I$ maps $\mathbf{H}_t$ bijectively onto $\mathbf{H}_{t-m}$ and $(L + \lambda I)^{-1}$ is bounded independently of λ.

We now draw a second important consequence of this theorem:

Using Rellich.

Choose λ so large that the operators

$$(L+\lambda I)^{-1} \quad \text{and} \quad (L^*+\lambda I)^{-1}$$

exist as bounded operators from $\mathbf{H}_0 \to \mathbf{H}_m$. Follow these operators with the injection $\iota_m : \mathbf{H}_m \to \mathbf{H}_0$ and set

$$M := \iota_m \circ (L+\lambda I)^{-1}, \qquad M^* := \iota_m \circ (L^*+\lambda I)^{-1}.$$

Since ι_m is compact (Rellich's lemma) and the composite of a compact operator with a bounded operator is compact, we conclude

Theorem

The operators M and M^* are compact.

Using our theorem about compact selfadjiont operators.

Suppose that $L = L^*$. (This is usually expressed by saying that L is "formally self-adjoint". More on this terminology will come later.) This implies that $M = M^*$. In other words, M is a compact self adjoint operator, and we can apply the spectral theorem for compact operators to conclude that eigenvectors of M form a basis of $R(M)$ and that the corresponding eigenvalues tend to zero. Our theorem says that $R(M)$ is the same as $\iota_m(\mathbf{H}_m)$ which is dense in $\mathbf{H}_0 = L_2(\mathbb{T})$. We conclude that the eigenvectors of M form a basis of $L_2(\mathbb{T})$. If $Mu = ru$ then $u = (L + \lambda I)Mu = rLu + \lambda ru$ so u is an eigenvector of L with eigenvalue

$$\frac{1 - r\lambda}{r}.$$

We conclude that the eigenvectors of L are a basis of $\mathbf{H}_0$.

Only finitely many eigenvalues of L are negative.

The eigenvalues of L are

$$\frac{1 - r\lambda}{r}$$

and the eigenvectors of L are a basis of $\mathbf{H}_0$. We claim that only finitely many of these eigenvalues of L can be negative. Indeed, since we know that the eigenvalues r_n of M approach zero, the numerator in the above expression is positive, for large enough n, and hence if there were infinitely many negative eigenvalues μ_k, they would have to correspond to negative r_k and so these $\mu_k \to -\infty$. I claim that this is impossible:

Only finitely many eigenvalues of L are negative, continued.

We know that

$$\|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m} \quad (15).$$

Indeed, taking $s_k = -\mu_k$ as the λ in (15) we conclude that $u = 0$, if $Lu = \mu_k u$ if k is large enough, contradicting the definition of an eigenvector. So all but a finite number of the r_n are positive, and these tend to zero. To summarize:

Theorem

The eigenvectors of L are C^∞ functions which form a basis of $\mathbf{H}_0$. Only finitely many of the eigenvalues μ_k of L are negative and $\mu_n \to \infty$ as $n \to \infty$.

It is easy to extend the results obtained above for the torus in two directions. One is to consider functions defined in a **domain** = bounded open set $\mathcal{G}$ of $\mathbb{R}^n$ and the other is to consider functions defined on a compact manifold. In both cases a few elementary tricks allow us to reduce to the torus case. We sketch what is involved for the manifold case.

Let $E \to X$ be a vector bundle over a manifold. We assume that X is equipped with a density which we shall denote by $|dx|$ and that E is equipped with a positive definite (smoothly varying) scalar product, so that we can define the L_2 norm of a smooth section s of E of compact support:

$$\|s\|_0^2 := \int_M |s|^2(x)|dx|.$$

Suppose for the rest of this lecture that X is compact.

Let $\{U_i\}$ be a finite cover of X by coordinate neighborhoods over which E has a given trivialization, and ρ_i a partition of unity subordinate to this cover. Let ϕ_i be a diffeomorphism of U_i with an open subset of $\mathbb{T}^n$ where n is the dimension of X. Then if s is a smooth section of E, we can think of $(\rho_i s) \circ \phi_i^{-1}$ as an $\mathbb{R}^m$ or $\mathbb{C}^m$ valued function on $\mathbb{T}^n$, and consider the sum of the $\|\cdot\|_k$ norms applied to each component. We shall continue to denote this sum by $\|\rho_i f \circ \phi_i^{-1}\|_k$ and then define

$$\|f\|_k := \sum_i \|\rho_i f \circ \phi_i^{-1}\|_k$$

where the norms on the right are in the norms on the torus. These norms depend on the trivializations and on the partitions of unity.

These norms depend on the trivializations and on the partitions of unity. But any two norms are equivalent, and the $\| \;\|_0$ norm is equivalent to the "intrinsic" L_2 norm defined above. We define the Sobolev spaces $\mathbf{W}_k$ to be the completion of the space of smooth sections of E relative to the norm $\| \;\|_k$ for $k \geq 0$, and these spaces are well defined as topological vector spaces independently of the choices. Since Sobolev's lemma holds locally, it goes through unchanged.

Similarly Rellich's lemma: if s_n is a sequence of elements of $\mathbf{W}_\ell$ which is bounded in the $\| \|_\ell$ norm for $\ell > k$, then each of the elements $\rho_i s_n \circ \phi_i^{-1}$ belong to $\mathbf{H}_\ell$ on the torus, and are bounded in the $\| \|_\ell$ norm, hence we can select a subsequence of $\rho_1 s_n \circ \phi_1^{-1}$ which converges in $\mathbf{H}_k$, then a subsubsequence such that $\rho_i s_n \circ \phi_i^{-1}$ for $i = 1, 2$ converge etc. arriving at a subsequence of s_n which converges in $\mathbf{W}_k$.

A differential operator L mapping sections of E into sections of E is an operator whose local expression (in terms of a trivialization and a coordinate chart) has the form

$$Ls = \sum_{|p| \leq m} \alpha_p(x) D^p s$$

Here the a_p are linear maps (or matrices if our trivializations are in terms of $\mathbb{R}^m$).
Under changes of coordinates and trivializations the change in the coefficients are rather complicated, but the **symbol** of the differential operator

$$\sigma(L)(\xi) := \sum_{|p|=m} a_p(x) \xi^p \quad \xi \in T^* X_x$$

is well defined.

If we put a Riemann metric on the manifold, we can talk about the length $|\xi|$ of any cotangent vector.
If L is a differential operator from E to itself (i.e. $F{=}E$) we shall call L **even elliptic** if m is even and there exists some constant C such that

$$\langle v, \sigma(L)(\xi)v\rangle \geq C|\xi|^m|v|^2$$

for all $x \in X,\ v \in E_x,\ \xi \in T^*X_x$ and $\langle\ ,\ \rangle$ denotes the scalar product on E_x. Gårding's inequality holds. Indeed, locally, this is just a restatement of the (vector valued version) of Gårding's inequality that we have already proved for the torus. But Stage 4 in the proof extends unchanged (other than the replacement of scalar valued functions by vector valued functions) to the more general case.

We assume knowledge of the basic facts about differentiable manifolds, in particular the existence of an operator $d : \Omega^k \to \Omega^{k+1}$ with its usual properties, where Ω^k denotes the space of exterior k-forms. Also, if X is orientable and carries a Riemann metric then the Riemann metric induces a scalar product on the exterior powers of T^*X and also picks out a volume form. So there is an induced scalar product $(\ ,\ ) = (\ ,\ )_k$ on Ω^k and a formal adjoint δ of d

$$\delta : \ \Omega^k \to \Omega^{k-1}$$

which satisfies

$$(d\psi, \phi) = (\phi, \delta\phi)$$

where ϕ is a $(k+1)$-form and ψ is a k-form.

The Hodge operator.

Then

$$\Delta := d\delta + \delta d$$

is a second order differential operator on Ω^k and satisfies

$$(\Delta\phi, \phi) = \|d\phi\|^2 + \|\delta\phi\|^2$$

where $\|\phi\|\|^2 = (\phi, \phi)$ is the intrinsic L_2 norm (so $\|\ \| = \|\ \|_0$ in terms of the notation of the preceding section).

Furthermore, if

$$\phi = \sum_I \phi_I dx^I$$

is a local expression for the differential form ϕ, where

$$dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \quad I = (i_1, \ldots, i_k)$$

then a local expression for Δ is

$$\Delta\phi = -\sum g^{ij} \frac{\partial \phi_I}{\partial x^i \partial x^j} + \cdots$$

where

$$g^{ij} = \langle dx^i, dx^j \rangle$$

and the $\cdots$ are lower order derivatives. In particular Δ is elliptic.

Let $\phi \in \Omega^k$ and suppose that

$$d\phi = 0.$$

Let $\mathcal{C}(\phi)$, the **cohomology class** of ϕ be the set of all $\psi \in \Omega^k$ which satisfy

$$\phi - \psi = d\alpha, \quad \alpha \in \Omega^{k-1}$$

and let

$$\overline{\mathcal{C}(\phi)}$$

denote the closure of $\mathcal{C}$ in the L_2 norm. It is a closed subspace of the Hilbert space obtained by completing Ω^k relative to its L_2 norm. Let us denote this space by L_2^k, so $\overline{\mathcal{C}(\phi)}$ is a closed subspace of L_2^k.

Theorem

If $\phi \in \Omega^k$ *and* $d\phi = 0$*, there exists a unique* $\tau \in \overline{\mathcal{C}(\phi)}$ *such that*

$$\|\tau\| \leq \|\psi\| \ \ \forall\ \psi \in \mathcal{C}(\phi).$$

Furthermore, τ *is smooth,and*

$$d\tau = 0 \quad \textit{and} \quad \delta\tau = 0.$$

Proof.

If we choose a minimizing sequence for $\|\psi\|$ in $\mathcal{C}(\phi)$ we know it is Cauchy, cf. the proof of the existence of orthogonal projections in a Hilbert space. So we know that τ exists and is unique. For any $\alpha \in \Omega^{k+1}$ we have

$$(\tau, \delta\alpha) = \lim(\psi, \delta\alpha) = \lim(d\psi, \alpha) = 0$$

as ψ ranges over a minimizing sequence. The equation $(\tau, \delta\alpha) = 0$ for all $\alpha \in \Omega^{k+1}$ says that τ is a weak solution of the equation $d\tau = 0$.

We claim that

$$(\tau, d\beta) = 0 \quad \forall \; \beta \in \Omega^{k-1}$$

which says that τ is a weak solution of $\delta\tau = 0$.

Indeed, for any $t \in \mathbb{R}$,

$$\|\tau\|^2 \leq \|\tau + td\beta\|^2 = \|\tau\|^2 + t^2\|d\beta\|^2 + 2t(\tau, d\beta)$$

so $\quad -2t(\tau, d\beta) \leq t^2\|d\beta\|^2.$
If $(\tau, d\beta) \neq 0$, we can choose

$$t = -\epsilon\frac{(\tau, d\beta)}{|(\tau, d\beta)|}, \quad \epsilon > 0$$

so

$$|(\tau, d\beta)| \leq \frac{1}{2}\epsilon|d\beta|^2.$$

As ϵ is arbitrary, this implies that $(\tau, d\beta) = 0$.

So $(\tau, \Delta\psi) = (\tau, [d\delta + \delta d]\psi) = 0$ for any $\psi \in \Omega^k$. Hence τ is a weak solution of $\Delta\tau = 0$ and so is smooth. The space $\mathcal{H}^k$ of weak, and hence smooth solutions of $\Delta\tau = 0$ is finite dimensional by the general theory. It is called the space of **harmonic forms**. We have seen that there is a unique harmonic form in the cohomology class of any closed form, and that the cohomology groups are finite dimensional. $\square$

In fact, the general theory tells us that

$$L_2^k = \bigoplus_\lambda E_\lambda^k$$

(Hilbert space direct sum) where E_λ^k is the eigenspace with eigenvalue λ of Δ. Each E_λ is finite dimensional and consists of smooth forms, and the $\lambda \to \infty$. The eigenspace E_0^k is just $\mathcal{H}^k$, the space of harmonic forms. Also, since

$$(\Delta\phi, \phi) = \|d\phi\|^2 + \|\delta\phi\|^2$$

we know that all the eigenvalues λ are non-negative.

Since $d\Delta = d(d\delta + \delta d) = d\delta d = \Delta d$, we see that

$$d : E_\lambda^k \to E_\lambda^{k+1}$$

and similarly

$$\delta : E_\lambda^k \to E_\lambda^{k-1}.$$

For $\lambda \neq 0$, if $\phi \in E_\lambda^k$ and $d\phi = 0$, then $\lambda\phi = \Delta\phi = d\delta\phi$ so $\phi = d(1/\lambda)\delta\phi$ so d restricted to the E_λ is exact, and similarly so is δ. Furthermore, on $\bigoplus_k E_\lambda^k$ we have $\lambda I = \Delta = (d+\delta)^2$ so

$$E_\lambda^k = dE_\lambda^{k-1} \oplus \delta E_\lambda^{k+1}.$$

This decomposition is orthogonal since $(d\alpha, \delta\beta) = (d^2\alpha, \beta) = 0$.

As a first consequence we see that

$$L_2^k = \mathcal{H}^k \oplus \overline{d\Omega^{k-1}} \oplus \overline{\delta\Omega^{k+1}}$$

(the Hodge decomposition). If H denotes projection onto the first component, then Δ is invertible on the image of $I - H$ with an inverse there which is compact. So if we let N denote this inverse on im $I - H$ and set $N = 0$ on $\mathcal{H}^k$ we get:

$$\begin{aligned} \Delta N &= I - H \\ Nd &= dN \\ \delta N &= N\delta \\ \Delta N &= N\Delta \\ NH &= 0 \end{aligned}$$

which are the fundamental assertions of Hodge theory, together with the assertion proved above that $H\phi$ is the unique minimizing element in its cohomology class.

We have seen that

$$d+\delta : \bigoplus_k E_\lambda^{2k} \to \bigoplus_k E_\lambda^{2k+1} \text{ is an isomorphism for } \lambda \neq 0 \quad (16)$$

which of course implies that

$$\sum_k (-1)^k \dim E_\lambda^k = 0$$

This shows that the index of the operator $d+\delta$ acting on $\bigoplus L_2^k$ is the Euler characteristic of the manifold. (The index of any operator is the difference between the dimensions of the kernel and cokernel).

Let $P_{k,\lambda}$ denote the projection of L_2^k onto E_λ^k. So

$$e^{-t\Delta} = \sum e^{-\lambda t} P_{k,\lambda}$$

is the solution of the heat equation on L_2^k. As $t \to \infty$ this approaches the operator H projecting L_2^k onto $\mathcal{H}_k$. Letting Δ_k denote the operator Δ on L_2^k we see that

$$\mathrm{tr} e^{-t\Delta_k} = \sum e^{-\lambda_k}$$

where the sum is over all eigenvalues λ_k of Δ_k counted with multiplicity. It follows from (16) that the alternating sum over k of the corresponding sum over non-zero eigenvalues vanishes. Hence

$$\sum (-1)^k \mathrm{tr}\; e^{-t\Delta_k} = \chi(X)$$

is independent of t. The index theorem computes this trace for small values of t in terms of local geometric invariants.

The operator $d + \delta$ is an example of a Dirac operator whose general definition we will not give here. The corresponding assertion and local evaluation is the content of the celebrated Atiyah-Singer index theorem, one of the most important theorems discovered in the twentieth century.

The resolvent

In order to connect what we have done here with notation that will come later, it is convenient to let $A = -L$ so that now the operator

$$(zI - A)^{-1}$$

is compact as an operator on $\mathbf{H}_0$ for z sufficiently positive. (I have dropped the ι_m which should come in front of this expression.)

The operator A now has only finitely many positive eigenvalues, with the corresponding spaces of eigenvectors being finite dimensional. In fact, the eigenvectors $\lambda_n = \lambda_n(A)$ (counted with multiplicity) approach $-\infty$ as $n \to \infty$ and the operator $(zI - A)^{-1}$ exists and is a bounded (in fact compact) operator so long as $z \neq \lambda_n$ for any n. Indeed, we can write any $u \in \mathbf{H}_0$ as

$$u = \sum_n a_n \phi_n$$

where ϕ_n is an eigenvector of A with eigenvalue λ_n and the ϕ form an orthonormal basis of $\mathbf{H}_0$.

Then

$$(zI - A)^{-1}u = \sum \frac{1}{z - \lambda_n} a_n \phi_n.$$

The operator $(zI - A)^{-1}$ is called the **resolvent** of A at the point z and denoted by

$$R(z, A)$$

or simply by $R(z)$ if A is fixed. So

$$R(z, A) := (zI - A)^{-1}$$

for those values of $z \in \mathbb{C}$ for which the right hand side is defined.

If z and are complex numbers with $\mathrm{Re} z > \mathrm{Re} a$, then the integral

$$\int_0^\infty e^{-zt} e^{at} dt$$

converges, and we can evaluate it as

$$\frac{1}{z-a} = \int_0^\infty e^{-zt} e^{at} dt.$$

If $\mathrm{Re} z$ is greater than the largest of the eigenvalues of A we can write

$$R(z, A) = \int_0^\infty e^{-zt} e^{tA} dt$$

where we may interpret this equation as a shorthand for doing the integral for the coefficient of each eigenvector, or as an operator valued integral. We will spend a lot of time in this course generalizing this formula and deriving many consequences from it.

Summary,

1. Review of Sobolev spaces.
   - Distributions and Schwartz's theorem.
2. Gårding's inequality.
   - Differential operators.
   - Rellich's lemma
   - Some numerical inequalities.
   - Elliptic operators.
   - Statement and proof of Gårding's inequality.
3. Consequences of Gårding's inequality.
4. Extension of the basic lemmas to manifolds.
   - Example: Hodge theory.
5. The resolvent.