Math212a1404
Applications of the spectral theorem for compact self-adjoint operators, 1.

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Contents

Introduction.

In the handout of the preceding lecture, we proved that the functions e^{inx} constitute an orthonormal basis of $L_2(\mathbb{T})$ by elementary means. In today's lecture we will give a much more complicated proof of this fact using the spectral theorem that we proved in the last lecture. But this complicated proof will extend to much more general situations.

Review: The key theorem on compact self-adjoint transformations.

The main theorem we proved last time:

Theorem

Let T be a compact self-adjoint operator on a pre-Hilbert space. Then $R(T)$ has an orthonormal basis $\{\phi_i\}$ consisting of eigenvectors of T and if $R(T)$ is infinite dimensional then the corresponding sequence $\{r_n\}$ of eigenvalues converges to 0.

We briefly recall the proof:

Non-negative bounded symmetric operators.

For a non-negative bounded symmetric operator T, we proved, using two different applications of the Cauchy-Schwarz inequality, that

$$\|Tv\| \leq \|T\|^{\frac{1}{2}}(Tv,v)^{\frac{1}{2}}.$$

It follows from this that

$$(Tv,v) = 0 \quad \Rightarrow \quad Tv = 0$$

and that if we have a sequence $\{v_n\}$ of vectors with $(Tv_n, v_n) \to 0$ then $\|Tv_n\| \to 0$ and so

$$(Tv_n, v_n) \to 0 \quad \Rightarrow \quad Tv_n \to 0.$$

For any bounded symmetric operator T we can apply this to $rI - T$ if $r \geq \|T\|$. The proof of the theorem then broke up into 8 easy steps:

Proof step 1.

We know that T is bounded. If $T = 0$ there is nothing to prove. So assume that $T \neq 0$ and let

$$m_1 := \|T\| > 0.$$

By the definition of $\|T\|$ we can find a sequence of vectors $u_n \in \mathbf{S}$ such that $\|Tu_n\| \to \|T\|$. By the definition of compactness we can find a subsequence of this sequence so that $Tu_{n_i} \to w$ for some $w \in V$. On the other hand, the transformation T^2 is self-adjoint and bounded by $\|T\|^2$. Hence $\|T\|^2 I - T^2$ is non-negative, and

$$((\|T\|^2 I - T^2)u_n, u_n) = \|T\|^2 - \|Tu_n\|^2 \to 0.$$

So we know from the above that

$$\|T\|^2 u_n - T^2 u_n \to 0.$$

We know that

$$m_1^2 u_n - T^2 u_n = \|T\|^2 u_n - T^2 u_n \to 0.$$

Applied to the subsequence u_{n_i} this says that

$$m_1^2 u_{n_i} \to Tw$$

since $Tu_{n_i} \to w$ and T is continuous. So we have proved that

$$u_{n_i} \to \frac{1}{m_1^2} Tw.$$

Proof step 2.

$$u_{n_i} \to \frac{1}{m_1^2} Tw.$$

Applying T to this we get

$$Tu_{n_i} \to \frac{1}{m_1^2} T^2 w$$

or

$$T^2 w = m_1^2 w.$$

Also $\|w\| = \|T\| = m_1 \neq 0$. So $w \neq 0$. So w is an eigenvector of T^2 with eigenvalue m_1^2. We have

$$0 = (T^2 - m_1^2 I)w = (T + m_1 I)(T - m_1 I)w.$$

Proof step 3.

We know that $0 = (T^2 - m_1^2 I)w = (T + m_1 I)(T - m_1 I)w$.
If $(T - m_1 I)w = 0$, then w is an eigenvector of T with eigenvalue m_1 and we normalize it by setting

$$\phi_1 := \frac{1}{\|w\|} w.$$

Then $\|\phi_1\| = 1$ and

$$T\phi_1 = m_1\phi_1.$$

If $(T - m_1 I)w \neq 0$ then $y := (T - m_1 I)w$ is an eigenvector of T with eigenvalue $-m_1$ and again we normalize by setting

$$\phi_1 := \frac{1}{\|y\|} y.$$

So we have found a unit vector $\phi_1 \in R(T)$ which is an eigenvector of T with eigenvalue $r_1 = \pm m_1$.

Proof step 4.

Now let

$$V_2 := \phi_1^{\perp}.$$

If $x \in V_2$ so that $(x, \phi_1) = 0$, then

$$(Tx, \phi_1) = (x, T\phi_1) = r_1(x, \phi_1) = 0.$$

In other words,

$$T(V_2) \subset V_2$$

and we can consider the linear transformation T restricted to V_2 which is again compact. If we let m_2 denote the norm of the linear transformation T when restricted to V_2 then $m_2 \leq m_1$ and we can apply the preceding procedure to find a unit eigenvector ϕ_2 with eigenvalue $\pm m_2$.

Proof step 5.

We proceed inductively, letting

$$V_n := \{\phi_1, \dots, \phi_{n-1}\}^{\perp}$$

and find an eigenvector ϕ_n of T restricted to V_n with eigenvalue $\pm m_n \neq 0$ if the restriction of T to V_n is not zero. So there are two alternatives:

- after some finite stage the restriction of T to V_n is zero. In this case $R(T)$ is finite dimensional with orthonormal basis $\phi_1, \ldots, \phi_{n-1}$. Or
- The process continues indefinitely so that at each stage the restriction of T to V_n is not zero and we get an infinite sequence of eigenvectors and eigenvalues r_i with $|r_i| \geq |r_{i+1}|$.

The first case is one of the alternatives in the theorem, so we need to look at the second alternative.

Proof step 6.

We first prove that $|r_n| \to 0$. If not, there is some $c > 0$ such that $|r_n| \geq c$ for all n (since the $|r_n|$ are decreasing). If $i \neq j$,then by the Pythagorean theorem we have

$$\|T\phi_i - T\phi_j\|^2 = \|r_i\phi_i - r_j\phi_j\|^2 = r_i^2\|\phi_i\|^2 + r_j^2\|\phi_j\|^2.$$

Since $\|\phi_i\| = \|\phi_j| = 1$ this gives

$$\|T\phi_i - T\phi_j\|^2 = r_i^2 + r_j^2 \geq 2c^2.$$

Hence no subsequence of the $T\phi_i$ can converge, since all these vectors are at least a distance $c\sqrt{2}$ apart. This contradicts the compactness of T.
To complete the proof of the theorem we must show that the ϕ_i form a basis of $R(T)$.

Proof step 7.

So if v is any element of our pre-Hilbert space V, and we set $w := Tv$, we must show that the Fourier series of w with respect to the ϕ_i converges to w. We begin with the Fourier coefficients of v relative to the ϕ_i which are given by

$$a_n = (v, \phi_n).$$

Then the Fourier coefficients of w are given by

$$b_i = (w, \phi_i) = (Tv, \phi_i) = (v, T\phi_i) = (v, r_i\phi_i) = r_i a_i.$$

So

$$w - \sum_{i=1}^n b_i\phi_i = Tv - \sum_{i=1}^n a_i r_i \phi_i = T(v - \sum_{i=1}^n a_i\phi_i).$$

Proof step 7, continued.

$$w - \sum_{i=1}^{n} b_i\phi_i = Tv - \sum_{i=1}^{n} a_i r_i \phi_i = T(v - \sum_{i=1}^{n} a_i\phi_i).$$

Now $v - \sum_{i=1}^{n} a_i\phi_i$ is orthogonal to $\phi_1, \dots, \phi_n$ so belongs to V_{n+1}. So

$$\|T(v - \sum_{i=1}^{n} a_i\phi_i)\| \le |r_{n+1}| \|(v - \sum_{i=1}^{n} a_i\phi_i)\|.$$

By the Pythagorean theorem,

$$\|(v - \sum_{i=1}^{n} a_i\phi_i)\| \le \|v\|.$$

Proof step 8, final step.

We know that

$$\|T(v-\sum_{i=1}^n a_i\phi_i)\| \leq |r_{n+1}|\|(v-\sum_{i=1}^n a_i\phi_i)\|.$$

and

$$\|(v-\sum_{i=1}^n a_i\phi_i)\| \leq \|v\|.$$

Putting the two previous inequalities together we get

$$\|w-\sum_{i=1}^n b_i\phi_i\| = \|T(v-\sum_{i=1}^n a_i\phi_i)\| \leq |r_{n+1}|\|v\| \to 0.$$

This proves that the Fourier series of w converges to w concluding the proof of the theorem.

The functions e^{inx} are eigenvectors of $\frac{d}{dx}$.

Each of the functions $e^{inx}$is an eigenvector of the operator

$$D = \frac{d}{dx}$$

in that

$$D\left(e^{inx}\right) = ine^{inx}.$$

So they are also eigenvalues of the operator D^2 with eigenvalues $-n^2$.

The operator D^2 looks as if it is self-adjoint, but...

Also, on the space of twice differentiable periodic functions the operator D^2 satisfies

$$(D^2 f, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f''(x)\overline{g(x)}dx = f'(x)\overline{g(x)}\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\overline{g'(x)}dx$$

by integration by parts. Since f' and g are assumed to be periodic, the end point terms cancel, and integration by parts once more shows that

$$(D^2 f, g) = (f, D^2 g) = -(f', g').$$

But of course D and certainly D^2 is not defined on $\mathcal{C}(\mathbb{T})$ since some of the functions belonging to this space are not differentiable.

Other troubles.

Furthermore, the eigenvalues of D^2 are tending to $-\infty$ rather than to zero. So somehow the operator D^2 must be replaced with something like its inverse. In fact, we will work with the inverse of $D^2 - I$, but first some preliminaries.

Other spaces.

We will let $\mathcal{C}^2([-\pi,\pi])$ denote the functions defined on $[-\pi,\pi]$ and twice differentiable there, with continuous second derivatives up to the boundary. We denote by $\mathcal{C}([-\pi,\pi])$ the space of functions defined on $[-\pi,\pi]$ which are continuous up to the boundary. We can regard $\mathcal{C}(\mathbb{T})$ as the subspace of $\mathcal{C}([-\pi,\pi])$ consisting of those functions which satisfy the boundary conditions $f(\pi) = f(-\pi)$ (and then extended to the whole line by periodicity).

We regard $\mathcal{C}([-\pi,\pi])$ as a pre-Hilbert space with the same scalar product that we have been using:

$$(f,g)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx.$$

If we can show that every element of $\mathcal{C}([-\pi,\pi])$ is a sum of its Fourier series (in the pre-Hilbert space sense) then the same will be true for $\mathcal{C}(\mathbb{T})$. So we will work with $\mathcal{C}([-\pi,\pi])$.

The map $D^2 - I : \mathcal{C}^2([-\pi,\pi]) \to \mathcal{C}([-\pi,\pi])$.

We can consider the operator $D^2 - I$ as a linear map

$$D^2 - I : \mathcal{C}^2([-\pi,\pi]) \to \mathcal{C}([-\pi,\pi]).$$

This map is surjective, meaning that given any continuous function g we can find a twice differentiable function f satisfying the differential equation

$$f'' - f = g.$$

In fact we can find a whole two dimensional family of solutions because we can add any solution of the homogeneous equation

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In fact we can find a whole two dimensional family of solutions because we can add any solution of the homogeneous equation

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to f and still obtain a solution. We could write down an explicit solution for the equation $f'' - f = g$, but we will not need to. It is enough for us to know that the solution exists, which follows from the general theory of ordinary differential equations.

The general solution of the homogeneous equation is given by

$$h(x) = ae^x + be^{-x}.$$

Let $M \subset \mathcal{C}^2([-\pi,\pi])$ be the subspace consisting of those functions which satisfy the "periodic boundary conditions"

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$$f(\pi) = f(-\pi), \quad f'(\pi) = f'(-\pi).$$

Given any f we can always find a solution of the homogeneous equation such that $f - h \in M$. Indeed, we need to choose the complex numbers a and b such that if h is as given above, then

$h(\pi) - h(-\pi) = f(\pi) - f(-\pi)$, and $h'(\pi) - h'(-\pi) = f'(\pi) - f'(-\pi)$.

Collecting coefficients and denoting the right hand side of these equations by c and d we get the linear equations

$$(e^{\pi} - e^{-\pi})(a - b) = c, (e^{\pi} - e^{-\pi})(a + b) = d$$

which has a unique solution.

$T : \mathcal{C}([-\pi, \pi]) \to M$.

So there exists a unique operator

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with the property that

$$(D^2 - I) \circ T = I.$$

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If we start with an element g of M, then $g'' - g$ belongs to $\mathcal{C}([-\pi, \pi])$ and by uniqueness, $T(g'' - g) = g$. So the image of T is M. In symbols, $R(T) = M$.

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$$Tw = \lambda w$$

then

$$D^2 w = (D^2 - I)w + w = \frac{1}{\lambda}[(D^2 - I) \circ T]w + w = \left(\frac{1}{\lambda} + 1\right) w.$$

So w must be an eigenvector of D^2; it must satisfy

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If $\mu = 0$ then $w =$ a constant is a solution. If $\mu = r^2 > 0$ then w is a linear combination of e^{rx} and e^{-rx}. No non-zero such combination can belong to M. Indeed, the periodicity conditions on $ae^{rx} + be^{-rx}$ (to belong to M) say that

$$(a-b)(e^{r\pi} - e^{-r\pi}) = 0, (a+b)(e^{r\pi} - e^{-r\pi}) = 0$$

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which has only the solution $a = b = 0$.
If $\mu = -r^2$ then the solution is a linear combination of e^{irx} and e^{-irx} and the above argument shows that r must be such that $e^{ir\pi} = e^{-ir\pi}$ so $r = n$ is an integer.

Proving the theorem.

Thus our theorem, which asserts that

$$T \quad \text{is self adjoint and compact}$$

will show that the e^{inx} are a basis of M, and a little more work, similar to what we did in the last lecture would show that they are in fact also a basis of $\mathcal{C}([-\pi,\pi])$.

Let us get to work on proving our theorem.

Self-adjointness of T.

It is easy to see that T is self adjoint. Indeed, let $f = Tu$ and $g = Tv$ so that f and g are in M and

$$(u, Tv) = ([D^2-1]f, g) = -(f', g') - (f, g) = (f, [D^2-1]g) = (Tu, v)$$

where we have used integration by parts and the boundary conditions defining M for the two middle equalities.

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We now turn to the proof that T is compact.

We have already verified that for any $f \in M$ we have

$$([D^2 - 1]f, f) = -(f', f') - (f, f).$$

Taking absolute values we get (a special case of) **Gårding's inequality**:

$$\|f'\|^2 + \|f\|^2 \leq |([D^2 - 1]f, f)|. \tag{2}$$

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(We actually get equality here, but the more general version of this that we will develop later will be an inequality.)
Let $u = [D^2 - 1]f$ and use the Cauchy-Schwarz inequality

$$|([D^2 - 1]f, f)| = |(u, f)| \leq \|u\|\|f\|$$

on the right hand side of (2) to conclude that

$$\|f\|^2 \leq \|u\|\|f\|$$

or

$$\|f\| \leq \|u\|.$$

Use Gårding's inequality

$$\|f'\|^2 + \|f\|^2 \leq |([D^2 - 1]f, f)|$$

again to conclude that

$$\|f'\|^2 \leq \|u\|\|f\| \leq \|u\|^2$$

by the preceding inequality,

$$\|f\| \leq \|u\|.$$

We have $f = Tu$, and let us now suppose that u lies on the unit sphere i.e. that $\|u\| = 1$. Then we have proved that

$$\|f\| \leq 1, \quad \text{and} \quad \|f'\| \leq 1. \tag{3}$$

We wish to show that from any sequence of functions satisfying the two conditions

$$\|f\| \leq 1, \quad \text{and} \quad \|f'\| \leq 1.$$

we can extract a subsequence which converges. Here convergence means, of course, with respect to the norm given by

$$\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx.$$

In fact, we will prove something stronger: that given any sequence of functions satisfying our two conditions we can find a subsequence which converges in the uniform norm

$$\|f\|_\infty := \max_{x\in[-\pi,\pi]} |f(x)|.$$

Convergence the uniform norm implies convergence in L_2.

Notice that

$$\|f\| = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx\right)^{\frac{1}{2}} \leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(\|f\|_\infty)^2dx\right)^{\frac{1}{2}} = \|f\|_\infty$$

so convergence in the uniform norm implies convergence in the L_2 norm we have been using.

To prove our result, notice that for any $-\pi \leq a < b \leq \pi$ we have

$$|f(b) - f(a)| = \left| \int_a^b f'(x)dx \right| \leq \int_a^b |f'(x)|dx = 2\pi(|f'|, \mathbf{1}_{[a,b]})$$

where $\mathbf{1}_{[a,b]}$ is the function which is one on $[a, b]$ and zero elsewhere. Apply Cauchy-Schwarz to conclude that

$$|(|f'|, \mathbf{1}_{[a,b]})| \leq \| \, |f'| \, \| \cdot \|\mathbf{1}_{[a,b]}\|.$$

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$$|(|f'|, \mathbf{1}_{[a,b]})| \leq \| \; |f'| \; \| \cdot \|\mathbf{1}_{[a,b]}\|.$$

But

$$\|\mathbf{1}_{[a,b]}\|^2 = \frac{1}{2\pi}|b - a| \;\; \text{and} \;\; \| \; |f'|\| = \|f'\| \leq 1.$$

We have proved that

$$|f(b) - f(a)| \leq (2\pi)^{\frac{1}{2}} |b - a|^{\frac{1}{2}}.$$

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$$|f(b)-f(a)| \leq (2\pi)^{\frac{1}{2}}|b-a|^{\frac{1}{2}}.$$

Take b to be a point where $|f|$ takes on its maximum value, so that $|f(b)| = \|f\|_\infty$. Let a be a point where $|f|$ takes on its minimum value. (If necessary interchange the role of a and b to arrange that $a<b$ or observe that the condition $a<b$ was not needed in the above proof.) Then the above inequality implies that

$$\|f\|_\infty - \min|f| \leq (2\pi)^{\frac{1}{2}}|b-a|^{\frac{1}{2}}.$$

We have proved that

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$$\|f\|_\infty - \min|f| \le (2\pi)^{\frac{1}{2}}|b-a|^{\frac{1}{2}}.$$

But

$$1 \ge \|f\| = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2(x)dx\right)^{\frac{1}{2}} \ge \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(\min|f|)^2dx\right)^{\frac{1}{2}} = \min|f|$$

and $|b-a| \le 2\pi$ so

$$\|f\|_\infty \le 1 + 2\pi.$$

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Thus the values of all the $f \in T[S]$ are all uniformly bounded - (they take values in a disk of radius $1 + 2\pi$ about the origin) and they are equicontinuous in that

$$|f(b) - f(a)| \leq (2\pi)^{\frac{1}{2}} |b - a|^{\frac{1}{2}}.$$

holds. This is enough to guarantee that out of every sequence of such f we can choose a uniformly convergent subsequence.

Recall how the proof of this goes: Since all the values of all the f are bounded, at any point we can choose a subsequence so that the values of the f at that point converge, and, by passing to a succession of subsequences (and passing to a diagonal), we can arrange that this holds at any countable set of points. In particular, we may choose say the rational points in $[-\pi, \pi]$. Suppose that f_n is this subsequence. Then

$$|f(b) - f(a)| \leq (2\pi)^{\frac{1}{2}} |b - a|^{\frac{1}{2}}.$$

implies that the f_n form a Cauchy sequence in the uniform norm and hence converge in the uniform norm to some continuous function.

Indeed, for any ϵ choose δ such that

$$(2\pi)^{\frac{1}{2}}\delta^{\frac{1}{2}} < \frac{1}{3}\epsilon,$$

choose a finite number of rational points which are within δ distance of any point of $[-\pi,\pi]$ and choose N sufficiently large that $|f_i - f_j| < \frac{1}{3}\epsilon$ at each of these points, r. when i and j are $\geq N$. Then at any $x \in [-\pi,\pi]$

$$|f_i(x) - f_j(x)| \leq |f_i(x) - f_i(r)| + |f_j(x) - f_j(r)| + |f_i(r) - f_j(r)| \leq \epsilon$$

since we can choose r such that that the first two and hence all of the three terms is $\leq \frac{1}{3}\epsilon$.

The purpose of the next few sections is to begin the study of a vast generalization of the results we obtained for the operator D^2. We will prove the corresponding results for any "elliptic" differential operator (definitions below).

I plan to study differential operators acting on vector bundles over manifolds. But it requires some effort to set things up, and I want to get to the key analytic ideas which are essentially repeated applications of integration by parts. So I will start with elliptic operators L acting on functions on the torus $\mathbb{T} = \mathbb{T}^n$, where there are no boundary terms when we integrate by parts. Then an immediate extension gives the result for elliptic operators on functions on manifolds, and also for boundary value problems such as the Dirichlet problem.

The treatment here rather slavishly follows the treatment by Bers and Schechter in *Partial Differential Equations* by Bers, John and Schechter AMS (1964).

What are currently known as Sobolev spaces (I will give the definition in the next few slides) were first introduced by Hans Lewy in his work on the initial value problem for the wave equation. This work was described by J. Hadamard in an appendix devoted to Lewy's work in Hadamard's well known book on the Cauchy problem published in 1932.

The space $P(\mathbb{T})$ and its scalar products.

Recall that $\mathbb{T}$ now stands for the n-dimensional torus. Let $\mathbf{P} = \mathbf{P}(\mathbb{T})$ denote the space of trigonometric polynomials. These are functions on the torus of the form

$$u(x) = \sum a_\ell e^{i\ell\cdot x}$$

where $\ell = (\ell_1, \ldots, \ell_n)$ is an n-tuplet of integers and the sum is finite. For each integer t (positive, zero or negative) we introduce the scalar product

$$(u, v)_t := \sum_\ell (1 + \ell \cdot \ell)^t a_\ell \overline{b}_\ell. \qquad (4)$$

when $v = \sum b_\ell e^{i\ell x}$.

$$(u, v)_t := \sum_\ell (1 + \ell \cdot \ell)^t a_\ell \overline{b}_\ell, \quad (4)$$

In fact, in (4) we could let t be any real number, and we shall do so once or twice in the course of some arguments, but we will concentrate attention on integer values of t.

Also observe, that an element of $\mathbf{P}(\mathbb{T})$ is nothing other than a collection $\{a_\ell\}$ where only a finite number of the a_ℓ are non-zero.

$$(u,v)_t := \sum_\ell (1+\ell\cdot\ell)^t a_\ell \overline{b_\ell}.$$

For $t=0$ this is the scalar product

$$(u,v)_0 = \frac{1}{(2\pi)^n}\int_{\mathbb{T}} u(x)\overline{v(x)}dx.$$

This differs by a factor of $(2\pi)^{-n}$ from the scalar product that is used by Bers and Schecter.

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We will denote the norm corresponding to the scalar product $(\ ,\ )_s$ by $\|\ \|_s$.

The operator $(1+\Delta)^t$.

If

$$\Delta := -\left(\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^n)^2}\right)$$

the operator $(1+\Delta)$ satisfies

$$(1+\Delta)u = \sum (1+\ell\cdot\ell)a_\ell e^{i\ell\cdot x}.$$

We can think of $(1+\Delta)$ as the operator on $\mathbf{P}(\mathbb{T})$ which simply multiplies each a_ℓ by the factor $(1+\ell\cdot\ell)$. With this definition, it makes sense to talk of $(1+\Delta)^t$ for any t: It is simply the operator which multiplies each a_ℓ by $(1+\ell\cdot\ell)^t$. Remember that there are (as yet) no convergence problems as all sums are finite sums.

Relations between the norms.

We have

$$((1+\Delta)^t u, v)_s = (u, (1+\Delta)^t v)_s = (u, v)_{s+t}$$

and

$$\|(1+\Delta)^t u\|_s = \|u\|_{s+2t}. \tag{5}$$

The generalized Schwarz inequality.

We then get the generalized Cauchy-Schwarz inequality

$$|(u,v)_s| \leq \|u\|_{s+t}\|v\|_{s-t} \tag{6}$$

for any t, as a consequence of the usual Cauchy-Schwarz inequality.

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for any t, as a consequence of the usual Cauchy-Schwarz inequality. Indeed,

$$\begin{aligned} \left|\sum_\ell (1+\ell\cdot\ell)^s a_\ell \overline{b}_\ell\right| &= \left|\sum_\ell (1+\ell\cdot\ell)^{\frac{s+t}{2}} a_\ell (1+\ell\cdot\ell)^{\frac{s-t}{2}} \overline{b}_\ell\right| \\ &= \left|((1+\Delta)^{\frac{s+t}{2}} u, (1+\Delta)^{\frac{s-t}{2}} v)_0\right| \\ &\leq \|(1+\Delta)^{\frac{s+t}{2}} u\|_0 \|(1+\Delta)^{\frac{s-t}{2}} v\|_0 \\ &= \|u\|_{s+t}\|v\|_{s-t}. \end{aligned}$$

The generalized Cauchy-Schwarz inequality reduces to the usual Cauchy-Schwarz inequality when $t = 0$.

Clearly we have

$$\|u\|_s \leq \|u\|_t \quad \text{if } s \leq t.$$

If D^p denotes a partial derivative,

$$D^p = \frac{\partial^{|p|}}{\partial(x^1)^{p_1} \cdots \partial(x^n)^{p_m}}$$

then

$$D^p u = \sum (i\ell)^p a_\ell e^{i\ell \cdot x}.$$

Notation.

In these equations we are using the following notations:

- If $p = (p_1, \dots, p_n)$ is a vector with non-negative integer entries we set
$$|p| := p_1 + \cdots + p_n.$$

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In these equations we are using the following notations:

- If $p = (p_1, \ldots, p_n)$ is a vector with non-negative integer entries we set

$$|p| := p_1 + \cdots + p_n.$$

- If $\xi = (\xi_1, \ldots, \xi_n)$ is a (row) vector we set

$$\xi^p := \xi_1^{p_1} \cdot \xi_2^{p_2} \cdots \xi_n^{p_n}$$

It is then clear that

$$\|D^p u\|_t \leq \|u\|_{t+|p|} \tag{7}$$

and similarly

$$\|u\|_t \leq (\text{constant depending on } t) \sum_{|p|\leq t} \|D^p u\|_0 \quad \text{if } t \geq 0. \tag{8}$$

In particular,

Theorem

The norms

$$u \mapsto \|u\|_t$$

$t \geq 0$ *and*

$$u \mapsto \sum_{|p|\leq t} \|D^p u\|_0$$

are equivalent.

The Sobolev spaces $\mathbf{H}_t$.

We let $\mathbf{H}_t$ denote the completion of the space $\mathbf{P}$ with respect to the norm $\| \ \|_t$. Each $\mathbf{H}_t$ is a Hilbert space, and we have natural embeddings

$$\mathbf{H}_t \hookrightarrow \mathbf{H}_s \ \text{ if } s < t.$$

The equation

$$\|(1+\Delta)^t u\|_s = \|u\|_{s+2t}$$

says that $(1+\Delta)^t$,initially defined on $\mathbf{P}$, extends to a map

$$(1+\Delta)^t : \mathbf{H}_{s+2t} \to \mathbf{H}_s$$

and is an isometry.

The duality between $\mathbf{H}_t$ and $\mathbf{H}_{-t}$.

From the generalized Cauchy-Schwarz inequality we also have a natural pairing of $\mathbf{H}_t$ with $\mathbf{H}_{-t}$ given by the extension of $(\ ,\ )_0$, so

$$|(u, v)_0| \leq \|u\|_t \|v\|_{-t}. \tag{9}$$

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To repeat this important point: The scalar product $(\ ,\ )_0$ initially defined on $\mathbf{P}(\mathbb{T})$ satisfies (9). But (9) allows us to extend $(\ ,\ )_0$ to a bicontinuous pairing between $\mathbf{H}_t$ and $\mathbf{H}_{-t}$.

$\mathbf{H}_{-t}$ is the dual space of $\mathbf{H}_t$.

In fact, this pairing allows us to identify $\mathbf{H}_{-t}$ with the space of continuous linear functions on $\mathbf{H}_t$. Indeed, if ϕ is a continuous linear function on $\mathbf{H}_t$ the Riesz representation theorem tells us that there is a $w \in \mathbf{H}_t$ such that $\phi(u) = (u, w)_t$. Set

$$v := (1 + \Delta)^t w.$$

Then

$$v \in \mathbf{H}_{-t}$$

and

$$(u, v)_0 = (u, (1 + \Delta)^t w)_0 = (u, w)_t = \phi(u).$$

We record this fact as

$$\mathbf{H}_{-t} = (\mathbf{H}_t)^* . \tag{10}$$

As an illustration of (10), observe that the series

$$\sum_{\ell} (1 + \ell \cdot \ell)^s$$

converges for

$$s < -\frac{n}{2}.$$

This means that if define v by taking

$$b_\ell \equiv 1$$

then $v \in \mathbf{H}_s$ for $s < -\frac{n}{2}$.

The Dirac delta function.

This means that if define v by taking

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then $v \in \mathbf{H}_s$ for $s < -\frac{n}{2}$.
If u, given by $u(x) = \sum_\ell a_\ell e^{i\ell\cdot x}$, is any trigonometric polynomial, then

$$(u, v)_0 = \sum a_\ell = u(0).$$

So the natural pairing (9) allows us to extend the linear function sending $u \mapsto u(0)$ to all of $\mathbf{H}_t$ if $t > \frac{n}{2}$. We can now give v its "true name": it is the Dirac "delta function" δ (on the torus) where

$$(u, \delta)_0 = u(0).$$

We initially defined $\mathbf{H}_t$ as the completion of the space $\mathbf{P}(\mathbb{T})$ with respect to the norm $\|\cdot\|_t$. We now find that for $t > \frac{n}{2}$ it makes sense to talk of the value of $u \in \mathbf{H}_t$ at the point 0. (Or at any other point - see Sobolev's lemma to be proved below.) So u "has become" a function.

With this in mind, we might want to think of elements of $\mathbf{H}_t$ as "generalized functions".

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So $\delta \in H_{-t}$ for $t > \frac{n}{2}$, and the preceding equation is usually written symbolically as

$$\frac{1}{(2\pi)^n} \int_{\mathbb{T}} u(x)\delta(x)dx = u(0);$$

but the true mathematical interpretation is as given above.

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We set

$$\mathbf{H}_\infty := \bigcap \mathbf{H}_t, \quad \mathbf{H}_{-\infty} := \bigcup \mathbf{H}_t.$$

Sobolev's Lemma.

The space $\mathbf{H}_0$ is just $L_2(\mathbb{T})$, and we can think of the space $\mathbf{H}_t, \quad t>0$ as consisting of those functions having "generalized L_2 derivatives up to order t". Certainly a function of class C^t belongs to $\mathbf{H}_t$. With a loss of degree of differentiability the converse is true:

Lemma

[Sobolev.] *If $u \in \mathbf{H}_t$ and*

$$t \geq \left[\frac{n}{2}\right] + k + 1$$

then $u \in C^k(\mathbb{T})$ and

$$\sup_{x \in \mathbb{T}} |D^p u(x)| \leq \text{const.} \|u\|_t \quad \text{for } |p| \leq k. \tag{11}$$

Proof of Sobolev's lemma

Proof.

By applying the lemma to $D^p u$ it is enough to prove the lemma for $k = 0$. So we assume that $u \in \mathbf{H}_t$ with $t \geq [n/2] + 1$. Then

$$\left(\sum |a_\ell|\right)^2 \leq \left(\sum (1 + \ell \cdot \ell)^t |a_\ell|^2\right) \sum (1 + \ell \cdot \ell)^{-t} < \infty,$$

since the series $\sum (1 + \ell \cdot \ell)^{-t}$ converges for $t \geq [n/2] + 1$. So for this range of t, the Fourier series for u converges absolutely and uniformly. The right hand side of the above inequality gives the desired bound. □

Distributions aka generalized functions.

A **distribution** on $\mathbb{T}^n$ is a linear function T on $C^\infty(\mathbb{T}^n)$ with the continuity condition that

$$\langle T, \phi_k \rangle \to 0$$

whenever

$$D^p \phi_k \to 0$$

uniformly for each fixed p. If $u \in \mathbf{H}_{-t}$ we may define

$$\langle u, \phi \rangle := (\phi, \overline{u})_0$$

and since $C^\infty(\mathbb{T})$ is dense in $\mathbf{H}_t$ we may conclude

Schwartz's theorem.

Lemma

$\mathbf{H}_{-t}$ is the space of those distributions T which are continuous in the $\| \ \|_t$ norm, i.e. which satisfy

$$\|\phi_k\|_t \to 0 \quad \Rightarrow \langle T, \phi_k \rangle \to 0.$$

We then obtain

Theorem

[Laurent Schwartz.] *$\mathbf{H}_{-\infty}$ is the space of all distributions. In other words, any distribution belongs to $\mathbf{H}_{-t}$ for some t.*

Proof of Schwartz's theorem.

Proof.

Suppose that T is a distribution that does not belong to any $\mathbf{H}_{-t}$. This means that for any $k > 0$ we can find a C^∞ function ϕ_k with

$$\|\phi_k\|_k < \frac{1}{k}$$

and

$$|\langle T, \phi_k \rangle| \geq 1.$$

But by Sobolev's Lemma we know that $\|\phi_k\|_k < \frac{1}{k}$ implies that $D^p\phi_k \to 0$ uniformly for any fixed p contradicting the continuity property of T. □

Rellich's lemma.

In the next lecture we will be doing a lot of inequalities and estimates to get at the application of our spectral theorem for compact operators to elliptic equations. A key idea where the compactness comes in is a "lemma" due to Rellich. I am afraid that this result will get lost in the forest. Also, it ties in with what we have been doing. So here it is:

Lemma

[Rellich's lemma.] *If $s < t$ the embedding $\mathbf{H}_t \hookrightarrow \mathbf{H}_s$ is compact.*

We must show that the image of the unit ball B of $\mathbf{H}_t$ in $\mathbf{H}_s$ can be covered by finitely many balls of radius ϵ.

Proof.

Choose N so large that $(1+\ell\cdot\ell)^{(s-t)/2} < \frac{\epsilon}{2}$ when $\ell\cdot\ell > N$.

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Proof.

Choose N so large that $(1+\ell\cdot\ell)^{(s-t)/2} < \frac{\epsilon}{2}$ when $\ell\cdot\ell > N$. Let Z_t be the subspace of $\mathbf{H}_t$ consisting of all u such that $a_\ell = 0$ when $\ell\cdot\ell \leq N$.This is a space of finite codimension, and hence the unit ball of $Z_t^\perp \subset \mathbf{H}_t$ can be covered by finitely many balls of radius $\frac{\epsilon}{2}$. The space $Z_t^\perp$ consists of all u such that $a_\ell = 0$ when $\ell\cdot\ell > N$. The image of $Z_t^\perp$ in $\mathbf{H_s}$ is the orthogonal complement of the image of Z_t.

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$$\|u\|_s^2 \leq (1+N)^{s-t}\|u\|_t^2 \leq \left(\frac{\epsilon}{2}\right)^2.$$

So the image of $B\cap Z_t$ is contained in a ball of radius $\frac{\epsilon}{2}$. Every element of the image of B can be written as a(n orthogonal) sum of an element in the image of $B\cap Z_t^\perp$ and an element of $B\cap Z_t$ and so the image of B is covered by finitely many balls of radius ϵ. □

Summary.

1. Review.
2. Relation of Fourier's Fourier series to the operator $\frac{d}{dx}$.
   - Gårding's inequality, special case.
3. Sobolev spaces.
   - Distributions and Schwartz's theorem.
   - Rellich's lemma.