282X TOPICS IN INVARIANT DESCRIPTIVE SET THEORY

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[These are partial notes from a graduate course on invariant descriptive set theory, with an emphasis on forcing applications.

We followed [Gao09] for the basics of Polish groups and their actions, and [LZ20] for pins and their applications.]

1. Classification by countable structures and turbulence

Recall that if E on X is the isomorphism equivalence relation for countable models of some countable language, then E admits an **absolute** complete classification $x \mapsto c(x)$, satisfying the following conditions

(1) The map $c\colon X \to I$ is defined in some set theoretic way ($c(x) = A \iff \psi(x, A)$).
(2) In any generic extension, ψ still defines a map which is a complete classification of E.
(3) For $x \in X$, if $V \models \psi(x, A)$ then $V[G] \models \psi(x, A)$.

The third condition says that the calculation of the invariant of x, $A = c(x)$, does not change as we move to a generic extension. This is the crucial aspect making such classifications "reasonable". For example, it prevents the classification $x \mapsto [x]_E$ from being "reasonable" when E is not a countable equivalence relation. In a sense, it is a generous way of saying that the invariant $c(x)$ can be *computed* given x. That is, the computation is local, and does not depend on things unrelated to x, like some set theoretic truths in the universe.

For an isomorphism relation E, we saw that the map

$$M \mapsto \varphi_M$$

is an absolute complete classification as above. Furthermore, in this case the invariants ϕ_M can be coded by hereditarily countable sets.

In more concrete scenarios the map is even simpler. For example, consider the second Friedman-Stanley jump $=^{++}$ defined on the space $(\mathbb{R}^\omega)^\omega$ by

$$(\mathbb{R}^\omega)^\omega \ni x \mapsto \{\{x(n,m) : m \in \omega\} : n \in \omega\}.$$

$=^{++}$ is defined so that this is a complete classification. It clearly sastisfies the absoluteness requirements. The invariants here are countable sets of countable sets of reals.

Below we show that some Borel equivalence relations are not classifiable by countable structures. This will be done by showing that they do not admit an absolute classification as above.

Remark 1.1. Let E and F be analytic equivalence relation on Polish spaces X and Y respectively. Suppose F admits an absolute complete classification $y \mapsto B_y$ as

Date: Spring 2021.

above. Assume further that E is Borel reducible to F. Then E admits an absolute complete classification as well (with the same type of invariants).

Proof. Let $f\colon X \to Y$ be a Borel reduction from E to F. Define $A_x = B_{f(x)}$. Then $x \mapsto A_x$ is as required. (By the usual absoluteness arguments, the Borel map f remains a reduction in generic extensions. Also for $x \in V$, $f(x)$ is the same as calculated in V or in $V[G]$.) $\square$

1.1. **An early argument of H. Friedman.** Let I be the ideal of all zero-density subsets of ω. Let E be the equivalence relation $2^\omega/I$, that is, $x \mathrel{E} y \iff x \Delta y \in I$. Let $\mathbb{P}$ be Cohen forcing for adding a single subset of ω. That is, the set of all finite functions from ω to 2, ordered by reverse inclusion.

Proposition 1.2 (Friedman [Fri00])**.** *Let M be a countable model. Then there are two Cohen generics a, b over M such that*

- $a \mathrel{E} b$
- $M[a] \cap M[b] = M$;

Corollary 1.3. *Suppose F is classifiable by countable structures and $f\colon E \to F$ is a Borel homomorphism from E to F. That is*

$$\forall x, y \in X \; x \mathrel{E} y \implies f(x) \mathrel{F} f(y).$$

Then there is a non-meager set $C \subseteq X$ such that $f(x) \mathrel{F} f(y)$ for any $x, y \in C$.

In this case we say that E is **generically F-ergodic**.

Proof. Fix an absolute complete classification $y \mapsto B_y$ for F. Define $A_x = B_{f(x)}$ for $x \in 2^\omega$. Then the map $x \mapsto A_x$ satisfies that $x \mathrel{E} y \implies A_x = A_y$, together with the absoluteness properties as above. Fix a, b as in the proposition above, and let $A = A_a = A_b$. Then $A \in M[a] \cap M[b] = M$.

Fix a condition $p \in M$ in the Cohen forcing such that $p \Vdash A_{\dot{x}} = \check{A}$, where $\dot{x}$ is the name for the Cohen generic real in 2^ω. Let $C \subseteq 2^\omega$ be the set of all Cohen generics over M which agree with p. This is a non-meager set (comeager in the open set defined by p).

Now for any $x, y \in C$, $A_x = A = A_y$. So $B_{f(x)} = B_{f(y)}$, and therefore $f(x) \mathrel{F} f(y)$. $\square$

Note that each E-equivalence class is meager in 2^ω (exercise). It follows immediate that a Borel homomorphism from E to F as above is not injective on the classes, and therefore not a reduction.

Corollary 1.4. *The equivalence relation E is not classifiable by countable structures.*

The following is the main technical lemma in the proof of Proposition 1.2.

Lemma 1.5 (Friedman [Fri00])**.** *Suppose p, q are conditions in $\mathbb{P}$ of length n and ϕ, ψ are statements such that p does not decide ϕ. Then there exists two extensions (of the same length) $p* \leq p$ and $q* \leq q$ such that*

- $p * \setminus p$ *and* $q * \setminus q$ *differ by at most 1 coordinate, and*
- *either* $(p* \Vdash \phi$ *and* $q* \Vdash \neg\psi)$ *or* $(p* \Vdash \neg\phi$ *and* $q* \Vdash \psi)$.

Proof. Assume otherwise, that there are conditions p, q of length n such that p does not decide ϕ and for any respective extensions $p*, q*$, if they decide ϕ and ψ respectively and $p * \setminus p$ and $q * \setminus q$ differ by at most 1 coordinate, then $p* \Vdash \phi \iff q* \Vdash \psi$.

Note that for any two extensions $p*, q*$ of the same length which differ by only one coordinate, we may assume they decide ϕ and ψ respectively. This is because we can extend them in the same way to some $p*^\frown r$ and $q*^\frown r$ which decide ϕ and ψ respectively.

Fix two extensions p^* and p^{**} of p of the same length, deciding opposite values for ϕ. Consider a *walk*, $t_0, ..., t_n$ such that

- $p^* = p^\frown t_0$, $p^{**} = p^\frown t_n$;
- $t_{i+1} \setminus p$ and $t_i \setminus p$ differ by 1 coordinate;
- $p^\frown t_i$ decides ϕ and $q^\frown t_i$ decides ψ.

(For the third condition we may extend p^* and p^{**} first.)

By assumption,

$$p^\frown t_i \Vdash \phi \iff q^\frown t_i \Vdash \psi \iff p^\frown t_{i+1} \Vdash \phi.$$

We conclude that $p^* = p_0 \Vdash \phi \iff p^{**} = p_n \Vdash \phi$, a contradiction. □

The generics a and b are constructed recursively, $a = \bigcup_i p_i$, $b = \bigcup_i q_i$, as follows. Fix some enumeration of the dense open subset of $\mathbb{P}$ in M, as well as an enumeration of the $\mathbb{P}$-names in M and the members of M. We construct the generics by diagonalizing, making sure that no new subset of M lands in both $M[a]$ and $M[b]$.

Suppose we have p_i, q_i, $Z \in M$, $\tau \in M$ a $\mathbb{P}$-name for a subset of Z.

First we may replace p_i, q_i with $p_i^\frown r$ and $q_i \frown r$ to meet some dense open set. Also take r to be of length $\geq i$.

Suppose p_i does not force τ to be in M. Then there is some $z \in Z$ such that p_i does not decide $\check{z} \in \tau$. We may find extensions p_{i+1}, q_{i+1}, such that $p_{i+1} \setminus p_i$ and $q_{i+1} \setminus q_i$ differ by at most one coordinate, and

$$p_{i+1} \Vdash \check{z} \in \tau \iff q_{i+1} \Vdash \neg(\check{z} \in \tau).$$

(Apply the lemma with $\phi = \psi = \check{z} \in \tau$.)

This process assures that $a\ E\ b$ and for any $Z \in M$, for any $A \subseteq Z$, if $A \in M[a] \cap M[b]$ then $A \in M$. To conclude Proposition 1.2 the following (trivial) lemma suffices:

Lemma 1.6. Suppose M_1, M_2 are two extensions of M (with the same ordinals as M) such that for any $Z \in M$ and any $A \subseteq Z \in M_1 \cap M_2$ is already in M. Then $M_1 \cap M_2 = M$.

Proof. By induction on the rank of sets. □

Finally, note that the equivalence relation E is an orbit equivalence relation. First, note that the symmetric difference operation Δ makes $\mathcal{P}(\omega)$ into a Polish group, with the Cantor space topology 2^ω. The equivalence relation E is induced by the action of the subgroup I of all subset of zero density. This is *not* a Polish subgroup. However, it is a **Polishable** group. That is, there is some topology (not inherited from Cantor space) making I into a Polish group.

Exercise 1.7. Show that the metric

$$d(x,y)=\sup_{n\in\omega}\frac{|(x\Delta y)\cap\{0,...,n\}|}{n+1}$$

makes the group I into a Polish group.

1.2. Turbulence.

The standard, and most useful, way to show an equivalence relation is not classifiable by countable structure, is Hjorth's turbulence condition for group actions. (See [Hjo00] and [Kec02]).

Let G be a Polish group, X a Polish space and $a\colon G \curvearrowright X$ a continuous action. Let $\{U_n\}$ be countable basis for X and $\{V_n\}$ a countable nbhd basis for at 1_G in G.

Definition 1.8. Fix $U\subseteq X$ open and $V\subseteq G$ an open nbhd of 1_G. For $x,y\in U$, a **U-V-walk** from x to y is a sequence $x=x_0,...,x_l=y$ such that

- $x_i\in U$;
- $x_{i+1}=g_i\cdot x_i$ for some $g_i\in V$.

For $x\in U$, the **local U-V-orbit of** x, denoted $\mathcal{O}(x,U,V)$, is the set of all $y\in U$ for which there is a U-V-walk from x to y.

Definition 1.9. Say that the action a is **turbulent** if

(1) The orbits are dense and meager;
(2) for any $x\in X$, for any open $x\in U\subseteq X$ and open $1_G\in V\subseteq G$, the local orbit $\mathcal{O}(x,U,V)$ is somewhere dense in U.

Remark 1.10. Recall: if E on X is an equivalence relation such that every equivalence class is dense and meager, then any Borel E-invariant map $f\colon X\to\mathbb{R}$ maps a comeager subset of X to a single real. That is, E is generically $=_{\mathbb{R}}$-ergodic.

Theorem 1.11 ([Hjo00], [Kec02])**.** Suppose $a\colon G\curvearrowright X$ is turbulent. Let F be an equivalence relation which is classifiable by countable structures. Then E_a is generically F-ergodic.

In particular, since the E_a orbits are meager, E_a is not Borel reducible to any equivalence relation which is classifiable by countable structures.

Exercise 1.12. Let $I\subseteq\mathcal{P}(\omega)$ be the density zero ideal, viewed as a Polish group as above. Show that the action of I on 2^ω by $x\cdot y=x\Delta y$ is turbulent.

We will follow the approach from [LZ20], showing that the turbulent condition provides two E_a-equivalent Cohen-generics $x,y\in X$ such that $V[x]\cap V[y]=V$.

Theorem 1.13. Let a,G,X be as above. The following are equivalent. Let $\mathbb{P}_X$ and $\mathbb{P}_G$ be Cohen forcing in the Polish spaces X and G respectively, and $\dot{x}$, $\dot{g}$ the names for the generic elements.

(1) The action a is **generically turbulent**. (That is, for a comeager set of $x\in X$, the localy orbits $\mathcal{O}(x,U,V)$ are somewhere dense.)
(2) $\mathbb{P}_X\times\mathbb{P}_G\Vdash V[\dot{x}]\cap V[\dot{g}\cdot\dot{x}]=V$.

First note that Theorem 1.13 together with the arguments in the preceding section immediately imply Theorem 1.11.

Proof of Theorem 1.13.

Exercise 1.14.
- Show that if $g\in G$ is generic over V and $h\in G\cap V$, then $h\cdot g$ is also generic over V.

- Show that if $x \in X$ is $\mathbb{P}_X$-generic over V and $g \in G$ is $\mathbb{P}_G$-generic over V then $g \cdot x$ is $\mathbb{P}_X$-generic over V. Hint: In $V[g]$, the map $x \mapsto g \cdot x$ is a homeomorphism of X. The map $p \mapsto g \cdot p$ is an automorphism of $\mathbb{P}_X$ which sends the generic $\dot{x}$ to $g \cdot \dot{x}$.

First we show that (1) implies (2) (which is the direction necessary to prove Theorem 1.11). Assume σ and τ are $\mathbb{P}_X$-names such that $\sigma[x] = \tau[g \cdot x]$. We may assume that σ is forced to be a subset of V, and show that $\sigma[x] \in V$.

Fix $(p, q) \in \mathbb{P}_X \times \mathbb{P}_G$ such that $x \in p$, $g \in q$ and (p, q) forces that $\sigma[\dot{x}] = \tau[\dot{g} \cdot \dot{x}]$. Fix an open nbhd $1 \in W \subseteq G$ such that for any $h \in W$, $g \cdot h^{-1} \in q$.

Claim 1.15. Suppose $x \in p$ is $\mathbb{P}_X$-generic over V and $h \in W$ is in V such that $h \cdot x \in p$. Then $\sigma[x] = \sigma[h \cdot x]$.

Proof. By assumption, $x, h \cdot x$ are in p and g, gh^{-1} are in q. So

$$\sigma[h \cdot x] = \tau[(gh^{-1})h \cdot x] = \tau[g \cdot x] = \sigma[x].$$

□

Suppose now $x_0, ..., x_l$ is a p-$W \cap V$-walk of $\mathbb{P}_X$-generics. That is, each $x_i \in p$ is $\mathbb{P}_X$-generic over V, and $x_{i+1} = h_i \cdot x_i$ for some $h_i \in W$ from V. Then $\sigma[x_0] = \sigma[x_l]$.

Since the action is generically turbulent, it follows that local orbits $\mathcal{O}(x, p, W)$ are dense, when x is Cohen-generic. Fix any condition t extending p. Fix an open set $t' \subseteq t$ such that $\mathcal{O}(x, t, W)$ is dense in t'. The local orbit where W is replaced by $W \cap V$ is still dense in t', since $G \cap V$ is dense in G.

Finally, for any t'' extending t', we may find some generic $x'' \in t''$ such that $\sigma[x''] = \sigma[x]$. It follows that t' decides all values $\check{v} \in \tau$, for $v \in V$. We showed that any $t \in \mathbb{P}_X$ extending p there is an extension t' deciding σ. So p forces that $\sigma[\dot{x}] \in V$.

Next we prove (2) implies (1). Assume for contradiction that the action is not generically turbulent, yet part (2) holds. We may find a $\mathbb{P}_X$-generic $x \in X$ such and some open U, W such that $\mathcal{O}(x, U, W)$ is nowhere dense. Fix a generic $g \in W$ over $V[x]$ such that $g \cdot x \in U$. Then

$$\mathcal{O}(x, U, W) = \mathcal{O}(g \cdot x, U, W).$$

In particular, the nowhere dense set $\mathcal{O}(x, U, W)$ is in $V[x] \cap V[g \cdot x] = V$. This is a contradiction, as $x \in \mathcal{O}(x, U, W)$, and is $\mathbb{P}_X$-generic over V. □

Exercise 1.16. Consider the subgroup c_0 of $\mathbb{R}^\omega$ of all sequences which converge to zero.

- Show that c_0 is a Polish group with the distance $d(x, y) = \sup_n |x(n) - y(n)|$.
- Show that the action of c_0 on $\mathbb{R}^\omega$ by translations is turbulent.
- Show that the orbit equivalence relation induced by the acion above is Borel bireducible with $2^\omega / I$, where I is the zero density ideal. (See [Kan08, p. 75])

Exercise 1.17. Let $p \geq 1$. Define d_p on $\mathbb{R}^\omega$ by $d_p(x, y) = (\sum_n |x(n) - y(n)|^p)^{\frac{1}{p}}$. Let l^p be the subgroup of $\mathbb{R}^\omega$ of all x with $d_p(x, \bar{0}) < \infty$. Then l^p is a Polish group with the distance d_p, and the action of l^p on $\mathbb{R}^\omega$ by translations is turbulent.

2. CLI subgroups of S_∞

Recall that any closed subgroup of S_∞ is of the form $\mathrm{Aut}(\mathcal{M})$, where $\mathcal{M}$ is an L-structure on ω (for some relational language L).

Theorem 2.1 (Gao)**.** Let L be a countable language and M an L-structure. The following are equivalent.

(1) $\mathrm{Aut}(\mathcal{M})$ is CLI;
(2) the scott sentence of $\mathcal{M}$, φ_M, does not have uncountable models.

Proof. Note that the Baire space metric d is left-invariant. So $\mathrm{Aut}(\mathcal{M})$ is CLI if and only if d on $\mathrm{Aut}(\mathcal{M})$ is complete if and only if $\mathrm{Aut}(\mathcal{M})$ is d-closed.

Exercise 2.2. The d-closure of $\mathrm{Aut}(\mathcal{M})$ in ω^ω is precisely those functions $f\colon \omega \to \omega$ which form an elementary embedding from M to itself. Hint: if f is an elementary embedding from M to M, the for any turple $\bar{a} \in \omega^{<\omega}$, $(M, \bar{a})$ and $(M, f(\bar{a}))$ are $\equiv_\infty$-equivalent, so there is an automorphism of M sending $\bar{a}$ to $f(\bar{a})$.

Assume now that φ_M has an uncountable model, $\mathcal{N}$. By downwards LS (which holds for $L_{\omega_1,\omega}$), we may find a countable $\mathcal{M}_1 \prec \mathcal{N}$. Applying downwards LS again, we may find $\mathcal{M}_2 \prec \mathcal{N}$ such that $\mathcal{M}_2$ strictly contains $\mathcal{M}_1$. It follows then that $\mathcal{M}_1 \prec \mathcal{M}_2$.

As both $\mathcal{M}_1$ and $\mathcal{M}_2$ are countable models of φ_M, they are both isomorphic to $\mathcal{M}$. By composing these isomorphisms with the inclusion $\mathcal{M}_1 \prec \mathcal{M}_2$, we get an elementary embedding $f\colon \mathcal{M} \to \mathcal{M}$ which is not onto. That is, f is in the closure of $\mathrm{Aut}(\mathcal{M})$ but not in $\mathrm{Aut}(\mathcal{M})$, so $\mathrm{Aut}(\mathcal{M})$ is not CLI.

Conversely, assume now that $\mathrm{Aut}(\mathcal{M})$ is not CLI. Then there is an elementary embedding $f\colon \mathcal{M} \to \mathcal{M}$ which is not onto. Let $\mathcal{M}'$ be the structure defined on the image of $\mathcal{M}$ under f, so the $f\colon \mathcal{M} \to \mathcal{M}'$ is an isomorphism. Then $\mathcal{M}' \prec \mathcal{M}$ ($\mathcal{M}' \models \psi(f(\bar{a})) \iff \mathcal{M} \models \psi(\bar{a}) \iff \mathcal{M} \models \psi(f(\bar{a}))$, is a non trivial elementary extension. As $\mathcal{M}'$ is isomorphic to $\mathcal{M}$, we conclude that there is a non-trivial elementary extension $\mathcal{M} \prec \mathcal{M}^*$ with $\mathcal{M}^* \equiv \mathcal{M}$. And this is true if $\mathcal{M}$ is replaced by any model isomorphic to it.

Define recursively a sequence of elementary extensions $M_0 \prec M_1 \prec \ldots \prec M_\alpha$, for $\alpha < \omega_1$ such that:

- $\mathcal{M}_0 = \mathcal{M}$, and each $\mathcal{M}_\alpha$ is isomorphic to $\mathcal{M}$.
- $\mathcal{M}_{\alpha+1} = (\mathcal{M}_\alpha)^*$;
- $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$ for limit α.

The key point is that at limit stages $\mathcal{M}_0 \prec \mathcal{M}_\alpha$, and therefore $\mathcal{M}_\alpha \models \varphi_M$. Now as $\mathcal{M}_\alpha$ is countable and satisfies the Scott sentence of $\mathcal{M}$, it is isomorphic to $\mathcal{M}$, and we can find an extension $(\mathcal{M}_\alpha)^*$.

Finally, let $\mathcal{N} = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$. Then again $\mathcal{M} = \mathcal{M}_0 \prec \mathcal{N}$, and so $\mathcal{N} \models \varphi_M$ is an uncountable model. □

2.1. More on Pinned equivalence relations.

Suppose E is an equivalence relation with $x \mapsto A_x$ an absolute complete classification. Say that a set A is a **potential invariant** for E if there is some x, in some generic extension, such that $A = A_x$. Say that A is trivial if there is x (in the ground model) such that $A = A_x$ (that is, A is an invariant.)

Example 2.3. For $=^+$, all sets of reals are potential invariants. The trivial ones are the countable sets of reals.

Claim 2.4. There is a correspondence between E-pins and potential E-invariants as follows.

Given a potential E-invariant A. Fix a poset $\mathbb{P}$ and a name σ such that $\mathbb{P} \Vdash A_\sigma = A$. Then $(\mathbb{P}, \sigma)$ is an E-pin.

Conversely, given an E-pin $(\mathbb{P}, \sigma)$, fix a $\mathbb{P}$-generic G over V, and let $A = A_{\sigma[G]}$. If H is $\mathbb{P}$-generic over $V[G]$, then $A = A_{\sigma[H]}$ as well, so $A \in V[G] \cap V[H] = V$ is a potential E-invariant.

Note that A is trivial if and only if $(\mathbb{P}, \sigma)$ is trivial.

Corollary 2.5. The following are equivalent.

(1) E is pinned;
(2) if A is a potential invariant for E, there is an x (in the ground model) such that $A = A_x$.

In particular, $\mathbb{R}$ is a non-trivial invariant for $=^+$, which we used to show it is unpinned. In fact any uncountable set of reals also shows this. We will see that interesting uncountable sets of reals (say of size $\aleph_1$) can be used to show non-pinnedness. On the other hand, in models such as the Solovay model, there are no sets of reals which are "much more exotic" than $\mathbb{R}$.

2.2. **CLI group actions.** Recall:

Theorem 2.6 (Hjorth)**.** If G is a CLI Polish group, then all of its induced orbit equivalence relations are pinned.

This theorem has a converse:

Theorem 2.7 (Thompson)**.** If G is not CLI, then it admits an action with a non-pinned orbit equivalence relation.

Following [KMPZ20], we give a simple proof of the special case when G is a closed subgroup of S_∞.

Fix a closed subgroup $G \leq S_\infty$. Consider the **Bernoulli shift action** (see [KMPZ20, p. 13]) of G on $\mathbb{R}^\omega$ defined by

$$g \cdot x(n) = x(g^{-1}(n)).$$

Let $X \subseteq \mathbb{R}^\omega$ be the Polish subspace of injective sequences. We will view such an injective sequence $x \in X$, as a labelling of the model $\mathcal{M}$ on ω with the reals $x(0), x(1), \ldots$. This gives us an isomorphic model on $\{x(n) : n \in \omega\}$, call this model $\mathcal{N}_x$. (That is, $\mathcal{N}_x \models (\phi(a(k)) \iff \mathcal{M} \models \phi(k))$.) Let E on X be the orbit equivalence relation induced by the action of G on X.

The map $x \mapsto \mathcal{N}_x$ is a complete classification of E. Furthermore, it is absolute. Clearly it is definable and it remains a complete classification in any model of ZF. Furthermore, for $x \in V$, the calculation of $\mathcal{N}_x$ from x is the same in V and $V[G]$.

Let $G = \mathrm{Aut}(\mathcal{M})$ for some $\mathcal{M}$. Assume now that G is not CLI. By Gao's theorem, there is a model $\mathcal{K}$ of $\varphi_\mathcal{M}$ of size $\aleph_1$. Note that in any generic extension collapsing $\aleph_1$ to be countable, $\mathcal{K}$ is then a countable model of $\varphi_\mathcal{M}$, and therefore isomorphic to $\mathcal{M}$.

Fix a sequence $\langle a_\alpha : \alpha < \omega_1 \rangle$ of distinct reals, and let $\mathcal{N}$ be the model on $\{a_\alpha : \alpha < \omega_1\}$ defined from a bijection between the domain of $\mathcal{K}$ and ω_1.

Claim 2.8. $\mathcal{N}$ is a potential invariant for E, and is not trivial.

After collapsing $\aleph_1$, as $\mathcal{N}$ is isomorphic to $\mathcal{M}$, there is an enumeration $x \in X$ of $\{a_\alpha : \alpha < \omega_1\}$ such that $\mathcal{N} = \mathcal{N}_x$. Furthermore, $\mathcal{N}$ is not $\mathcal{N}_x$ for any $x \in V$, as its domain is not countable.

3. Pinned cardinality

Recall:

- Suppose E on X admits an absolute complete classification $x \mapsto A_x$. A potential E-invariant is a set A such that $A = A_x$ for some x in a generic extension (most likely a collapse). A is trivial if $A = A_x$ for some x in the ground model. E is not pinned if and only if there is a non-trivial potential invariant.
- Hjorth showed that $=^+$ is not Borel reducible to an action of a CLI group, as $=^+$ is not pinned.
- Kechris then asked if "E is unpinned" is equivalent to $=^+ \leq_B E$. This was refuted by Zapletal. We give a proof below.

Suppose E is not pinned as witnessed by A. Zapletal [Zap11] introduced the following idea: *how much do we need to collapse in order to find the x for which $A = A_x$?*

Remark 3.1. Recall that potential invariants are in one-to-one correspondence with pins. Furthermore, if f is a Borel reduction from E to F, $x \mapsto A_x$, $y \mapsto B_y$ are complete classifications of E and F respectively, then f maps an E-pin $(\mathbb{P}, \sigma)$ to an F-pin, $(\mathbb{P}, f(\sigma))$.

Thus f sends potential $=^+$-invariants to potential E-invariants. More specifically: let A be an E-invariant, $\dot{x}$ is $\mathbb{P}$-name such that $\mathbb{P} \Vdash A = A_{\dot{x}}$. If x, x' are mutually generic, we conclude that $B = B_{f(x)} = B_{f(x')}$ is in $V[x] \cap V[x'] = V$. So B is an F-invariant such that $\mathbb{P} \Vdash B = B_{f(\dot{x})}$.

Note furthermore that in any generic extension $V \subseteq V'$, B is a trivial potential invariant if and only if A is a trivial potential invariant.

Remark 3.2. This approach is closely related to the presentation in [URL17].

3.1. Zapletal's counter example [Zap11].

Fix a Borel graph $\mathcal{G}$ on $\mathbb{R}$ such that $\mathcal{G}$ has uncountable cliques, yet it does not have perfect cliques.

For example, consider $\mathcal{G}$ defined by

$$x \, \mathcal{G} \, y \iff x \leq_T y \text{ or } y \leq_T x.$$

Exercise 3.3. Show that

(1) $\mathcal{G}$ has a clique of size $\aleph_1$;
(2) $\mathcal{G}$ does not have cliques of size $\aleph_2$.

Let $\mathcal{C} \subseteq \mathbb{R}^\omega$ be the set of all $x \in \mathbb{R}^\omega$ such that $\{x(n) : n \in \omega\}$ is a $\mathcal{G}$-clique. Consider the Borel equivalence relation E defined as $=^+ \restriction \mathcal{C}$, the restriction of $=^+$ to the (invariant) Borel set $\mathcal{C}$.

Note that the map $x \mapsto \{x(n) : n \in \omega\}$ remains an absolute complete classification of E. The potential-invariants are precisely the sets of reals $A \subseteq \mathbb{R}$ which form a $\mathcal{G}$-clique. In particular, all the potential invariants are of size $\leq \aleph_1$. Also, any uncountable $\mathcal{G}$-clique $A \subseteq \mathbb{R}$ is a non-trivial potential invariant. Therefore E is not pinned.

Proposition 3.4. $=^+$ is not Borel reducible to E.

Proof. As question of Borel reducibility are absolute, we may work in a model with the continuum has size $\geq \aleph_2$. Let $A = \mathbb{R}$, a potential $=^+$-invariant. In any generic extension $V \subseteq V'$, A is trivial if and only if V' collapse $|\mathbb{R}|^V$. In particular, A is not trivial in a $\mathrm{Col}(\aleph_1, \aleph_0)$-extension of V.

Assume towards a contradiction that there is a Borel reduction $f \colon \mathbb{R}^\omega \to \mathcal{C}$ from $=^+$ to E. This gives a definable map from potential $=^+$-invariants to potential E-invariants as described above. Let B be the E-invariant corresponding to A. Then after collapsing $\aleph_1$, B is trivialized, and therefore so is A, a contradiction. □

3.2. **Pinned cardinals.**

Definition 3.5 ([LZ20])**.** Let E be an analytic equivalence relation on a Polish space X. Let $\mathbb{P}$ be a poset and σ a $\mathbb{P}$-name. Recall:

- $(\mathbb{P}, \sigma)$ is an E-pin if $\mathbb{P} \times \mathbb{P}$ forces that σ_l is E-equivalent to σ_r, where σ_l and σ_r are the interpretation of σ using the left and right generics respectively.
- An E-pin $\langle \mathbb{P}, \sigma \rangle$ is **trivial** if there is some $x \in X$ such that $\mathbb{P} \Vdash \sigma \mathrel{E} \check{x}$.
- E is **pinned** if all E-pins are trivial.

Furthermore:

- Given two E-pins $\langle \mathbb{P}, \sigma \rangle$ and $\langle \mathbb{Q}, \tau \rangle$, say that they are $\bar{E}$-equivalent, $\langle \mathbb{P}, \sigma \rangle \mathrel{\bar{E}} \langle \mathbb{Q}, \tau \rangle$, if $\mathbb{P} \times \mathbb{Q} \Vdash \sigma \mathrel{E} \tau$. We call the $\bar{E}$-classes **virtual E-classes**.
- The **pinned cardinal of** E, $\kappa(E)$, is the smallest κ such that every E-pin is $\bar{E}$-equivalent to an E-pin with a post of size $< \kappa$.
- Let $\lambda(E)$ be the cardinality of $\bar{E}$-classes.
- Let $\lambda(E, \mathbb{P})$ be the cardinality of $\bar{E}$-classes for pins of the form $(\mathbb{P}, \sigma)$.

Lemma 3.6 ([LZ20])**.** If E is Borel reducible to F then

- $\kappa(E) \leq \kappa(F)$;
- $\lambda(E) \leq \lambda(F)$;
- $\lambda(E, \mathbb{P}) \leq \lambda(F, \mathbb{P})$.

Exercise 3.7. $\kappa(E) < \infty \iff \lambda(E) < \infty$.

Remark 3.8. Suppose E admits an absolute complete classification $x \mapsto A_x$. For a potential E-invariant A, let $\kappa(A)$ be the smallest κ such that A is trivial in a $\mathrm{Col}(\kappa, \aleph_0)$-extension. Then $\kappa(E)$ is the smallest cardinal κ such that $\kappa(A) < \kappa$ for all potential invariants A. $\lambda(E)$ is the number of potential invariants. $\lambda(E, \mathbb{P})$ is the number of potential invariants which are trivialized by $\mathbb{P}$.

Remark 3.9. $\kappa(E) < \infty$ if and only if $\lambda(E) < \infty$.

Example 3.10.
- $\kappa(=^+) = \mathfrak{c}^+$, $\lambda(=^+) = 2^{\mathfrak{c}}$.
- $\kappa(=^{++}) = (2^{\mathfrak{c}})^+$, $\lambda(=^+) = 2^{2^{\mathfrak{c}}}$.

It follows that $=^{++}$ is not Borel reducible to $=^+$.

Exercise 3.11. Find $\kappa(=^{+\alpha})$ and $\lambda(=^{+\alpha})$. Conclude that $=^{+\alpha+1} \not\leq_B =^{+\alpha}$.

Fact 3.12. If E is a Borel equivalence relation, then $\kappa(E) < \beth_{\omega_1}$. In fact, if E is $\mathbf{\Pi}^0_\alpha$ then $\kappa(E) \leq \beth_\alpha^+$.

This follows from a result of Stern:

Lemma 3.13 (Stern, see [Hjo98])**.** Suppose τ is an invariant $\mathbb{P}$-name for a $\mathbf{\Pi}^0_\alpha$ set. That is $\mathbb{P}$ forces that τ is a code for a $\mathbf{\Pi}^0_\alpha$ set and $\mathbb{P} \times \mathbb{P}$ forces that the left and right interpretations produce the same Borel set.

Then, in a $\mathrm{Col}(\omega, \beth_\alpha^+)$ extension $V[G]$, there is a code for a Borel set B, such that in any further extension, for any x,

$$V[x] \models \mathbb{P} \Vdash x \in \tau \iff x \in B.$$

Assuming the lemma, let E be a $\mathbf{\Pi}^0_\alpha$ equivalence relation. Let $(\mathbb{P}, \sigma)$ be an E-pin. Let τ be the E-name for $[\sigma]_E$, the E-class of σ. Then τ is an invariant $\mathbb{P}$-name for a $\mathbf{\Pi}^0_\alpha$ set.

By the lemma, there is a $\mathrm{Col}(\omega, \beth_\alpha^+)$-name μ so that if $G \subseteq \mathrm{Col}(\omega, \beth_\alpha^+)$ is generic over V and B is the Borel set defined by μ, then in any further generic extension, for any x,

$$V[x] \models \mathbb{P} \Vdash x \in \tau \iff x \in B.$$

In particular, if $H \subseteq \mathbb{P}$ is generic over $V[G]$, we conclude that the set B is not empty in $V[G][H]$, as $\sigma[H] \in B$.

By absoluteness, B is non empty in $V[G]$ as well. Fix a name μ so that $\mu[G] \in B$. WLOG $\mathrm{Col}(\omega, \beth_\alpha^+) \Vdash \mu \in B$.

Then if $G \times H$ is $\mathrm{Col} \times \mathbb{P}$-generic, then $\mu[G] \in \tau[H] = [\sigma[H]]_E$. So $\mu[G] \mathrel{E} \sigma[H]$. We conclude that $(\mathrm{Col}(\omega, \beth_\alpha^+), \mu)$ is a pin, which is $\bar{E}$-related to $(\mathbb{P}, \sigma)$.

Some ideas for Stern's lemma:

First note that by the invariance assumption,

$$\exists p (p \Vdash x \in \tau) \iff \mathbb{P} \Vdash x \in \tau.$$

Suppose first τ is a code for an open set. Let B be the union of all basic open sets U such that $\mathbb{P} \Vdash U \subseteq \tau$. Then B (in V) is open and $x \in B$ if and only if there is a basic open U forced in τ with $x \in U$, if and only if x is forced to be in τ.

Assume τ is a name for a G_δ set. That is, a name for a countable sequence of codes for open sets. For $i < \omega$ and $p \in \mathbb{P}$ define $X(i,p)$ to be the set of all basic open U such that for some $q \leq p$, q forces U is contained in the i'th open set in τ.

Each $X(i,p)$ can be coded as a subset of ω, in V. Therefore there are at most $\beth_1$ many. (Even though there may be many more conditions in $\mathbb{P}$.)

It follows that in an extension collapsing $\beth_1$, the set $\{X(i,p) : i \in \omega, p \in \mathbb{P}\}$ is countable, and produces a Borel set

$$B = \bigcap_{i,p} \bigcup X(i,p),$$

As required.

3.3. **Proofs of the Friedman-Stanley jump theorem.**

Exercise 3.14. Suppose E is a countable Borel equivalence relation. Show that E^+ is Borel bireducible with $=^+$. Also $E^\omega \leq_B =^+$.

Theorem 3.15 (Friedman-Stanley)**.** If E is a Borel equivalence relation with more than one equivalence class. Then $E <_B E^+$.

We know that $E \leq_B E^+$, and it remains to show that $E^+ \not\leq_B E$.

<u>Proof 1</u>: Assume for a contradiction that there is some E with $E^+ \sim_B E$. We may assume that $=_\mathbb{N} \leq_B E$. But then, inductively, $=^{+\alpha} \leq_B E$ for all ordinals α. Contradicting the fact that $\kappa(E) < \beth_{\omega_1}$.

Exercise 3.16. Let $(E^+)^\mathbb{N}$ is Borel bireducible with E^+.

<u>Proof 2</u>:

Exercise 3.17 (Larson-Zapletal [LZ20])**.** If $\lambda(E) < \infty$, then $\lambda(E^+) = 2^{\lambda(E)}$.

If E is Borel, $\lambda(E) < \infty$, so by Cantor's theorem, $\lambda(E^+) > \lambda(E)$.
(Closer to) the original proof:

Theorem 3.18 (H. Friedman)**.** Suppose E is a Borel equivalence relation on a Polish space X. Then for any Borel function $f\colon X^\omega \to X$, if

$$x\ E^+\ y \implies f(x)\ E\ f(y),$$

then there is $x \in X^\omega$ and $n < \omega$ such that $f(x)\ E\ x(n)$.

Assume now that there is a Borel E and $g\colon X^\omega \to X$, a reduction of E^+ to E. Define a Borel $f\colon (X^\omega)^\omega \to X^\omega$ so that for $\bar{x} = \langle x_n : n < \omega\rangle \in (X^\omega)^\omega$, $f(\bar{x})$ enumerates

$$\{g(x_n) : \forall k(g(x_n))\ \not{E}\ (x_n)_k\}\,.$$

Check: f is a Borel homomorphism from E^{++} to E. We show that f is a Borel diagonalizer from $E^+ \to E$, contradicting Friedman's theorem.

Given $\bar{x} = \langle x_n : n < \omega\rangle \in (X^\omega)^\omega$.

If $g(x_n)\ E\ (x_n)_k$ for some k, then $g(x_n)$ is not in the set enumerated by $f(\bar{x})$. In act nothing in the E-class of $g(x_n)$ is there, as g is injective on the classes. So $f(\bar{x})$ is not E^+-related to x_n, as required.

If $g(x_n)\ \not{E}\ (x_n)_k$ for all k, then again $f(x)$ cannot be E^+-related to x_n.

There is a slight issue: that f is not defined in the case where the displayed formula above is empty. This can be fixed as follows. Fix some $z \in X$ such that $g(\langle z, z, ...\rangle) \neq z$. This is possible as g is injective between E^+ classes and E-classes.

Now in the definition above consider instead $f(\bar{x})$ to enumerate

$$\{g(x_n) : \forall k(g(x_n))\ \not{E}\ (x_n)_k\} \cup \{g(\langle z, z, ...\rangle)\}.$$

Check that this works.

Proof of Friedman's theorem:

Consider first the case when E is $=_{\mathbb{R}}$, the equality relation on the reals. The statement says that there is no way to choose for each countable set a real outside of it, in a way that is definable and independent of the enumeration of this set.

Indeed, suppose $f\colon \mathbb{R}^\omega \to \mathbb{R}$ is a function such that for any $x \in \mathbb{R}^\omega$ and any $n < \omega$, $f(x) \neq x(n)$, $f(x) = f(y)$ whenever x and y enumerate the same set of reals, and these properties are absolute in forcing extensions.

Let $x, y \in \mathbb{R}^\omega$ be two mutually generic enumerations of the ground model reals $\mathbb{R}^V$. Then $f(x) = f(y)$ is in $V[x] \cap V[y] = V$. Since x enumerates $\mathbb{R}^V$, there must be some n such that $f(x) = x(n)$, a contradiction.

We now prove the theorem in the general case.

Proof. Assume towards a contradiction that there is such an f satisfying that for all $x \in X^\omega$ and for all $n < \omega$, $f(x)\ \not{E}\ x(n)$. Note that this is all absolute.

Since E is Borel, there is a set of Z of E-pins such that for any E-pin $(\mathbb{Q}, \sigma)$, there is some E-pin $(\mathbb{Q}', \sigma')$ in Z which is $\tilde{E}$-equivalent to $(\mathbb{Q}, \sigma)$.

Let $\mathbb{P}$ be a poset which adds generic filters $\dot{G}(\mathbb{Q}, \sigma)$ for all $(\mathbb{Q}, \sigma) \in Z$ and collapses Z to be countable. Let $\dot{x}$ be a name for an element of X^ω which enumerates the set $\left\{\sigma[\dot{G}(\mathbb{Q}, \sigma)] : (\mathbb{Q}, \sigma) \in Z\right\}$. Let τ be the name for $f(\dot{x})$.

Suppose G_1, G_2 are $\mathbb{P}$-generic over V, let $G_i(\mathbb{Q}, \sigma)$ be the realization of $\dot{G}(\mathbb{Q}, \sigma)$ according to G_i. Then $\sigma[G_1(\mathbb{Q}, \sigma)]$ and $\sigma[G_2(\mathbb{Q}, \sigma)]$ are E-related, since $(\mathbb{Q}, \sigma)$ is an E-pin. It follows that x_1 and x_2 are E^+-related, and so $f(x_1)$ and $f(x_2)$ are E-related. We conclude that $(\mathbb{P},\tau)$ is an E-pin.

There is therefore some $(\mathbb{Q},\sigma)$ in Z which is $\tilde{E}$-equivalent to $(\mathbb{P},\tau)$. Let G be $\mathbb{P}$-generic over V, $x=\dot{x}[G]$, and $G(\mathbb{Q},\sigma)$ be the interpretation of $\dot{G}(\mathbb{Q},\sigma)$ according to G. There is some n such that $x(n)=\sigma[G(\mathbb{Q},\sigma)]$. So $f(x)\ E\ \sigma[G(\mathbb{Q},\sigma)]=x(n)$, in contradiction. $\square$

4. Pinned equivalence relations in the Solovay model

In contrast to the ZFC situation:

Theorem 4.1 ([LZ20])**.** *In the Solovay model W, the following are equivalent for a Borel equivalence relation E.*

- *E is not pinned;*
- *$=^+$ is Borel reducible to E.*

4.1. **An application.** Let us see how this result can be used to provide Borel reductions from $=^+$.

Let $\mathbb{C}(X)$ be Cohen forcing corresponding the the Polish space X. For example, $\mathbb{C}(2^\omega)$ is identified with the poset of finite partial functions from ω to 2, and $\mathbb{C}((2^\omega)^\omega)$ is identified with the finite support product of countably many copies of $\mathbb{C}(2^\omega)$.

Theorem 4.2 ([KSZ13])**.** *Let $C\subseteq(2^\omega)^\omega$ be a comeager set. Then $=^+\restriction C$ is Borel bireducible with $=^+$.*

That is, the equivalence relation $=^+$, with its standard presentation on $(2^\omega)^\omega$, retains its full complexity on any comeager set.

Proof. It suffices to show that $=^+$ is Borel reducible to $=^+\restriction C$. The Solovay-model dichotomy will be used as follows.

Lemma 4.3. *Provably in ZF+DC, $=^+\restriction C$ is not pinned.*

Given the lemma, it therefore follows that $=^+\restriction C$ is not pinned in the Solovay model, and therefore $=^+\leq_B=^+\restriction C$.

We now turn to prove the lemma. Fix a sufficiently elementary countable model M containing the code for C. Then for $x\in(2^\omega)^\omega$ which is $\mathbb{C}((2^\omega)^\omega)$-generic over M, x is in C. Therefore we may replace C with the set of all $x\in(2^\omega)^\omega$ which are $\mathbb{C}((2^\omega)^\omega)$-generic over M.

Fix an injective map $\varphi\colon 2^\omega\to(2^\omega)^\omega$ so that for any distinct $a_1,...,a_n\in 2^\omega$, $(\varphi(a_1),...,\varphi(a_n))$ is $\mathbb{C}((2^\omega)^\omega)^n$-generic over M.

Let A be the union of all reals $x\in 2^\omega$ appearing in a sequence $\varphi(a)$ for some $a\in 2^\omega$. A is uncountable.

Lemma 4.4. *In a collapse extension, there is an enumeration $g\in(2^\omega)^\omega$ of A such that g is $\mathbb{C}((2^\omega)^\omega)$-generic over M.*

Given the lemma, we conclude that A is a potential invariant for $=^+\restriction C$. Since A is not countable, it is not an invariant in the ground model, and therefore $=^+\restriction C$ is not pinned. $\square$

4.2. **Some ideas from the proof.** Let κ be an inaccessible cardinal, $\mathbb{P} = \mathrm{Col}(\omega, <\kappa)$, $G \subseteq \mathbb{P}$ generic over V, and $W = L(\mathbb{R})^{V[G]}$.

Remark 4.5. If E is a Borel equivalence relation whose code is in V, and $(\mathbb{Q}, \sigma)$ is an E-pin in V, then $(\mathbb{Q}, \sigma)$ is a trivial pin in W.

Proof. Since $\kappa(E) < \beth_{\omega_1}$, we may assume that $|\mathbb{Q}| < \beth_{\omega_1}$. So there is a $\mathbb{Q}$-generic filter H over V, in $V[G]$. In particular $\sigma[H] = x$ is in W. Now show that, in W, $\mathbb{Q} \Vdash \sigma \; E \; \check{x}$. □

In W, let E be a Borel equivalence relation on a Polish space X. By the usual absorption trick, we may assume that the code for E is in V. Fix $(\mathbb{Q}, \sigma)$, an E-pin in W. Again assume that $(\mathbb{Q}, \sigma)$ is definable in W with parameters in V alone. We also use σ to denote the corresponding $\mathbb{P} * \dot{\mathbb{Q}}$-name in V.

Remark 4.6. If $((\mathbb{P} * \dot{\mathbb{Q}}) \restriction r, \sigma)$ is an E-pin in V, for some $r \in \mathbb{P} * \dot{\mathbb{Q}}$, then $(\mathbb{Q}, \sigma)$ is trivial in W.

Proof. By the previous remark, if $((\mathbb{P} * \dot{\mathbb{Q}}) \restriction r, \sigma)$ is an E-pin in V, then it is trivial in W. That is, there is $x \in W$ such that in W, $(\mathbb{P} * \dot{\mathbb{Q}}) \restriction r \Vdash \sigma \; E \; \check{x}$. Check now that, in W, also $\mathbb{Q} \Vdash \sigma \; E \; \check{x}$. (Recall that if $(\mathbb{R}, \tau)$ is a pin then for *any* two generics for $\mathbb{R}$ the interpretations of τ are equivalent.) □

Assume now that E is not pinned in W. Fix a non-trivial E-pin, $(\mathbb{Q}, \sigma)$. By the remark above, $((\mathbb{P} * \dot{\mathbb{Q}}) \restriction r, \sigma)$ is not an E-pin, for any r.

Exercise 4.7. Show that $(\mathbb{P} * \dot{\mathbb{Q}}) \times (\mathbb{P} * \dot{\mathbb{Q}}) \Vdash \sigma_{left} \; E \; \sigma_{right}$.

Fix now a countable model M, sufficiently large.

Consider now the map defines as follows. Given a filter $G \subseteq \mathbb{P}$ which is generic over M, take some ("left most branch in the tree to construct a generic") generic filter $H \subseteq \dot{Q}[G]$ over $M[G]$, and take $\sigma[G * H]$.

Remark 4.8. If $\mathbb{R}^{M[G]} = \mathbb{R}^{M[\tilde{G}]}$, $G \mapsto x \in X$ and $\tilde{G} \mapsto \tilde{x} \in X$, then $x \; E \; \tilde{x}$.

Proof. First note that W is the same as computed in $M[G]$ or $M[\tilde{G}]$. Furthermore, $(\mathbb{Q}, \sigma)$ is the same when interpreted by G or $\tilde{G}$. This is because $(\mathbb{Q}, \sigma)$ is definable using only parameters in V. Now $x = \sigma[G * H]$ and $\tilde{x} = \sigma[\tilde{G} * \tilde{H}]$ for some $H, \tilde{H}$. Since $(\mathbb{Q}, \sigma)$ is an E-pin in W, we conclude that $x \; E \; \tilde{x}$. □

So our map $G \mapsto x = \sigma[G * H]$ (for some H) is a homomorphism from $=^+$ to E.

Exercise 4.9. Suppose $G_1 \times G_2$ is $\mathbb{P}^2$-generic over M. Let $G_i \mapsto x_i$ according to the map above. Then $x_1 \not\!E \; x_2$.

This follows from the previous exercise, and the usual arguments with pins showing that we may replace the H_1, H_2 coming from the definition of the map to some H_1, H_2 so that $G_1 * H_1$, $G_2 * H_2$ are mutually generic.

So our map is close to being a reduction from $=^+$ to E. (After composing with the reduction $=^+ \to =^+$ sending an arbitrary set of reals the set of reals $\mathbb{R}^{M[G]}$ for some generic $G \subseteq \mathbb{P}$ over M.)

Remark 4.10. One could hope to end here. That is, maybe we can find a reduction $=^+ \to =^+$ that sends distinct sets of reals to mutually generic filters. That is not possible.

Exercise 4.11. Assume $D \subseteq \mathbb{R}^\omega$ is such that for any $x, y \in D$, if $x \neq y$ then $\{x(n) : n \in \omega\} \cap \{y(n) : n \in \omega\} = \emptyset$. Then $=^+ \not\leq_B =^+ \restriction D$.

Recall that if $G_1 \times G_2$ are $\mathbb{P}^2$ generic over M, then there is a $\mathbb{P}$-generic G_{12} such that $\mathbb{R}^{M[G_{12}]} = \mathbb{R}^{M[G_1 \times G_2]}$. We use this notation below.

Lemma 4.12. Suppose $G_1 \times G_2 \times G_3$ are $\mathbb{P}^3$-generic over M. Let $G_{12} \mapsto x_{12}$ and $G_{23} \mapsto x_{23}$. Then $x_{12} \not E x_{23}$.

Proof. Assume not. Then it must be that $\mathbb{P}^3 \Vdash x_{13}\ E\ x_{23}$. (This follows from the homogeneity of $\mathbb{P}$. Also if $\tilde{G}$ is the application of an automorphism of $\mathbb{P}$ to G, then $M[G]$ and $M[\tilde{G}]$ are equal, and so are the corresponding Solovay models computed by them. As before, the elements in X, $x, \tilde{x}$ corresponding to $G, \tilde{G}$ are E-related.)

Force now with $\mathbb{P}^4$, let $G_1 \times G_2 \times G_3 \times G_4$ be generic. Then

$$x_{12}\ E\ x_{23}\ E\ x_{34}.$$

So $\mathbb{P}^4 = (\mathbb{P}^2)^2$ forces that

$$x_{12}\ E\ x_{34}.$$

Identifying $\mathbb{P}$ with $\mathbb{P}^2$, we got a $\mathbb{P}^2$-generic $G_1 \times G_2$ such that

$$x_1\ E\ x_2,$$

contradicting Exercise 4.9. □

So we have a homomorphism from $=^+$ to E, such that on pairs $a, b \in \mathbb{R}^\omega$ of the form $a = \mathbb{R}^{M[G_{12}]}$ and $b = \mathbb{R}^{M[G_{23}]}$ where $G_1 \times G_2 \times G_3$ is $\mathbb{P}^3$-generic, E is a reduction.

Finally we get the desired reduction from $=^+$ to E by composing the map above with a Borel reduction $f\colon =^+ \to_B =^+$ satisfying

(1) $f(a) = \mathbb{R}^{M[G]}$ for some $\mathbb{P}$-generic G over M;
(2) if $a, b \in \mathbb{R}^\omega$ are of the form $a = x \cup z$, $b = y \cup z$, where x, y, z are pairwise disjoint, then there are generic filter $G_1 \times G_2 \times G_3$ for $\mathbb{P}^3$ such that $f(a) = \mathbb{R}^{M[G_{12}]}$ and $f(b) = \mathbb{R}^{M[G_{23}]}$;
(3) if $a \subseteq b$, write $a = x$ and $b = x \cup y$ for disjoint x, y, then there are filters $G_1 \times G_2$ such that $f(a) = \mathbb{R}^{M[G_1]}$ and $f(b) = \mathbb{R}^{M[G_{12}]}$.

The lemma above dealt with case (2), to show that the map is a reduction. A very similar argument gives the following lemma, showing that we get a reduction in case (3).

Lemma 4.13. Suppose $G_1 \times G_2$ are $\mathbb{P}^2$-generics over M. Let $G_1 \mapsto x_1$ and $G_{12} \mapsto x_{12}$. Then $x_1 \not E x_{12}$.

5. Symmetric models

We saw that studying properties of equivalence relations in different models of set theory can have implications for Borel reducibility. One type of examples was moving to models with the right cardinal arithmetic assumptions to get an inequality between pinned cardinals. Another was going to the Solovay model, and asking questions about the equivalence relation there.

It turns out that some "60's style" symmetric models, those where weak fragments of choice fail (even DC), can be used to analyse Borel reducibility and Borel homomorphisms between equivalence relation.

5.1. **Abstract nonsense.** Suppose A is a set in some generic extension of V. Let $V(A)$ be the minimal transitive model of ZF extending V and containing the set A. For example, this model can be written as the class directed union of $L(A, x)$ for $x \in V$. We will sometimes call this the model **generated by** A (over V).

There are similar ways of forming symmetric models: $L(A)$, $\mathrm{HOD}_{\mathrm{tc}\{A\}}$, $\mathrm{HOD}_{V,\mathrm{tc}\{A\}}$. For the A's we study here, all these models will have the same relevant properties. Also in our examples $V(A)$ and $\mathrm{HOD}_{V,\mathrm{tc}\{A\}}$ will coincide.

We view $V(A)$ as a (minimal) definable closure of A:

Fact 5.1. The following holds in $V(A)$. For any set X, there is some formula ψ, parameters $\bar{a}$ from the transitive closure of A and $v \in V$ such that X is the unique set satisfying $\psi(X, A, \bar{a}, v)$. Equivalently, there is a formula φ such that $X = \{x : \varphi(x, A, \bar{a}, v)\}$.

We will be particularly interested in sets definable from A and parameters in V alone.

5.2. E_0^ω **as an unpinned equivalence relation.** First let us see how pinned equivalence relations become unpinned. Let E_0^ω on $(2^\omega)^\omega$ be defined by pointwise-equivalence. Note that the map

$$x \mapsto A_x = \langle [x(n)]_{E_0} : n < \omega \rangle$$

is a complete classification for E_0^ω satisfying the required absoluteness properties. The invariants are countable sequences of E_0-classes. (Recall, being an E_0-class is absolute. Same for other countable equivalence relations.)

Fix $x \in (2^\omega)^\omega$ a Cohen generic over V. Let $A_n = [x(n)]_{E_0}$ and $A = \langle A_n : n < \omega \rangle$.

Proposition 5.2. In $V(A)$, $\prod_{n \in \omega} A_n = \emptyset$.

Lemma 5.3. Suppose $Z \in V(A)$, $Z \subseteq V$ is definable using $x_0, ..., x_n$ and A. Then $Z \in V[x_0, ..., x_n]$.

Proof. Fix a formula ϕ and a parameter $v \in V$ such that, in $V(A)$,

$$z \in Z \iff \phi(z, A, x_0, ..., x_n, v).$$

Claim 5.4. Suppose p, q are conditions agreeing on $x_0, ..., x_n$, then p, q cannot force conflicting statements about $\phi^{V(\dot{A})}(\check{z}, \dot{A}, \dot{x}_0, ..., \dot{x}_n, \check{v})$.

Proof. Assume to the contrary, that $p \Vdash \phi^{V(\dot{A})}(z, ...)$ and $q \Vdash \neg\phi^{V(\dot{A})}(z, ...)$. Without loss of generality, assume that our generic $x \in (2^\omega)^\omega$ extends p. Let x' be the result of making finite changes to $x_{n+1}, x_{n+2}, ...$ so that x' extends q. Note that x' is generic over V. Furthermore, $\dot{A}[x] = \dot{A}[x'] = A$, and $x'_i = x_i$ for $i \leq n$.

Working in $V[x]$, we conclude that

$$\phi^{V(A)}(z, A, x_0, ..., x_n, v).$$

However, working in $V[x']$ we conclude that

$$\neg\phi^{V(A)}(z, A, x_0, ..., x_n, v),$$

a contradiction. $\square$

Finally, we can define Z in $V[x_0, ..., x_n]$ as all $z \in V$ such that there is some p in the Cohen forcing which agrees with $x_0, ..., x_n$ and such that $p \Vdash \phi^{V(\dot{A})}(\check{z}, \dot{A}, \dot{x}_0, ..., \dot{x}_n, \check{v})$. $\square$

Corollary 5.5. E_0^ω is not pinned in $V(A)$.

Proof. Note that A is a potential invariant for E_0^ω, as $A = A_x$. Furthermore, if $A = A_y$ then $y \in \prod_n A_n$. So A is not an invariant in $V(A)$. □

Recall that if E is a countable equivalence relation then "E is pinned" is provable in ZF.

Corollary 5.6. E_0^ω is not Borel reducible to any countable Borel equivalence relation.

5.3. **Definability of potential invariants.** Let E and F be Borel (or analytic) equivalence relations on Polish spaces X and Y respectively. Assume furthermore that $f\colon X \to Y$ is a Borel homomorphism from E to F. That is, $x\ E\ y \implies f(x)\ E\ f(y)$. Then, for any E-pin $(\mathbb{P}, \tau)$, $(\mathbb{P}, f(\tau))$ is an F-pin. In other words, there is a map, definable from the code for f, sending E-pins to F-pins, and this holds uniformly in any generic extension. Furthermore, if f is a Borel reduction, then the corresponding map on pins is a reduction from $\bar{E}$ to $\bar{F}$.

Assume further that E and F admit absolute classifications $x \mapsto A_x$ and $y \mapsto B_y$ respectively. Then E-pins correspond to potential invariants: sets A such that in some generic extension $A = A_x$. In this case, a Borel homomorphism f from E to F corresponds to a definable map, using the parameters for f, E, F and the complete classifications, sending potential E-invariants to potential F-invariants.

For concreteness, let us write it here. The map sends a potential E-invariant A to the unique set B satisfying

in some (equivalently, any) $x \in X$ in a generic extension,
if $A = A_x$ then $B = B_{f(x)}$.

Furthermore, if $f\colon X \to Y$ is a reduction of E to F, then the map $A \mapsto B$ is injective. Moreover, in this case A can be defined from its image B as follows

in some (equivalently, any) $y \in Y$ in a generic extension,
if $B = B_y$ and $x \in X$ is such that $f(x)\ F\ y$, then $A = A_x$.

Corollary 5.7. Suppose $E \leq_B F$ and A is an E-invariant in some generic extension, and B is the F-invariant corresponding to it, then $V(A) = V(B)$, where $V(A)$ is the minimal transitive extension of V containing A.

Example 5.8. Consider E_0 on 2^ω. An absolute complete classification in this case is $x \mapsto A_x = [x]_{E_0}$. Suppose $x \in 2^\omega$ is Cohen generic, let $A = A_x$.

Lemma 5.9 (Levy)**.** If $y \subseteq V$ is definable using A and parameters in V alone, then $y \in V$.

Corollary 5.10. $E_0 \not\leq_B =_{\mathbb{R}}$.

In fact, this consideration, and Levy's homogeneity argument, can be seen to be equivalent to the generic-ergodicity of E_0.

Lemma 5.11. Suppose E, F, X, Y, $x \mapsto A_x$, $y \mapsto B_y$ as above. Let $\mathbb{P}$ be Cohen forcing for X. Let $x \in X$ be $\mathbb{P}$-generic over V, $A = A_x$. The following are equivalent.

(1) Any partial Borel homomorphism, defined on x, $f\colon E \to F$ sends a non-meager set to a single F class;
(2) If $B \in V[x]$ is an F-invariant definable in $V(A)$ using A and parameters in V alone, then $B \in V$.

Remark 5.12. If E is generically-ergodic, that is, if $Z \subseteq X$ is invariant under E ($[Z]_E = Z$), then Z is either meager or comeager, then clause (1) is equivalent to "any Borel homomorphism sends a comeager set to a single F-class". This is referred to as "E is generically F-ergodic".

Proof. Assume first (2), and let f be a Borel homomorphism defined on x. Let $y = f(x)$ and $B = B_y$. Then by (2) we conclude that B is in V. Fix a condition p forcing that "$B_{f(\dot{x})} = \check{B}$". Then any Cohen-generic x' which agrees with p (a non-meager set) satisfies $B_{f(x')} = B$, so $f(x')\ F\ f(x)$.

Assume now (1). Let $B \in V[x]$ be an F-invariant definable from A and parameters in V alone. In particular there is a name τ such that $B = B_{\tau[x]}$. Fix a condition p forcing the above. Fix a countable model M containing enough. The map $x \mapsto \tau^{M[x]}[x]$ is now a Borel homomorphism defined on a comeager subset of p. By (1), there is a condition p (with $x \in p$) such that $(p,p) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau[x_l]\ F\ \tau[x_r]$. We conclude that $B = B_{\tau[x]} \in V$. □

This analysis of homomorphisms can go further.

Proposition 5.13. Suppose $f\colon E_0^\omega \to E_0$ is a Borel homomorphism. Then, on comeager sets, there is an n and a homomorphism $g\colon E_0^n \to E_0$ such that f factors through g and the projection $\pi_n\colon E_0^\omega \to E_0^n$.

Proof. The point is this. Let $A = A_x$ be the generic invariant for E_0^ω. Suppose $B = [y]_{E_0}$ is in $V(A)$, definable from A. Then, for some n, y is definable from $x_0, ..., x_n$ (in $V[x_0, ..., x_n]$).

Let $y = \tau[x] = \sigma[x_0, ..., x_n]$, where σ is a namefor Cohen forcing in $(2^\omega)^{n+1}$. Fix a countable model M and define a map sending Cohen generics $x_0, ..., x_n$ to $g(x_0, ..., x_n) = \sigma[x_0, ..., x_n]$. Then g is a a Borel homomorphism from E_0^{n+1} to E_0, defined on a comeager set.

On a set of generics (extending a condition forcing the above), the homomorphism f agrees, up to E_0 equivalence, with the composition of π_{n+1} and g. □

5.4. **On the basic Cohen model.** Recall that $=^+$ is defined on $\mathbb{R}^\omega$ so that the map

$$x \mapsto A_x = \{x(n) : n \in \omega\}$$

is a complete classification (which satisfies the absoluteness properties). Let $x \in \mathbb{R}^\omega$ be Cohen generic, and $A = A_x$. Then the "basic Cohen model" can be presented as $V(A)$. It is naturally connected to $=^+$.

Definition 5.14. For any $Z \in V(A)$ and $\bar{a} \subseteq A$, say that $\bar{a}$ is a **support** for Z if there is a formula ϕ, a parameter $v \in V$ such that Z is defined as the unique solution to

$$\phi(Z, A, \bar{a}, v)$$

in $V(A)$. Equivalently, there is a formula ψ such that in $V(A)$, Z is defined as

$$\{z : \phi(z, A, \bar{a}, v)\}.$$

Lemma 5.15. Suppose $Z \in V(A)$ is defined in $V(A)$ as

$$\{z : \phi(z, A, \bar{a}, v)\}.$$

and assume further that $Z \subseteq V[\bar{a}]$. Then $Z \in V[\bar{a}]$.

Proof. It suffices to prove the lemma with $\bar{a}$ is empty. The usual homogeneity arguments show that for $z \in V$

$$V[x] \models \phi^{V(A)}(z, A, v) \iff V \models \mathbb{P} \Vdash \phi^{V(\dot{A})}(\check{z}, \dot{A}, \check{v}).$$

So Z is defined in V as $\left\{z : \mathbb{P} \Vdash \phi^{V(\dot{A})}(\check{z}, \dot{A}, \check{v})\right\}$. □

Lemma 5.16 (Continuity Lemma)**.** Let ϕ be a formula, $\bar{a} = a_0, ..., a_{n-1}$ a finite sequence of distinct members of A, and $v \in V$. Suppose $\phi(A, \bar{a}, v)$ holds in $V(A)$. Then there are open sets $U_0, ..., U_{n-1}$ such that $a_i \in U_i$ and for any $\bar{b} = b_0, ..., b_{n-1}$ consisting of distinct elements from A, if $b_i \in U_i$ for all $i \leq n-1$, then $\phi(A, \bar{b}, v)$ holds in $V(A)$.

See [Fel71, p.133], also [CMRS20, p.19] for a proof in this presentation of the basic Cohen model.

Lemma 5.17 ($\approx$ Mostowski [Mos39]?)**.** Suppose both $\bar{a}_1, \bar{a}_2$ are supports for $Z \in V(A)$, then $\bar{a}_1 \cap \bar{a}_2$ is a support for Z.

Proof. Fix formulas ϕ_i and parameters v_i such that Z is defined as the unique solution to $\phi_i(Z, A, \bar{a}_i, v_i)$ in $V(A)$. We may write $\bar{a}_i = \bar{a}, \bar{b}_i$ where $\bar{b}_1, \bar{b}_2$ are disjoint. We need to show that $\bar{a}$ is a support for Z. Let l_i be the length of $\bar{b}_i$. (We assume the sequences $\bar{a}, \bar{b}_i$ all contain distinct elements.)

Apply the continuity lemma to the statement

$$\phi_i(X, A, \bar{a}, \bar{b}_i, v_i) \text{ both uniquely define } X \text{ in } V(A),$$

and get open sets U_i, V_j such that the same holds for any other sequences $\bar{b}'_1, \bar{b}'_2$ with $b'_1(i) \in U_i$ and $b'_2(j) \in V_j$. Let $v = (v_1, v_2)$ and define $\psi(Z, A, \bar{a}, v)$ as

there exists two disjoint sequences $\bar{c}_1, \bar{c}_2$ of length l_1, l_2 such that $c_1(i), c_2(j)$ are in U_i, V_j respectively, and $\phi_1(X, A, \bar{a}, \bar{c}_1, v_1) \iff \phi_2(X, A, \bar{a}, \bar{c}_2, v_2)$ uniquely define X, and $\phi_1(Z, A, \bar{a}, \bar{c}_1, v_1)$ holds.

Then Z is the unique solution to $\psi(Z, A, \bar{a}, v)$ in $V(A)$. □

Corollary 5.18. For every set $Z \in V(A)$ there is a **minimal support** $\bar{a} \subseteq A$. Furthermore, the map $Z \mapsto \bar{a}$ is definable in $V(A)$.

Proposition 5.19. The basic Cohen model $V(A)$ satisfies "choice for sets of well orderable sets". That is, if Y is a set such that each $y \in Y$ is well-orederable, then there is a choice sequence $c \colon Y \to \bigcup Y$.

Proof. For $y \in Y$, let $s(y)$ be the minimal $a \subseteq A$ such that for any $t \in y$, the support of t is contained in a. The map $y \mapsto s(y)$ is definable in $V(A)$. Since each $y \in Y$ is well-orderable, then $s(y)$ is finite. In particular, $s(y)$ can be canonically ordered as $\bar{a}(y)$.

For $y \in Y$, choose some "minimal" formula ϕ and parameter v (according to a fixed well ordering in V) such that there is some $t \in y$ defined as the unique solution to $\phi(t, A, \bar{a}(y), v)$. Again the map $y \mapsto \bar{a}, \phi, v$ is definable in $V(A)$, and therefore the map $c(y)$ =the unique solution to $\phi(-, A, \bar{a}, v)$ is definable in $V(A)$, and is the desired choice function. □

As an overkill:

Corollary 5.20. $=^+$ is not Borel reducible to E_0^ω.

Proof. Otherwise, we would get that $V(A) = V(B)$ for some $B = \langle B_n : n < \omega\rangle$, where each B_n is an E_0-class, and B is definable from A alone. Each B_n is therefore a countable set of reals in $V(A)$.

By the argument above, there is a choice function $c \in \prod_n B_n$, which is in fact definable without parameters. As c is coded by a real, it follows that $c \in V$. So we conclude that $V(A) \subseteq V$, a contradiction. □

5.5. **Forcing over the Cohen model.**

Lemma 5.21 (Monro [Mon73])**.** Work in the basic Cohen model $V(A)$, let $\mathbb{P}$ be the poset of all finite partial functions $p\colon A \times A \to 2$. Then forcing with $\mathbb{P}$ adds no subsets of V to $V(A)$.

In particular, no sets of ordinals are added. This can only happen when forcing over a model where AC fails.

Proof. Suppose τ is a $\mathbb{P}$-name for a subset of V, $\tau \in V(A)$. Let $\bar{a}$ be a support for τ. We show that for any condition $p \in \mathbb{P}$, if p forces $v \in \tau$ then $p \restriction \bar{a} \times \bar{a}$ forces the same.

Let the domain of p be $\bar{a}, \bar{b}$, with $\bar{b}$ disjoint from $\bar{a}$. By the Continuity Lemma, there are infinitely many distinct tuples $\bar{b}'$, disjoint from $\bar{a}$, such that $p[\bar{b}'] \Vdash v \in \tau$ as well, where $p[\bar{b}']$ is the condition with domain $\bar{a}, \bar{b}'$ defined on $\bar{b}'$ as p is defined on $\bar{b}$. Note that $p[\bar{b}']$ extends $p \restriction \bar{a} \times \bar{a}$.

Now for any q extending $p \restriction \bar{a} \times \bar{a}$, there is some $\bar{b}'$ such that $p[\bar{b}'] \Vdash v \in \tau$ and $p[\bar{b}']$ is compatible with q (take $\bar{b}'$ disjoint from the domain of q). It follows that $p \restriction \bar{a} \times \bar{a}$ forces $v \in \tau$. □

Let $A = A^1$, fix $G \subseteq \mathbb{P}$ generic over $V(A^1)$ and define

$$A^2_a = \left\{b \in A^1 : G(a, b) = 1\right\},\ A^2 = \left\{A^2_a : a \in A^1\right\}.$$

Note that $V(A^2) = V(A^1)(A^2)$. In fact, denoting $V(A^1) = M$, then $M(A^2)$ over M looks just as M is over V.

Lemma 5.22. If $Z \in V(A^2)$ is definable from A^2, a finite set $\bar{a} \subseteq A^2$, and parameters in $V(A^1)$, and furthermore $Z \subseteq V(A^1)$, then $Z \in V(A^1)[\bar{a}]$.

This follows from homogeneity, as before, just using M instead of V.

Generally, forcing does not change the ordinals, so any set of ordinals is trivially a subset of the ground model. The point here is that any set of sets of ordinals is a subset of the ground model $V(A^1)$, by Lemma 5.21. For example, any subset of $\mathbb{R}$ is a subset of the ground model.

Corollary 5.23. $=^{++}$ is not Borel reducible to $=^{+}$.

Proof. Note that A^2 is a complete invariant for $=^{++}$. Assume $=^{++}$ is Borel reducible to $=^{+}$, then we conclude that there is a set of reals B such that $V(A^2) = V(B)$, and B is definable from A^2. Since $V(A^2), V(A^1)$ have the same subsets of V, it follows that $B \subseteq V(A^1)$. By the previous lemma, in fact $B \in V(A^1)$, a contradiction. □

As before, this proof in fact provides a tight analysis of Borel homomorphisms. Let $u\colon =^{++} \to_B =^{+}$ be a Borel homomorphism sending $x \in (\mathbb{R}^\omega)^\omega$ to a sequence $y \in \mathbb{R}^\omega$ enumerating the $\omega \times \omega$-many reals. (That is, u takes a set of sets of reals and gives the union.)

Proposition 5.24. Any Borel homomorphism $f\colon =^{++}\to=^{+}$ factors, on a comeager set, through u and some Borel homomorphism from $=^{+}$ to $=^{+}$.

This fails with the topology on $(\mathbb{R}^{\omega})^{\omega}$, but holds when viewing $=^{++}$ as an equivalence relation on $\mathbb{R}^{\omega}\times(2^{\omega})^{\omega}$ as follows. Given $(x,u)\in\mathbb{R}^{\omega}\times(2^{\omega})^{\omega}$, let $A_{(x,u)}(n)=\{x(i)\,:\,u(n)(i)=1\}$, and $A_{(x,u)}=\left\{A_{(x,u)}(n)\,:\,n\in\omega\right\}$. Now define F on $\mathbb{R}^{\omega}\times(2^{\omega})^{\omega}$ so that $(x,u)\mapsto A_{(x,u)}$ is a complete classification. Then F is Borel bireducible with $=^{++}$ and satisfies the above proposition.

5.6. **Between the jumps.**

Definition 5.25 (Hjorth-Kechris-Louvaeu [HKL98])**.** Fix $n\geq 3$ and $0\leq k\leq n-2$. Let $\mathcal{P}^{n,k}_{*}(\omega)$ be the collection of pairs (A,R) such that:

1. A is a hereditarily countable set in $\mathcal{P}^{n}(\omega)$;
2. R is a ternary relation on $A\times A\times(\mathcal{P}^{k}(\omega)\cap\mathrm{tc}(A))$ such that
   - for any $a,b\in A$ there is some r such that $R(a,b,r)$ holds;
   - given any $a\in A$, for any $b,b'\in A$ and any r, if $R(a,b,r)$ and $R(a,b',r)$ both hold then $b=b'$.

Fact 5.26 (See [HKL98, p. 95, 98, 99])**.** There is a $L_{\omega_1,\omega}$ theory whose models precisely codes sets in $\mathcal{P}^{n,k}_{*}$.

The equivalence relation $\cong^{*}_{n,k}$ is defined as the isomorphism relation of countable structures coding pairs (A,R) in $\mathcal{P}^{n,k}_{*}(\omega)$. In other words, $\cong^{*}_{n,k}$ is defined precisely so that it admits a natural (absolute) classification with $\mathcal{P}^{n,k}_{*}(\omega)$ as a set of complete invariants.

Given an invariant (A,R) for $\cong^{*}_{n,k}$, the set A is an invariant for $\cong_{n}$, while R provides a parametrization of A using lower rank sets, therefore "simplifying its complexity". Thus the invariants in $\mathcal{P}^{n,k}_{*}(\omega)$ should be viewed as sets with intermediate complexity, between $\mathcal{P}^{n-1}(\omega)$ and $\mathcal{P}^{n}(\omega)$. Indeed, the equivalence relations of Hjorth-Kechris-Louveau refine the Friedman-Stanley hierarchy:

$$=^{(n-1)+}_{\mathbb{R}}\sim_{B}\cong_{n}\leq_{B}\cong^{*}_{n+1,0}\leq_{B}\cdots\leq_{B}\cong^{*}_{n+1,n-1}\leq_{B}\cong_{n+1}\sim_{B}=^{n+}_{\mathbb{R}}.$$

Remark 5.27. Suppose (A,R) is in $\mathcal{P}^{n,0}_{*}(\mathbb{N})$. Given $a\in A$, the relation R allows to enumerate A, sending $b\in A$ to the smallest $j\in\mathbb{N}$ such that $R(a,b,j)$ holds. Thus invariants for $\cong^{*}_{n,0}$ should be viewed as sets in $\mathcal{P}^{n}(\mathbb{N})$, which can be enumerated definably in a parameter.

For example, a set in $\mathcal{P}^{3,0}_{*}$ is a set of sets of reals together with a (uniform) way to enumerate it, using a parameter. This is analogous to a countable equivalence relation: for example, invariants for E_0 are sets of reals (the entire equivalence class), together with a way to enumerate it, using any real from the class.

Proposition 5.28 ([HKL98])**.**

$$=^{+}<_{B}\cong^{*}_{3,0}<_{B}\cong^{*}_{3,1}<_{B}=^{++}.$$

The two outer irreducibility results are proven in [HKL98] using the analysis of the potential complexity of the above equivalence relations. The middle result is proven via forcing, using some variation of the "non-pinned" arguments.

Below are proofs of these 3 irreducibility results, from the point of view of studying models generated by their invariants.

First, these invariants can be simplified as follows.

Lemma 5.29. Suppose $(A,R) \in \mathcal{P}^{3,1}_*$, in some generic extension of V. Then $V(A,R) = V(B)$ for some set of reals B. (B is *not* necessarily definable from A, R.)

First, note that disjoint sets can be coded as lower rank sets.

Claim 5.30. Suppose A is a set of disjoint sets of reals. Then $V(A) = V(B)$ for a set of reals B. Furthermore, B and A are bi-definable from one another.

Proof. Let $\bar{A} = \bigcup A$, a set of reals, and let R be the equivalence relation on $\bar{A}$ defining its partition into A. Let $B = (\bar{A}, R)$. Then B can be coded as a set of reals, and $V(A) = V(B)$ □

We now prove the lemma. Fix $a \in A$. Then the relation R provides a map $A \mapsto \mathcal{P}^2(\omega)$

$$b \mapsto \{r \in \mathcal{P}(\omega) : R(a,b,r)\}.$$

By assumption, the image of A is a set of disjoint sets of reals, and therefore can be coded by a set of reals. Furthermore, A is definable from its image, together with the parameter a. The relation R can be similarly coded as a set of reals, since $R \subseteq A \times A \times \mathbb{R}$.

Corollary 5.31. $=^{++} \not\leq_B \cong^*_{3,1}$.

Proof. Let A^2 be the $=^{++}$-invariant from above. By Lemma 5.29, if $=^{++} \leq_B \cong^*_{3,1}$ then $V(A^2)$ is of the form $V(B)$ for some *set of reals* B. If $B \in V(A^2)$ is a set of reals, $B \subseteq V(A^1)$. Any such set B is definable in $V(A^2)$ using finitely many members $\bar{a}$ from A^2, and parameters in $V(A^1)$. We then conclude that $V(B) \subseteq V(A^1)[\bar{a}]$, and therefore $V(B) \neq V(A)$. □

5.7. **A proof that $\cong^*_{3,1}$ is not Borel reducible to $\cong^*_{3,0}$.** Let $V(A^1)$ be the basic Cohen model, and let $\mathbb{P}$ be the poset for adding a *single* subset $X \subseteq A^1$ with finite approximations. As before, forcing with $\mathbb{P}$ over $V(A^1)$ adds no sets of ordinals.

Suppose $X \subseteq A^1$ is $\mathbb{P}$-generic over $V(A^1)$. Define

$$A = \{X \Delta \bar{a} : \bar{a} \subseteq A \text{ is finite}\}.$$

A is an invariant for $\cong^*_{\mathbf{3,1}}$, seen as follows: define $R \subseteq A \times A \times (2^\omega)^{<\omega}$ by

$$(Z, Y, \bar{a}) \in R \iff Z = Y \Delta \bar{a}.$$

For each $a \in A$, $R(a,-,-)$ defines an injective map from A to $(2^\omega)^{<\omega}$. Thus (A,R), which is bi-definable with A, is in $\mathcal{P}^{3,1}_*$ (is a potential invariant for $\cong^*_{3,1}$). We now study $V(A)$. Note that $V(A) = V(A^1)[X]$.

Just as in the trivial example, when we proved $E_0 \not\leq_B =_\mathbb{R}$, we have

Claim 5.32 (In $V(A^1)$)**.** For any $p, q \in \mathbb{P}$ there is an automorphism π of $\mathbb{P}$, fixing $\dot{A}$, such that $\pi(q)$ is compatible with p.

(Equivalently, for any generic $G \subseteq \mathbb{P}$ and any $q \in \mathbb{P}$ there is a generic $\tilde{G}$ with $\dot{A}[\tilde{G}] = \dot{A}[G]$ and $q \in \tilde{G}$.) An immediate consequence is (the usual weak homogeneity argument)

Proposition 5.33. If $S \subseteq V(A^1)$ is in $V(A)$, definable from A and parameters in $V(A^1)$, then $S \in V(A^1)$.

Since any set of reals in $V(A)$ is a subset of $V(A^1)$, we conclude

Corollary 5.34. $\cong^*_{3,1}\not\leq_B =^+$.

To show that $\cong^*_{3,1}$ is not Borel reducible to $\cong^*_{3,0}$, it suffices to show:

Proposition 5.35. Suppose $B \in V(A)$, B is a set of sets of reals, definable from A and parameters in V, and suppose that B is countable (in $V(A)$), then $V(B) \neq V(A)$.

Proof. Otherwise, in particular $X \in V(B)$, then X is definable using B and some finitely many members of B, and some parameters from $V(A^1)$. Since B is definable from A, there is a formula ψ, finitely many parameters $U_1, .., U_k \in B$ and $z \in V(A^1)$ such that X is the unique set satisfying $\psi^{V(A)}(X, A, U_1, ..., U_k, z)$.

Fix some condition $r \in \mathbb{P}$ forcing the above and work in $V(A^1)$. For any $a \in A^1 \setminus \operatorname{dom} r$, let π_a be the permutation of $\mathbb{P}$ swapping the value of a. Then $\pi_a r = r$ and $\pi_a \dot{A} = \dot{A}$. In particular, for any such a, $r = \pi_a r$ forces that $\pi_a \dot{U}_j \in \dot{B}$ and $\dot{X}\Delta\{a\} = \pi_a \dot{X}$ is defined uniquely by $\psi(\dot{X}\Delta\{a\}, \dot{B}, \pi_a\dot{U}_1, ..., \pi_a\dot{U}_k, \check{z})$.

Thus the map sending a to $\pi_a U_1, ..., \pi_a U_k$ is injective between $A^1 \setminus \operatorname{dom} r$ and B^k. Since B is countable, so is B^k and therefore A^1 is countable. Since $\mathbb{P}$ adds no reals, this is a contradiction. □

This proof also shows that:

Proposition 5.36. $\cong^*_{3,1}\not\leq (\cong^*_{3,0})^\omega$.

It suffices to establish the following.

Proposition 5.37. Let $B = \langle B_k : k < \omega\rangle \in V(A)$ be a sequence of set of sets of reals such that B is definable from A with parameters in V. Assume further that for each $k < \omega$, B_k is countable. Then $V(B) \subsetneq V(A)$.

The proof is similar to the one above. For contradiction, we assume that $V(B) = V(A)$, and so in particular X is in $V(B)$. Thus X can be defined using B, $C_1, ..., C_m$ and $z \in V(A^1)$ where C_i is a finite subset of B_i. It follows that each $X\Delta\{a\}$, $a \in A^1$, can be defined from some finitely many elements of $B_1, ..., B_m$. Since $B_1 \cup ... \cup B_m$ is countable, we reach a contradiction. (The main point here is that the infinite union $\bigcup_k B_k$ may not be countable, yet ZF proves that a finite union of countable sets is countable!)

5.8. **A proof that $\cong^*_{3,0}$ is not Borel reducible to $=^+$.** Let A^1 be the Cohen set as above. Our model will simply be the Cohen model $V(A^1)$, only presented as generated by an interesting $\cong^*_{3,0}$-invariant. Consider A^1 as a subset of 2^ω. For any $n \in \omega$, finite $\bar{k} = k_0, ..., k_{m-1} \subseteq \omega$ such that $n \notin \bar{k}$, define $\pi^n_{\bar{k}} x$ as follows. $\pi^n_{\bar{k}} x = x$ if $x(n) = 0$. If $x(n) = 1$, $\pi^n_{\bar{k}} x$ swaps the values $x(l)$ for $l \in \bar{k}$. That is, $\pi^n_{\bar{k}}$ permutes finitely many indices for half of the space 2^ω, and fixes the other half. Let $\Pi = \langle \pi^n_{\bar{k}}\rangle_{n,\bar{k}}$ be the group of all compositions of finitely many $\pi^n_{\bar{k}}$'s. Note that $\pi^n_{\bar{k}} \circ \pi^n_{\bar{k}} = \mathrm{id}$ for any $n, \bar{k}$, so Π is a group. For a set of reals $B \subseteq 2^\omega$, define $\pi B = \{\pi x : x \in B\}$. Finally, define

$$A = \left\{\pi A^1 : \pi \in \Pi\right\}.$$

For any $B \in A$, A^1 can be recovered as πB for some $\pi \in \Pi$. In particular, given any $B \in A$, $\{\pi B : \pi \in \Pi\} = A$. Since Π is countable, A is a $\cong^*_{3,0}$-invariant.

We will work with the model $V(A)$, which is simply the Cohen model $V(A^1)$. To show irreducibility to $=^+$:

Lemma 5.38. Suppose B is a set of reals in $V(A)$, definable from A and parameters from V. Then $V(B) \subsetneq V(A)$.

To prove the lemma, we present A^1 as a generic extension of an intermediate model $V \subseteq V(\tilde{A}^1) \subseteq V(A^1)$, such that the extension does not add reals (or sets of ordinals).

Let

$$\tilde{A}^1 = \bigcup_{a \in A^1} [a]_{E_0} = \bigcup A.$$

That is, $\tilde{A}^1$ contains all reals in A^1 and all their finite changes. Let $\mathbb{Q}$ be the poset of all finite functions $p\colon \operatorname{dom} p \longrightarrow 2$, $\operatorname{dom} p \subseteq \tilde{A}^1$, such that if $p(x) = p(y) = 1$ then x, y are not E_0-related. $\mathbb{Q}$ adds a generic subset of $\tilde{A}^1$, choosing one element out of each E_0-class.

Proposition 5.39. $\mathbb{Q}$ does not add sets of ordinals when forcing over $V(\tilde{A}^1)$

This can be done is we saw for Monro's poset. Another point of view is this:

Exercise 5.40. A^1 is $\mathbb{Q}$-generic over $V(\tilde{A}^1)$.

Note that $\tilde{A}^1$ is bi-definable with $\left\{[a]_{E_0} : a \in A^1\right\}$. The poset $\mathbb{Q}$ can be thought of as adding a choice function through $\left\{[a]_{E_0} : a \in A^1\right\}$ with finite approximations. For any finite $\bar{a} \subseteq A^1$, the set $D \subseteq \mathbb{Q}$ of all p such that $[a_i]_{E_0} \in \operatorname{dom} p$, is a dense open set. In fact, any dense open subset of $\mathbb{Q}$ in $V(\tilde{A}^1)$ is of this form. So any choice function is generic.

Since any real in the Cohen model $V(A^1)$ is already in $V(\bar{a})$ for a finite $\bar{a} \subseteq A^1$, it follows that $V(A^1)$ and $V(\tilde{A}^1)$ have the same reals.

The permutations π^n_k as above generate permutations of $\mathbb{Q}$ as follows. For $p \in \mathbb{Q}$, for any $x \in \tilde{A}^1$, $\pi^n_k p$ swaps the values of $p(x), p(\pi^n_k x)$, as long as one of them is defined. This generates permutations π of $\mathbb{Q}$ for any $\pi \in \Pi$.

Claim 5.41. For any $p, q \in Q$, there is some $\pi \in \Pi$ such that $\pi p \parallel q$.

For any $\pi \in \Pi$, $\pi \dot{A} = \dot{A}$.

Now the usual weak homogeneity argument shows that if $B \subseteq V(A^1)$, is definable in $V(A) = V(A^1)(A)$ using A and parameters in $V(A^1)$ alone, then $B \in V(A^1)$. Since any set of sets of reals B is a subset of $V(A^1)$, this concludes Lemma 5.38.

Again we can see that even:

Proposition 5.42. $(\cong^*_{3,0})^\omega$ is not reducible to $\cong^*_{3,1}$.

The "interesting" $(\cong^*_{3,0})^\omega$-invariant is constructed as follows. Start with a sequence of mutually generic Cohen sets $\left\langle A^1_n : n < \omega\right\rangle$. The model $V(\left\langle A^1_n : n < \omega\right\rangle)$ is essentially the same as the basic Cohen model. Define $\tilde{A}^1_n$ to be the E_0-saturation of A^1_n. Then $\left\langle A^1_n : n < \omega\right\rangle$ is generic over $V(\left\langle \tilde{A}^1_n : n < \omega\right\rangle)$ for the forcing to add a sequence of subsets to $\left\langle \tilde{A}^1_n : n < \omega\right\rangle$, choosing one element out of each E_0-class, by finite approximations. Finally, define $A_n = \left\{\pi A^1_n : \pi \in \Pi\right\}$. Then $A = \langle A_n : n < \omega\rangle$ is an invariant for $(\cong^*_{3,0})^\omega$, and it can be shown that $V(A)$ is not generated by any set of reals. (As we did for A^2.)

6. More on pins

Next we will consider pins for non-Borel equivalence relation. (Recall, if E is $\mathbf{\Pi}^0_\alpha$, then $\kappa(E) \leq \beth^+_\alpha$.) Some interesting things here:

- (Hjorth [Hjo01]) If $a \colon G \curvearrowright X$ is a counter example to the topological Vaught conjecture, then $\kappa(E_a) = \infty$;
- (Larson-Zapletal [LZ20]) If $\kappa(E)$ is above a measurable cardinal, then $\kappa(E)$ is unbounded ($\kappa(E) = \infty$);
- (Larson-Zapletal [LZ20]) for every countable α there is an analytic E with $\kappa(E) = \kappa(\alpha)$, the α'th-Erdos cardinal.

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