A general Voronoi summation formula for $GL(n,\mathbb{Z})$

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Abstract

In [14] we derived an analogue of the classical Voronoi summation formula for automorphic forms on $GL(3)$, by using the theory of automorphic distributions. The purpose of the present paper is to apply this theory to derive the analogous formulas for $GL(n)$.

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Dedicated to Shing-Tung Yau on the occasion of his 60th birthday

1. Introduction

The Voronoi summation formulas for $GL(2)$ and $GL(3)$ have had numerous applications to problems in analytic number theory, perhaps most notably to recent subconvexity results. The formulas provide an identity between sums of the form

$$\sum a_m \, e^{2\pi i k\alpha} \, f(m) \quad = \quad \sum \tilde{a}_k \, S(m,\alpha) \, F(m) \tag{1.1}$$

where a_m are Fourier coefficients of the automorphic form, $\alpha \in \mathbb{Q}$, $S(k,\alpha)$ an exponential sum, and f, F a pair of test functions related by an integral transformation. Indeed, such a rubric covers the Poisson summation formula, itself a cornerstone tool in analytic number theory. For $GL(2)$ the exponential sum is a single exponential, whereas for $GL(3)$ it is a Kloosterman sum. One way to prove the $GL(2)$ formula is to use Mellin inversion of the functional equation of the standard L-function with twists. An analytic variant of this method, carried out

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by Duke-Iwaniec [4], involves an $n-1$-dimensional hyperkloosterman sum. That could be regarded as predicting the appearance of a hyperkloosterman sum in the Voronoi formula for $GL(n)$. However, the Mellin-inversion approach quickly runs into computational difficulties.

The argument that we follow here closely follows the one in [14], except that it is done for general $GL(n)$, and not just $n = 3$ as was the case there. In particular, all of the analytic details necessary to justify that formula are essentially covered in [14], though the formal aspects of the computation – while philosophically identical – are more involved, of course. We first execute the computation in classical terms, entirely analogously to the $GL(3)$ argument in [14], and then later perform the computation adelically. The latter has the advantage of providing a formula for general congruence groups.

Fourier coefficients of automorphic forms on $GL(n)$ are indexed by $(n-1)$-tuples $k = (k_1, \ldots, k_{n-1}) \in \mathbb{Z}^{n-1}$; for a cusp form, the coefficients vanish unless $k \in (\mathbb{Z}_{\neq 0})^{n-1}$. For full level Hecke cusp forms, the coefficients are uniquely determined by the L-function coefficients $\{a_{(1,1,\ldots,1,k_{n-1})}\}$, or dually, by the coefficients $\{a_{(k_1,1,1,\ldots,1)}\}$. That is a consequence of fairly complicated Hecke relations. However, the Voronoi formulas we state hold even for non-Hecke eigenforms, and our proof does not require the Hecke property. When the full-level assumption is dropped, there are more Fourier coefficients to take into account. For $GL(2)$ this is explained by the Jacquet-Langlands and Atkin-Lehner theory, but in the absence of a satisfactory theory of this type for $GL(n)$, one cannot at present pin down the general Fourier coefficients for $GL(n)$ in terms of the L-function coefficients. Thus we state the formulas for full level, although our second, adelic proof produces a general formula once one has further information of Atkin-Lehner type.

We say that $(\lambda, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ is the representation parameter of a cusp form on $GL(n)$ if its archimedean component embeds[1] into the principal series representation

$$V_{\lambda,\delta} = \Bigg\{ f : GL(n,\mathbb{R}) \to \mathbb{C} \;\Bigg|\; f\left(g \begin{pmatrix} a_1 & 0 & 0 \\ \star & \ddots & 0 \\ \star & \star & a_n \end{pmatrix}\right) = f(g) \cdot \prod_{1 \le j \le n} \left(|a_j|^{\frac{n+1}{2}-j-\lambda_j} \operatorname{sgn}(a_j)^{\delta_j} \right) \Bigg\}. \tag{1.2}$$

We do not assume that the archimedean component is a full principal series representation; in particular, it need not belong to the spherical principal series. The parameter $(\lambda, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ may not be uniquely determined by the archimedean component: indeed, when the archimedean component is an irreducible principal series representation, the λ_j , δ_j can be (simultaneously) freely permuted, a fact which provides useful flexibility in the range of test functions allowed in the Voronoi formula. Except for that flexibility, our formula is independent of the parameters up to permutation, as will be clear from its statement below.

Another ingredient of the Voronoi formula is the integral transform relating the two test functions; in the prototypical example of the Poisson summation

[1] The Casselman embedding theorem [1] guarantees that such an embedding exists.

formula the Fourier transform plays that role. We shall give two descriptions here, entirely analogous to those in [13] and [14]. The first is more concise but somewhat symbolic, in that it needs to be interpreted carefully to have meaning. Suppose that $f \in |x|^{\lambda_n} \operatorname{sgn}(x)^{\delta_n} \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the space of Schwartz functions on the real line. The transform F of f is then defined as

$$F(y) \;=\; \int_{\mathbb{R}^n} f\left(\frac{x_1 \cdots x_n}{y}\right) \cdot \prod_{1 \le j \le n} \big(e(-x_j)|x_j|^{-\lambda_j} \operatorname{sgn}(x_j)^{\delta_j}\, dx_j\big), \tag{1.3}$$

with an inversion formula that involves replacing the λ_j with $-\lambda_j$. Evidently this formula can be regarded as a generalization of the Fourier transform. For details on how the formula needs to be interpreted see [14, §5].

The second, equivalent description of the transform $f \mapsto F$ relates the signed Mellin transform

$$M_\delta f(s) \;=_{\text{def}}\; \int_{\mathbb{R}} f(x)|x|^{s-1} \operatorname{sgn}(x)^\delta \, dx \qquad \big(\delta \in \mathbb{Z}/2\mathbb{Z}\big) \tag{1.4}$$

of $F(s)$ to that of $f(-s)$:

$$M_\delta F(s) \;=\; (-1)^{n\delta} \Bigg(\prod_{1 \le j \le n} G_{\delta_j+\delta}(s - \lambda_j + 1)\Bigg) M_\delta f(-s)\,; \tag{1.5}$$

here $G_\delta(s)$, with $\delta \in \mathbb{Z}/2\mathbb{Z}$, denotes the Gamma factor

$$G_\delta(s) \;=\; \begin{cases} 2\,(2\pi)^{-s}\,\Gamma(s)\cos(\frac{\pi s}{2}) & \text{if } \delta = 0 \\ 2\,i\,(2\pi)^{-s}\,\Gamma(s)\sin(\frac{\pi s}{2}) & \text{if } \delta = 1 \end{cases} \tag{1.6}$$

that was introduced in [13, §4]. One can reconstruct the odd and even components of F from the formula (1.5) by means of the signed Mellin inversion formula

$$\begin{aligned} \frac{\big(f(x) + (-1)^\delta f(-x)\big)}{2} \;&=\; \\ &=\; \frac{(\operatorname{sgn} x)^\delta}{4\pi i} \int_{\operatorname{Re}\, s = s_0} M_\delta f(s)|x|^{-s}\, ds \qquad \big(\operatorname{Re}\, s_0 \gg 0\big)\,. \end{aligned} \tag{1.7}$$

This latter description of the transform $f \mapsto F$ is less suggestive than (1.3), but more useful in applications. One can argue as in [14, §4] that

$$f \in |x|^{\lambda_n} \operatorname{sgn}(x)^{\delta_n} \mathcal{S}(\mathbb{R}) \quad \Longrightarrow \quad F \in \textstyle\sum_{1 \le j \le n} |x|^{-\lambda_j} \operatorname{sgn}(x)^{\delta_j} \mathcal{S}(\mathbb{R})\,. \tag{1.8}$$

In the singular cases where two or more of the λ_j differ by an integer, the above formula must be interpreted as including powers of $\log|x|$ – see [13, §6] for details. It is important to note that this only affects the asymptotics of the functions f and F near zero, a point at which they are never directly evaluated in our formula below. We had remarked that the components (λ_j, δ_j) of the parameter (λ, δ) are freely permutable when the archimedean representation is an irreducible principal series representation. In that situation we can replace $|x|^{\lambda_n}\,\mathrm{sgn}(x)^{\delta_n}\mathcal{S}(\mathbb{R})$ on the left hand side of (1.8) by $|x|^{\lambda_j}\,\mathrm{sgn}(x)^{\delta_j}\mathcal{S}(\mathbb{R})$, and then sum over j. That makes the hypotheses on the test functions f and F completely symmetric.

The $GL(n)$ summation formula requires one more ingredient, the $(n-1)$-dimensional hyperkloosterman sum

$$S(a,b\,;q,c,d) \;=\; \sum_{x_j\in(\mathbb{Z}/\frac{c_1\cdots c_j q}{d_1\cdots d_j}\mathbb{Z})^*,\ \text{for}\ j\le n-2} e\Big(\tfrac{d_1x_1a}{q} + \tfrac{d_2x_2\overline{x_1}}{\frac{c_1q}{d_1}} + \cdots + \tfrac{d_{n-2}x_{n-2}\overline{x_{n-3}}}{\frac{c_1\cdots c_{n-3}q}{d_1\cdots d_{n-3}}} + \tfrac{b\overline{x_{n-2}}}{\frac{c_1\cdots c_{n-2}q}{d_1\cdots d_{n-2}}}\Big)\,, \quad (1.9)$$

where $e(u)$ is shorthand for $e^{2\pi iu}$ and $\overline{x_j}$, for $x_j\in(\mathbb{Z}/\frac{qc_1\cdots c_j}{d_1\cdots d_j}\mathbb{Z})^*$, denotes the reciprocal of x_j modulo $m_1\cdots m_j$. The sum is used only when $d_1\cdots d_j$ divides $c_1\cdots c_jq$ for each $j\le n-2$.

Theorem 1.10. *Under the assumptions on a cusp form for $GL(n,\mathbb{Z})$ stated above, with fixed values of $c_1,\dots,c_{n-2}\in\mathbb{Z}_{\neq0}$ and relatively prime $a,q\in\mathbb{Z}$,*

$$\begin{aligned}\textstyle\sum_{r\neq0} a_{c_{n-2},\dots,c_1,r}\; e\big(-\tfrac{ra}{q}\big)\,f(r) \;&=\\ =\; |q|\sum_{d_1|qc_1}\;\sum_{d_2|\frac{qc_1c_2}{d_1}}\cdots\sum_{d_{n-2}|\frac{qc_1\cdots c_{n-2}}{d_1\cdots d_{n-3}}}\;\sum_{r\neq0}&\frac{a_{r,d_{n-2},\dots,d_1}}{|rd_1\cdots d_{n-2}|}\;\times\\ \times\; S(r,\bar a;q,c,d)\;&F\!\left(\frac{r\,d_{n-2}^2\,d_{n-3}^3\cdots d_1^{n-1}}{q^n\,c_{n-2}\,c_{n-3}^2\cdots c_1^{n-2}}\right).\end{aligned}$$

A Voronoi summation formula for $GL(n)$ appears in Goldfeld-Li [5, 6]. Our formula precedes the Goldfeld-Li formula; see [7]. It is also more general in two respects: it applies not only to spherical principal series representations, and involves summation over the Fourier coefficients $a_{c_{n-1},\dots,c_1,r}$ for arbitrary nonzero $c_1, c_2, \dots, c_{n-2}$, not with $c_1=c_2=\cdots=c_{n-2}=1$ as in [5, 6]. The freedom to specify arbitrary non-zero c_j is potentially important; in the case of $GL(3)$, for example, Li's convexity breaking result [11] crucially depends on this additional freedom. Theorem (1.10) applies to cusp forms. Ivic [8] proves and uses a Voronoi formula for multiple divisor functions, which corresponds to non-cusp forms. His proof uses Poisson summation. In fact, Voronoi formulas for noncusp forms can always be derived using the formulas for cusp forms on smaller groups, though this procedure may be complicated because of their nonzero constant terms.

One application of our formula is to give a new proof of the functional equation for the standard L-function of a cusp form for $GL(n,\mathbb{Z})$, and more generally those twisted by Dirichlet characters. This was carried out in [13, 14] for $n\le 3$ by a general argument, an argument that also applies to our situation here, as we shall argue presently. The key point is that our Voronoi formula can be applied to the test functions

$$f(x) \;=\; |x|^{-s}\,\mathrm{sgn}(x)^{\eta}\,,\quad F(x)\;=\;\textstyle\prod_{j=1}^{n} G_{\eta+\delta_j}(s+\lambda_j)^{-1}\;|x|^{s}\,\mathrm{sgn}(x)^{\eta}\,,\quad (1.11)$$

even though they do not satisfy the hypotheses of the theorem as stated. They can, because of a deeper analytic fact: the relevant automorphic distributions in section 2 vanish to infinite order at 0 and ∞ in the sense of [13] (see proposition 3.6 below). The proof of this vanishing to infinite order involves arguments related to the main mechanism of proof in the paper [2] of Casselman-Hecht-Miličić. In effect, certain Fourier components of automorphic distributions are completely determined, as distributions, by their restriction to an open Schubert cell, just as is the case for Whittaker distributions in their paper. We could have relied on their result, but have chosen to develop the necessary tools ourselves in section 3, for the following reasons. Our tools are both stronger and more concrete than the corresponding arguments in [2], and we expect to use them in the future, in situations not covered by [2]. In fact, we prove a slightly more general version of an important lemma – lemma 3.57 below – than we need for the proof of the Voronoi formula. This more general version, which we also expect to use in the future, cannot be reduced to the Whittaker case.

We now consider the formula in theorem 1.10 with the choice of functions (1.11), taking c_j, d_j, and q to be positive – as we may, because the coefficient a_k is insensitive to the signs of the entries of k. That results in a general functional equation for additively twisted L-functions:

$$\begin{aligned}\sum_{r\neq 0} a_{c_{n-2},\dots,c_1,r}\, e(-\tfrac{ra}{q})\,|r|^{-s}\,\operatorname{sgn}(r)^{\eta} \;&=\; \prod_{j=1}^{n} G_{\eta+\delta_j}(s+\lambda_j)^{-1} \quad\times \\ \times\;|q|^{1-ns} \sum_{\substack{d_j \mid \frac{qc_1\cdots c_j}{d_1\cdots d_{j-1}} \\ \text{for all } j\leq n-2}} \;&\sum_{r\neq 0} a_{r,d_{n-2},\dots,d_1}\, S(r,\bar a;q,c,d)\;\times \\ \times\;|r|^{s-1}\,\operatorname{sgn}(r)^{\eta}&\left(\prod\nolimits_{j=1}^{n-2}|c_j|^{-(n-1-j)s}\,|d_j|^{(n-j)s-1}\right),\end{aligned}\tag{1.12}$$

or, in the special case of $c_1=\cdots=c_{n-2}=1$, more simply

$$\begin{aligned}\sum_{r\neq 0} a_{1,\dots,1,r}\, e(-\tfrac{ra}{q})\,|r|^{-s}\,\operatorname{sgn}(r)^{\eta} \;&=\; \prod_{j=1}^{n} G_{\eta+\delta_j}(s+\lambda_j)^{-1} \quad\times \\ \times\;|q|^{1-ns} \sum_{\substack{d_j \mid \frac{q}{d_1\cdots d_{j-1}} \\ \text{for all } j\leq n-2}} \;&\sum_{r\neq 0} a_{r,d_{n-2},\dots,d_1}\;\times \\ \times\;S(r,\bar a;q,(1,\dots,1),d)\,|r|^{s-1}\,&\operatorname{sgn}(r)^{\eta}\prod\nolimits_{j=1}^{n-2}|d_j|^{(n-j)s-1}\,.\end{aligned}\tag{1.13}$$

Now let χ be a primitive Dirichlet character modulo q, such that $\chi(-1)=(-1)^{\eta}$. A basic identity asserts

$$\sum\nolimits_{a\in\mathbb{Z}/q\mathbb{Z}} \chi(a)\,e(-\tfrac{ra}{q}) \;=\; \begin{cases} 0 & \text{if } (r,q)>1 \\ \chi(-r)^{-1}\,g_\chi & \text{if } (r,q)=1\,, \end{cases}\tag{1.14}$$

where g_χ denotes the Gauss sum $\sum_{a\in\mathbb{Z}/q\mathbb{Z}}\chi(a)e(\frac{a}{q})$ for χ. Thus, summing (1.13) over the residue classes modulo q, one derives the functional equation of the standard L-function twisted by the Dirichlet character τ,

$$\sum_{r=1}^{\infty} a_{r,1,\ldots,1}\,\overline{\chi(r)}\,r^{s-1} \;=\; q^{ns}\,g_\chi^{-n}\prod_{j=1}^{n} G_{\eta+\delta_j}(s+\lambda_j)\sum_{r=1}^{\infty} a_{1,\ldots,1,r}\,\chi(r)\,r^{-s}\,. \quad (1.15)$$

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2. Automorphic Distributions

We derive the Voronoi summation formula not from the datum of an automorphic form, but from the essentially equivalent datum of an automorphic distribution. For the connection between the two we refer the reader to [14].

We kept the definition (1.2) of the principal series representation with parameter (λ,δ) purposely vague: when the functions f in the definition are required to be smooth, one denotes the resulting space by $V_{\lambda,\delta}^{\infty}$, and when these "functions" are only required to be distributions, one obtains $V_{\lambda,\delta}^{-\infty}$, the larger space of distribution vectors. Of course these are not the only choices of a topology for $V_{\lambda,\delta}$. In all cases the group $GL(n,\mathbb{R})$ acts by left translation. By definition an automorphic distribution for an arithmetic subgroup $\Gamma\subset GL(n,\mathbb{Q})$, with representation parameter (λ,δ), is a Γ-invariant vector τ in $V_{\lambda,\delta}^{-\infty}$ – in other words,

$$\begin{aligned} &\tau \;\in\; C^{-\infty}\big(GL(n,\mathbb{R})\big) \quad \text{such that} \\ &\tau\left(\gamma\, g\begin{pmatrix} a_1 & 0 & 0 \\ & \ddots & \\ \star & \ddots & 0 \\ \star & \star & a_n \end{pmatrix}\right) \;=\; \tau(g)\cdot\prod\nolimits_{1\le j\le n}\Big(|a_j|^{\frac{n+1}{2}-j-\lambda_j}\,\mathrm{sgn}(a_j)^{\delta_j}\Big) \end{aligned} \quad (2.1)$$

for all $\gamma\in\Gamma$ and $a_1,\,a_2,\,\ldots,\,a_n\in\mathbb{R}^*$. In dealing with distributions, we adopt the same convention as in our other papers: "distributions transform like functions" – i.e., they are naturally dual to smooth, compactly supported measures.

Let $N(\mathbb{R})\subset G(\mathbb{R}) =_{\mathrm{def}} GL(n,\mathbb{R})$ denote the subgroup of unipotent upper triangular matrices, and $N'(\mathbb{R})=[N(\mathbb{R}),N(\mathbb{R})]$, $N''(\mathbb{R})=[N(\mathbb{R}),N'(\mathbb{R})]$ its first two derived subgroups. In the classical approach we shall work exclusively at full level; we therefore change notation from Γ to $G(\mathbb{Z})$. Analogously we let $N(\mathbb{Z})$, $N'(\mathbb{Z})$, $N''(\mathbb{Z})$ denote the groups of integral points in $N(\mathbb{R})$ and its derived subgroups. Since we are working at full level, no nonzero automorphic distributions can exist unless $\delta_1+\cdots+\delta_n\equiv 0 \pmod 2$. We shall also assume $\lambda_1+\cdots+\lambda_n=0$; this can be arranged by multiplying τ by an appropriate character of the center. For emphasis,

$$\sum_{i=1}^{n}\lambda_i \;=\; 0 \quad\text{and}\quad \sum_{i=1}^{n}\delta_i \;\equiv\; 0 \pmod 2\,. \quad (2.2)$$

The first of these assumptions is not necessary, but helps to simplify formulas and arguments.

Our computations involve integrating translates of τ, sometimes multiplied by smooth functions, over compact sets. This makes sense: let $\ell(g)$ denote the left g-translate of τ, formally

$$(\ell(g)\tau)(g_1) \quad = \quad \tau(g^{-1}g_1)\,; \tag{2.3}$$

then $g \mapsto \ell(g)\tau$ is a smooth function on $G(\mathbb{R})$ with values in the closed subspace $V_{\lambda,\delta}^{-\infty}$ of $C^{\infty}(\mathbb{R})$, from which it inherits the structure of complete, locally convex, Hausdorff topological vector space. One can integrate measurable functions with values in such a topological vector space over any finite measure space. For example, since τ is $G(\mathbb{Z})$-invariant under left translation, $n \mapsto \ell(n)\tau$ induces smooth, $V_{\lambda,\delta}^{-\infty}$-valued functions on the compact spaces $N'(\mathbb{R})/N'(\mathbb{Z})$, $N''(\mathbb{R})/N''(\mathbb{Z})$. Thus

$$\tau' \;=\; \int_{N'(\mathbb{R})/N'(\mathbb{Z})} \ell(n)\tau\,dn \qquad \text{and} \qquad \tau'' \;=\; \int_{N''(\mathbb{R})/N''(\mathbb{Z})} \ell(n)\tau\,dn \tag{2.4}$$

are well defined vectors in $V_{\lambda,\delta}^{-\infty}$, by construction invariant under, respectively, $N'(\mathbb{R})$ and $N''(\mathbb{R})$:

$$\tau' \in \big(V_{\lambda,\delta}^{-\infty}\big)^{N'(\mathbb{R})}\,, \qquad \tau'' \in \big(V_{\lambda,\delta}^{-\infty}\big)^{N''(\mathbb{R})}\,. \tag{2.5}$$

Both are also $N(\mathbb{Z})$-invariant,

$$\tau'\,,\ \tau'' \;\in\; \big(V_{\lambda,\delta}^{-\infty}\big)^{N(\mathbb{Z})}\,, \tag{2.6}$$

since $N(\mathbb{Z})$ leaves τ invariant, normalizes all of the four groups $N'(\mathbb{R})$, $N'(\mathbb{Z})$, $N''(\mathbb{R})$, $N''(\mathbb{Z})$, and preserves the measures on $N'(\mathbb{R})$ and $N''(\mathbb{R})$.

Let $B_-(\mathbb{R}) \subset G(\mathbb{R})$ denote the subgroup of lower triangular matrices. In view of (2.1), every distribution in $V_{\lambda,\delta}^{-\infty} \subset C^{-\infty}(N(\mathbb{R}))$ behaves in a C^{∞} manner under right translation by elements of $B_-(\mathbb{R})$. Since $N(\mathbb{R}) \cdot B_-(\mathbb{R})$ is open in $G(\mathbb{R})$, we can restrict such distributions from $G(\mathbb{R})$ to the subgroup $N(\mathbb{R})$ [14]. In particular this applies to τ'; we shall refer to its restriction to $N(\mathbb{R})$ as τ_{abelian}. In view of (2.5–2.6), and because $N'(\mathbb{R}) \subset N(\mathbb{R})$ is normal,

$$\tau_{\text{abelian}} \;=_{\text{def}}\; \tau'\big|_{N(\mathbb{R})} \;\in\; C^{-\infty}\big(N(\mathbb{Z})\backslash N(\mathbb{R})/N'(\mathbb{R})\big)\,. \tag{2.7}$$

The entries on the first superdiagonal provide coordinates $(x_1, \ldots, x_{n-1})$ for $N'(\mathbb{R})\backslash N(\mathbb{R}) \cong \mathbb{R}^{n-1}$. Under this identification, the image of $N(\mathbb{Z})$ in $N'(\mathbb{R})\backslash N(\mathbb{R})$ corresponds to $\mathbb{Z}^{n-1}$. Thus τ_{abelian}, as distribution on $N(\mathbb{Z})\backslash N(\mathbb{R})/N'(\mathbb{R}) \cong \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$, has the Fourier expansion

$$\tau_{\text{abelian}}(x) \quad = \quad \sum\nolimits_{k\in(\mathbb{Z}_{\neq 0})^{n-1}} c_k\, e(k\cdot x)\,. \tag{2.8}$$

Here $e(u) = e^{2\pi i u}$ as before, and $k \cdot x$ stands for the sum $k_1x_1 + \cdots + k_{n-1}x_{n-1}$. Note that the summation does not involve terms for which at least one of the k_j equals zero – this reflects the assumption of cuspidality; cf. [14].

When there are several choices of the representation parameter, as is usually the case, the coefficients c_k in the expansion (2.8) do depend on (λ,δ). However, they are related to the Fourier coefficients a_k of any automorphic form associated to τ by the formula

$$a_{k_1,k_2,\dots,k_{n-1}} \;=\; \prod_{j=1}^{n-1} \left((\operatorname{sgn} k_j)^{\delta_1+\delta_2+\cdots+\delta_j} \, |k_j|^{\lambda_1+\lambda_2+\cdots+\lambda_j} \right) c_{k_1,k_2,\dots,k_{n-1}} \,. \tag{2.9}$$

The a_k are independent of (λ,δ) – in fact, they coincide with the Hecke eigenvalues in the case of full level, as we are assuming. Alternatively one can show that the a_k do not depend on (λ,δ) by calculating the effect of the intertwining operators between the different principal series representations into which our automorphic representation can be embedded.

For $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}^{n-2}$, define

$$n_{x,y} \;=\; \begin{pmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 1 & x_2 & y_2 & 0 & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & & & & 0 \\ 0 & 0 & 0 & 1 & x_{n-2} & y_{n-2} \\ 0 & 0 & 0 & 0 & 1 & x_{n-1} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.10}$$

Every element of N can be uniquely decomposed as either $n''n_{x,y}$ or as $n_{x,y}n''$, for some $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}^{n-2}$, and $n'' \in N''(\mathbb{R})$; only the factor n'' depends on which order is chosen. Thus

$$\tau' \;=\; \int_{(\mathbb{R}/\mathbb{Z})^{n-2}} \ell(n_{0,y})\tau''\, dy_1 \cdots dy_{n-2}\,. \tag{2.11}$$

Corresponding to the datum (j,m,k), with $1 \le j \le n-2$, $m \in \mathbb{Z}_{\neq 0}$, and $k \in \mathbb{Z}^{n-1}$, we define

$$\begin{aligned} R_{j,m,k}\tau \;&=\; \int_{(\mathbb{R}/\mathbb{Z})^{n-2}} \int_{\left\{ \substack{x \in (\mathbb{R}/\mathbb{Z})^{n-1} \\ x_{j+1} = 0} \right\}} e(k\cdot x \,+\, my_j)\; \ell(n_{x,y})\tau''\, dx\, dy\,, \\ S_{j,m,k}\tau \;&=\; \int_{(\mathbb{R}/\mathbb{Z})^{n-2}} \int_{\left\{ \substack{x \in (\mathbb{R}/\mathbb{Z})^{n-1} \\ x_{j} = 0} \right\}} e(k\cdot x \,+\, my_j)\; \ell(n_{x,y})\tau''\, dx\, dy\,, \end{aligned} \tag{2.12}$$

and we also define $R_{0,m,k}\tau$ and $S_{n-1,m,k}\tau$, but only corresponding to $m=1$:

$$\begin{aligned} R_{0,1,k}\tau \;&=\; \int_{(\mathbb{R}/\mathbb{Z})^{n-2}} \int_{\left\{ \substack{x \in (\mathbb{R}/\mathbb{Z})^{n-1} \\ x_1 = 0} \right\}} e(k\cdot x)\; \ell(n_{x,y})\tau''\, dx\, dy\,, \\ S_{n-1,1,k}\tau \;&=\; \int_{(\mathbb{R}/\mathbb{Z})^{n-2}} \int_{\left\{ \substack{x \in (\mathbb{R}/\mathbb{Z})^{n-1} \\ x_{n-1} = 0} \right\}} e(k\cdot x)\; \ell(n_{x,y})\tau''\, dx\, dy\,. \end{aligned} \tag{2.13}$$

In these equations dx is shorthand for $dx_1 \dots dx_j\, dx_{j+2} \dots dx_{n-1}$ in the case of $R_{j,m,k}\tau$, and for $dx_1 \dots dx_{j-1}\, dx_{j+1} \dots dx_{n-1}$ in the case of $S_{j,m,k}\tau$; dy stands for $dy_1 \dots dy_{n-2}$ in all cases.

The $R_{j,m,k}\tau$ and $S_{j,m,k}\tau$ are integrals of continuous $V_{\lambda,\delta}^{-\infty}$-valued functions over tori, hence

$$R_{j,m,k}\tau\,,\; S_{j,m,k}\tau \;\in\; V_{\lambda,\delta}^{-\infty}\,. \tag{2.14}$$

Large subgroups of $N(\mathbb{Z})$ leave them invariant – see lemma 2.18 below. The $G(\mathbb{Z})$-invariance of τ implies certain relations among the $R_{j,m,k}\tau$ and $S_{j,m,k}\tau$. To state them, we need to consider the embeddings

$$\Phi_j \ : \ SL(2,\mathbb{R}) \ \hookrightarrow \ G(\mathbb{R}) \ = \ GL(n,\mathbb{R}) \qquad (\, 1 \leq j \leq n-1\,) \tag{2.15}$$

into the $2{\times}2$ diagonal blocks with "vertices" (j,j), $(j{+}1,j{+}1)$. On the infinitesimal level this means

$$\Phi_{j\,*}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \ = \ E_{j,j+1}\,, \ \ \Phi_{j\,*}\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \ = \ E_{j+1,j}\,, \ \ \Phi_{j\,*}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \ = \ E_{j,j} - E_{j+1,j+1}\,, \tag{2.16}$$

with $E_{r,s} =$ matrix with (r,s) entry equal to one, and the other entries equal to zero. The image of Φ_j normalizes

$$\begin{aligned} N_j(\mathbb{R}) \ \ = \ \ & \text{subgroup of } G(\mathbb{R}) \text{ whose Lie algebra is spanned} \\ & \text{by } \ \{\, E_{r,s} \,\big|\, 1 \leq r < s \leq n\,,\ (r,s) \neq (j,j+1)\,\}\,. \end{aligned} \tag{2.17}$$

In fact, $N_j(\mathbb{R})$ is the unipotent radical of a parabolic which contains $\operatorname{Im}\Phi_j$ as the semisimple part of its Levi component.

Lemma 2.18. *Both $R_{0,1,k}\tau$ and $S_{n-1,1,k}\tau$ are $N(\mathbb{Z})$-invariant, and*

$$R_{j,m,k}\tau \ \in \ \big(V^{-\infty}_{\lambda,\delta}\big)^{N_{j+1}(\mathbb{Z})}\,, \qquad S_{j,m,k}\tau \ \in \ \big(V^{-\infty}_{\lambda,\delta}\big)^{N_{j}(\mathbb{Z})} \qquad (\,1 \leq j \leq n-2\,).$$

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$,

$$\begin{aligned} d\,m = c\,k_2 \quad \Longrightarrow \quad & \ell\big(\Phi_1\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)\big)R_{0,1,\widetilde{k}}\,\tau \ = \ S_{1,m,k}\,\tau\,, \\ & \text{with} \ \ \widetilde{k}_2 = a\,k_2 - b\,m\,,\ \ \widetilde{k}_i = k_i \ \ \text{otherwise}\,; \end{aligned}$$

similarly

$$\begin{aligned} d\,m = -c\,k_{n-2} \quad \Longrightarrow \quad & \ell\big(\Phi_{n-1}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)\big)S_{n-1,1,\widetilde{k}}\,\tau \ = \ R_{n-2,m,k}\tau\,, \\ & \text{with} \ \ \widetilde{k}_{n-2} = a\,k_{n-2} + b\,m\,,\ \ \widetilde{k}_i = k_i \ \ \text{otherwise}\,; \end{aligned}$$

and, for $2 \leq j \leq n-2$,

$$\begin{aligned} a\,m = -c\,k_{j+1} \quad \Longrightarrow \quad & \ell\big(\Phi_j\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)\big)S_{j,m,k}\tau \ = \ R_{j-1,\widetilde{m},\widetilde{k}}\,\tau\,, \ \ \text{with} \\ & \widetilde{m} = -c\,k_{j-1}\,,\ \ \widetilde{k}_{j-1} = d\,k_{j-1}\,,\ \ \widetilde{k}_{j+1} = d\,k_{j+1} + b\,m\,,\ \ \widetilde{k}_i = k_i \ \ \text{otherwise}\,. \end{aligned}$$

Finally, for $1 \leq j \leq n-2$,

$$\begin{aligned} \ell\big(\Phi_{j+1}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)R_{j,m,k}\tau \ &= \ R_{j,m,\widetilde{k}}\,\tau \quad \text{with } \widetilde{k}_i = k_i + \delta_{i,j}\,m\,, \\ \ell\big(\Phi_{j}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)S_{j,m,k}\tau \ &= \ S_{j,m,\widetilde{k}}\,\tau \quad \text{with } \widetilde{k}_i = k_i - \delta_{i,j+1}\,m\,. \end{aligned}$$

In the relationship between $S_{j,m,k}\tau$ and $R_{j-1,\widetilde{m},\widetilde{k}}\,\tau$ the indices $(\widetilde{m},\widetilde{k})$ and (m,k) appear to play non-symmetric roles, since the former are defined in terms of the latter. In fact, it is possible to express any $R_{j-1,m,k}\tau$ in terms of $S_{j,\widetilde{m},\widetilde{k}}\,\tau$, with suitably chosen $(\widetilde{m},\widetilde{k})$. One can see this either directly, by inverting the map $(m,k)\mapsto(\widetilde{m},\widetilde{k})$, or applying the automorphism (3.10), which interchanges the roles of the $R_{j,m,k}\tau$ and the $S_{n-j,m,k}\tau$.

Proof of lemma 2.18. The passage from τ to $R_{j-1,m,k}\tau$ and $S_{j,m,k}\tau$ involves two integrations, first over $N''(\mathbb{R})/N''(\mathbb{Z})$, then over $\mathbb{R}^{2n-4}/\mathbb{Z}^{2n-4}$. They can be combined into one integration against characters of $N_j(\mathbb{R})$ which are trivial on $N_j(\mathbb{Z})$:

$$\begin{aligned} R_{j-1,m,k}\tau \ &= \int_{N_j(\mathbb{R})/N_j(\mathbb{Z})} \chi^R_{j-1,m,k}(n)\,\ell(n)\tau\,dn\,, \\ S_{j,m,k}\tau \ &= \int_{N_j(\mathbb{R})/N_j(\mathbb{Z})} \chi^S_{j,m,k}(n)\,\ell(n)\tau\,dn\,; \end{aligned} \tag{2.19}$$

the characters $\chi^R_{j-1,m,k}\,,\,\chi^S_{j,m,k}\,:\,N_j(\mathbb{R})\longrightarrow\mathbb{C}^*$ are determined by the equations

$$\chi^R_{j-1,m,k}\big|_{N''(\mathbb{R})} \equiv 1\,, \quad \chi^S_{j,m,k}\big|_{N''(\mathbb{R})} \equiv 1\,,$$

and for $n_{x,y}\in N_j(\mathbb{R})$, or equivalently $x_j=0$,

$$\chi^R_{j-1,m,k}(n_{x,y}) \ = \ \begin{cases} e(k\cdot x + m\,y_{j-1}) & \text{if } 2\le j\le n-1 \\ e(k\cdot x) & \text{if } j=1 \text{ and } m=1\,, \end{cases} \tag{2.20}$$

$$\chi^S_{j,m,k}(n_{x,y}) \ = \ \begin{cases} e(k\cdot x + m\,y_{j}) & \text{if } 1\le j\le n-2 \\ e(k\cdot x) & \text{if } j=n-1 \text{ and } m=1\,. \end{cases}$$

The kernels of both $\chi^R_{j-1,m,k}$ and $\chi^S_{j,m,k}$ contain $N_j(\mathbb{Z})$, and even all of $N(\mathbb{Z})$ in the exceptional cases of $\chi^R_{0,1,k}$ and $\chi^S_{n-1,1,k}$. That implies the initial assertions.

The Φ_j-image of $SL(2,\mathbb{Z})$ lies in $G(\mathbb{Z})$, it normalizes $N_j(\mathbb{R})$ and $N_j(\mathbb{Z})$, and conjugation by it preserves Haar measure on $N_j(\mathbb{R})$. For $\gamma\in SL(2,\mathbb{Z})$, define

$$A_\gamma \ : \ N_j(\mathbb{R}) \ \longrightarrow \ N_j(\mathbb{R})\,, \qquad A_\gamma(n) \ = \ \Phi_j(\gamma)\,n\,\Phi_j(\gamma^{-1})\,. \tag{2.21}$$

Using the change of variables $n\rightsquigarrow A_\gamma(n)$, we find

$$\begin{aligned} \ell\big(\Phi_j(\gamma^{-1})\big)\,R_{j-1,m,k}\tau \ &= \int_{N_j(\mathbb{R})/N_j(\mathbb{Z})} \chi^R_{j-1,m,k}(n)\,\ell\big(A_{\gamma^{-1}}(n)\big)\tau\,dn \\ &= \int_{N_j(\mathbb{R})/N_j(\mathbb{Z})} \chi^R_{j-1,m,k}\big(A_\gamma(n)\big)\,\ell(n)\tau\,dn\,, \end{aligned} \tag{2.22}$$

and analogously

$$\ell\big(\Phi_j(\gamma^{-1})\big)\,S_{j,m,k}\tau \ = \int_{N_j(\mathbb{R})/N_j(\mathbb{Z})} \chi^S_{j,m,k}\big(A_\gamma(n)\big)\,\ell(n)\tau\,dn\,. \tag{2.23}$$

Now suppose $\gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in SL(2,\mathbb{Z})$, $(x,y)\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-2}$ with $x_j=0$, so that $n_{x,y}\in N_j(\mathbb{R})$. A straightforward matrix computation shows

$$\begin{aligned} &A_\gamma(n_{x,y}) \ \equiv \ n_{\widetilde{x},\widetilde{y}} \quad \text{modulo} \ \big(\operatorname{Ker}\chi^R_{j-1,\,\cdot\,,\,\cdot}\big)\cap\big(\operatorname{Ker}\chi^S_{j,\,\cdot\,,\,\cdot}\big) \quad \text{with} \\ &\widetilde{x}_{j-1}=d\,x_{j-1}-c\,y_{j-1}\,,\ \ \widetilde{x}_{j+1}=d\,x_{j+1}+c\,y_j\,,\ \ \widetilde{x}_i=x_i \ \text{ otherwise}\,, \\ &\widetilde{y}_{j-1}=a\,y_{j-1}-b\,x_{j-1}\,,\ \ \widetilde{y}_{j}=a\,y_{j}+b\,x_{j+1}\,,\ \ \widetilde{y}_i=y_i \ \text{ otherwise}\,. \end{aligned} \tag{2.24}$$

These identities remain correct in the exceptional cases of $j=1$ and $j=n-1$ when terms with out-of-range indices are disregarded. We conclude

$$\begin{aligned} \chi^R_{j-1,m,k}\big(A_\gamma(n_{x,y})\big) \ &= \ e\big(\textstyle\sum_{|i-j|\geq 2} k_i\, x_i \ + \ (dk_{j-1}-bm)x_{j-1} \ + \\ &\qquad + \ (am-ck_{j-1})y_{j-1} + c\, k_{j+1}\, y_j + dk_{j+1}x_{j+1}\big), \\ \chi^S_{j,m,k}\big(A_\gamma(n_{x,y})\big) \ &= \ e\big(\textstyle\sum_{|i-j|\geq 2} k_i\, x_i \ + \ dk_{j-1}x_{j-1} \ - \ ck_{j-1}\, y_{j-1} \ + \\ &\qquad + \ (dk_{j+1}+bm)x_{j+1} \ + \ (am+ck_{j+1})y_j\big)\,. \end{aligned} \tag{2.25}$$

These identities, too, remain valid in the exceptional cases when properly interpreted. In particular,

$$\begin{aligned} &a\, m = c\, k_{j-1} \quad \Longrightarrow \quad \chi^R_{j-1,m,k}\big(A_\gamma(n)\big) = \chi^S_{j,\widetilde m,\widetilde k}(n) \ \text{ with } \ \widetilde m = c\, k_{j+1}\,, \\ &\qquad \widetilde k_{j-1} = d\, k_{j-1} - b\, m\,, \ \ \widetilde k_{j+1} = d\, k_{j+1}\,, \ \ \widetilde k_i = k_i \ \text{ otherwise}\,; \\ &a\, m = -c\, k_{j+1} \quad \Longrightarrow \quad \chi^S_{j,m,k}\big(A_\gamma(n)\big) = \chi^R_{j-1,\widetilde m,\widetilde k}(n) \ \text{ with } \ \widetilde m = -c\, k_{j-1}\,, \\ &\qquad \widetilde k_{j-1} = d\, k_{j-1}\,, \ \ \widetilde k_{j+1} = d\, k_{j+1} + b\, m\,, \ \ \widetilde k_i = k_i \ \text{ otherwise}\,. \end{aligned} \tag{2.26}$$

Once more this must be properly interpreted in the exceptional cases. The first of the five equalities in the lemma follows from the second half of (2.26), with γ^{-1} in place of γ, the second from the first half of (2.26), again with γ^{-1} in place of γ, and third follows directly from (2.26); in all three cases we also appeal to (2.19) and either (2.22) or (2.23). For the last two equalities, we apply (2.22), replacing j with $j+1$, as well as (2.23) and (2.25), in both cases with $a=d=1$, $b=-1$, $c=0$. □

Recall the definition of the elementary matrices $E_{r,s}$ below (2.16). It will be convenient to use the notation

$$h_j(t) \ = \ \Phi_j\big(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\big) \ = \ \exp(t\, E_{j,j+1})\,. \tag{2.27}$$

Lemma 2.18 asserts the $N_j(\mathbb{Z})$-invariance of $R_{j-1,m,k}\tau$ and $S_{j,m,k}\tau$. Together with $N_j(\mathbb{Z})$, $h_j(1)$ generates all of $N(\mathbb{Z})$. In effect, our next lemma clarifies the obstacle to $N(\mathbb{Z})$-invariance for $R_{j-1,m,k}\tau$ and $S_{j,m,k}\tau$.

Lemma 2.28. *The $R_{j,m,k}\tau$ do not depend on k_{j+1} and the $S_{j,m,k}\tau$ do not depend on k_j. For $1\leq j\leq n-2$, $\ell\big(h_{j+1}(-\frac{k_j}{m})\big)R_{j,m,k}\tau$ depends on k_j only modulo m, and $\ell\big(h_j(\frac{k_{j+1}}{m})\big)S_{j,m,k}\tau$ depends on k_{j+1} only modulo m.*

Proof. In the integral (2.12–2.13) defining $R_{j,m,k}\tau$, the variable x_{j+1} is set equal to zero, so effectively the exponential factor does not involve k_{j+1}. That makes $R_{j,m,k}\tau$ independent of k_{j+1}. According to lemma 2.18, for $1\leq j\leq n-2$, increasing the index k_j by m has the same effect on $R_{j,m,k}\tau$ as a translation by $h_{j+1}(1)$, so $\ell\big(h_{j+1}(-\frac{k_j}{m})\big)R_{j,m,k}\tau$ depends on k_j only modulo m. The analogous assertions about the $S_{j,m,k}\tau$ are proved the same way. □

The assertions of lemma 2.18 become more transparent when stated in terms of the re-normalized quantities $\ell(h_{j+1}(-\frac{k_j}{m}))R_{j,m,k}\tau$ and $\ell(h_j(\frac{k_{j+1}}{m}))S_{j,m,k}\tau$:

Lemma 2.29. *Define* $m_j = \gcd(m, k_j)$, *the greatest common divisor of* m *and* k_j, *and choose* $\overline{k}_j \in \mathbb{Z}$ *so that* $\overline{k}_j\, k_j \equiv m_j \pmod m$. *Then*

$$\ell\left(\Phi_1\left(\begin{smallmatrix} 0 & m_2/m \\ -m/m_2 & \overline{k}_2 \end{smallmatrix}\right)\right) R_{0,1,(k_1,m_2,k_3,\dots,k_{n-1})}\tau \;=\; \ell\big(h_1(\tfrac{k_2}{m})\big) S_{1,m,k}\tau\,,$$

and

$$\ell\left(\Phi_{n-1}\left(\begin{smallmatrix} 0 & -m_{n-2}/m \\ m/m_{n-2} & \overline{k}_{n-2} \end{smallmatrix}\right)\right) S_{n-1,1,(k_1,\dots,k_{n-3},m_{n-2},k_{n-1})}\tau \;=\; \;=\; \ell\big(h_{n-1}(-\tfrac{k_{n-2}}{m})\big) R_{n-2,m,k}\,\tau\,.$$

Finally, for $2 \le j \le n-2$,

$$\ell\left(\Phi_j\left(\begin{smallmatrix} 0 & -m_{j+1}/m \\ m/m_{j+1} & 0 \end{smallmatrix}\right)\right)\left(\ell\big(h_j(\tfrac{k_{j+1}}{m})\big) S_{j,m,k}\tau\right) \;=\; \ell\big(h_j(-\tfrac{\widetilde{k}_{j-1}}{\widetilde{m}})\big) R_{j-1,\widetilde{m},\widetilde{k}}\,\tau\,,$$

with $\widetilde{m} = m\, k_{j-1}/m_{j+1}$, $\widetilde{k}_{j-1} = k_{j-1}\,\overline{k}_{j+1}$, $\widetilde{k}_{j+1} = m_{j+1}$, $\widetilde{k}_i = k_i$ *otherwise*.

Proof. Each of these identities follows from its counterpart in lemma 2.18. Let $c = \frac{m}{m_2}$, $d = \frac{k_2}{m_2}$, and $a = \overline{k}_2$. By definition of $\overline{k}_2$, there exists $b \in \mathbb{Z}$ such that $ad = k_2\,\overline{k}_2/m_2 = 1 + b\,m/m_2 = 1 + bc$. Then $d\,m = c\,k_2$, so the first identity in lemma 2.18 asserts

$$\ell\left(\Phi_1\left(\begin{smallmatrix} k_2/m_2 & -b \\ -m/m_2 & \overline{k}_2 \end{smallmatrix}\right)\right) R_{0,1,(k_1,m_2,k_3,\dots,k_{n-1})}\tau \;=\; S_{1,m,k}\tau\,. \tag{2.30}$$

The first assertion of the current lemma follows, since

$$\left(\begin{smallmatrix} 1 & k_2/m \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} k_2/m_2 & -b \\ -m/m_2 & \overline{k}_2 \end{smallmatrix}\right) \;=\; \left(\begin{smallmatrix} 0 & m_2/m \\ -m/m_2 & \overline{k}_2 \end{smallmatrix}\right)\,. \tag{2.31}$$

For the verification of the second assertion we let $a = \overline{k}_{n-2}$, $c = -m/m_{n-2}$, $d = k_{n-2}/m_{n-2}$, and choose b so that $ad = k_{n-2}\overline{k}_{n-2}/m_{n-2} = 1 - bm/m_{n-2} = 1 + bc$. Then $d\,m = -c\,k_{n-2}$, and the second identity in lemma 2.18 implies

$$\ell\left(\Phi_{n-1}\left(\begin{smallmatrix} k_{n-2}/m_{n-2} & -b \\ m/m_{n-2} & \overline{k}_{n-2} \end{smallmatrix}\right)\right) S_{n-1,1,(k_1,\dots,k_{n-3},m_{n-2},k_{n-1})}\tau \;=\; R_{n-2,m,k}\tau\,. \tag{2.32}$$

We obtain the second assertion by applying $\ell(h_{n-1}(-\frac{k_{n-2}}{m}))$ to both sides.

For the verification of the third assertion we suppose that $2 \le j \le n-2$, set $a = k_{j+1}/m_{j+1}$, $c = -m/m_{j+1}$, $d = \overline{k}_{j+1}$, and we choose b so that $ad = k_{j+1}\overline{k}_{j+1}/m_{j+1} = 1 - bm/m_{j+1} = 1 + bc$. Then $a\,m$ does equal $-c\,k_{j+1}$, so we can apply the third identity in lemma 2.18, with

$$\widetilde{m} \;=\; m\,k_{j-1}/m_{j+1}\,, \quad \widetilde{k}_{j-1} \;=\; k_{j-1}\,\overline{k}_{j+1}\,, \quad \widetilde{k}_{j+1} \;=\; m_{j+1}\,, \tag{2.33}$$

which then reads as follows:

$$\ell\left(\Phi_j\left(\begin{smallmatrix} \overline{k}_{j+1} & -b \\ m/m_{j+1} & k_{j+1}/m_{j+1} \end{smallmatrix}\right)\right) S_{j,m,k} \;=\; R_{j-1,\widetilde{m},\widetilde{k}}\,\tau\,. \tag{2.34}$$

At this point, the matrix identity

$$\left(\begin{smallmatrix} 1 & -\widetilde{k}_{j-1}/\widetilde{m} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \overline{k}_{j+1} & -b \\ m/m_{j+1} & k_{j+1}/m_{j+1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -k_{j+1}/m \\ 0 & 1 \end{smallmatrix}\right) \;=\; \left(\begin{smallmatrix} 0 & -m_{j+1}/m \\ m/m_{j+1} & 0 \end{smallmatrix}\right) \tag{2.35}$$

completes the verification of the third assertion. □

Lemma 2.36. *For* $1 \le j \le n-2$, $m \in \mathbb{Z}_{\neq 0}$, $k \in \mathbb{Z}^{n-1}$,

$$\begin{aligned}&\ell\big(h_{j+1}(-\tfrac{k_j}{m})\big)R_{j,m,k}\tau \;=\\ &\quad= \sum\nolimits_{r\in\mathbb{Z}/m\mathbb{Z}} e\left(\tfrac{r\,k_j}{m}\right) \int_{\mathbb{R}} \ell(h_j(t))\,\ell\big(h_j(\tfrac{r}{m})\big) S_{j,m,(k_1,\,\dots,\,k_j,r,k_{j+2},\,\dots,\,k_{n-1})}\tau\, dt\,.\end{aligned}$$

The integrals converge in the strong distribution topology and depend on the index r *only modulo* m , *as indicated by the notation.*

Proof. In order to relate the $R_{j,m,k}\tau$ to the $S_{j,m,k}\tau$, we express them both in terms of a projection $P_{j,k,m}\tau$, whose definition is similar to that of $R_{j,k,m}\tau$ and $S_{j,k,m}\tau$:

$$P_{j,m,k}\tau \;=\; \int_{(\mathbb{R}/\mathbb{Z})^{n-2}} \int_{\left\{\substack{x\in(\mathbb{R}/\mathbb{Z})^{n-1}\\ x_j=x_{j+1}=0}\right\}} e(k\cdot x \,+\, my_j)\;\ell(n_{x,y})\tau''\,dx\,dy\,. \tag{2.37}$$

Evidently $P_{j,m,k}\tau$ does not depend on k_j or k_{j+1}. Recall the definition of the elementary matrices $E_{r,s}$ above (2.27), and note that τ'' is invariant under translation by $N''(\mathbb{R})$, which is normal in $N(\mathbb{R})$. Since $\exp(uE_{j,j+2})\,n_{x,y} \equiv n_{x,\widetilde{y}}$ modulo $N''(\mathbb{R})$, with $\widetilde{y}_i = y_i + \delta_{i,j}\,u$,

$$\begin{aligned}&\ell\big(\exp(u\,E_{j,j+2})\big)P_{j,m,k}\tau \;=\\ &\quad= \int_{(\mathbb{R}/\mathbb{Z})^{n-2}} \int_{\left\{\substack{x\in(\mathbb{R}/\mathbb{Z})^{n-1}\\ x_j=x_{j+1}=0}\right\}} e(k\cdot x \,+\, m(y_j-u))\;\ell(n_{x,y})\tau''\,dx\,dy\\ &\quad=\; e(-m\,u)\,P_{j,m,k}\tau\,.\end{aligned} \tag{2.38}$$

A simple computation, which can be reduced to the case of $GL(3,\mathbb{R})$, shows that

$$h_j(u)\,h_{j+1}(v) \;=\; h_{j+1}(v)\,h_j(u)\,\exp(u\,v\,E_{j,j+2})\,. \tag{2.39}$$

Comparing (2.12) to (2.37), we find

$$R_{j,m,k}\tau \;=\; \int_0^1 e(k_j\,t)\,\ell(h_j(t))\,P_{j,m,k}\tau\,dt\,, \tag{2.40}$$

and similarly

$$S_{j,m,k}\tau \;=\; \int_0^1 e(k_{j+1}\,t)\,\ell(h_{j+1}(t))\,P_{j,m,k}\tau\,dt\,. \tag{2.41}$$

This last identity exhibits $S_{j,m,k}\tau$ as one of the Fourier components of $P_{j,m,k}\tau$ with respect to the action of a circle group – recall that $P_{j,m,k}\tau$ does not depend on k_{j+1} whereas $S_{j,m,k}\tau$ does depend on k_{j+1} – and consequently

$$P_{j,m,k}\tau \;=\; \sum\nolimits_{i\in\mathbb{Z}} S_{j,m,(k_1,\,\dots,\,k_j,\,i,k_{j+2},\,\dots,\,k_{n-1})}\tau \tag{2.42}$$

is the sum of all the Fourier components. This sum converges in the strong distribution topology since the circle action is continuous with respect to that topology.

Using, in order, (2.40), (2.39), (2.38), (2.42) and the transformation behavior of the Fourier component $S_{j,m,k}\tau$ under the action of $h_{j+1}(t)$, we find

$$\begin{aligned}
\ell\big(h_{j+1}(-\tfrac{k_j}{m})\big)R_{j,m,k}\tau \ &= \int_0^1 e(k_j\,t)\,\ell\big(h_{j+1}(-\tfrac{k_j}{m})\,h_j(t)\big)P_{j,m,k}\tau\,dt \ = \\
&= \int_0^1 e(k_j\,t)\,\ell\big(h_j(t)\,h_{j+1}(-\tfrac{k_j}{m})\,\exp(\tfrac{k_j\,t}{m}\,E_{j,j+2})\big)P_{j,m,k}\tau\,dt \\
&= \int_0^1 \ell\big(h_j(t)\,h_{j+1}(-\tfrac{k_j}{m})\big)P_{j,m,k}\tau\,dt \\
&= \sum\nolimits_{i\in\mathbb{Z}} \int_0^1 \ell\big(h_j(t)\,h_{j+1}(-\tfrac{k_j}{m})\big)\,S_{j,m,(k_1,\,\dots,\,k_j,\,i,k_{j+2},\,\dots,\,k_{n-1})}\tau\,dt \qquad (2.43)\\
&= \sum\nolimits_{i\in\mathbb{Z}} \int_0^1 e\left(\tfrac{i\,k_j}{m}\right)\,\ell(h_j(t))\,S_{j,m,(k_1,\,\dots,\,k_j,\,i,k_{j+2},\,\dots,\,k_{n-1})}\tau\,dt \\
&= \sum_{r=0}^{m-1}\,\sum\nolimits_{i\in\mathbb{Z}} \int_0^1 e\left(\tfrac{r\,k_j}{m}\right)\,\ell(h_j(t))\,S_{j,m,(k_1,\,\dots,\,k_j,r+im,k_{j+2},\,\dots,\,k_{n-1})}\tau\,dt \\
&= \sum\nolimits_{r\in\mathbb{Z}/m\mathbb{Z}} e\left(\tfrac{r\,k_j}{m}\right)\,\int_{\mathbb{R}} \ell(h_j(t))\,S_{j,m,(k_1,\,\dots,\,k_j,r,k_{j+2},\,\dots,\,k_{n-1})}\tau\,dt\,.
\end{aligned}$$

At the final step we have used the identity

$$\ell(h_j(1))S_{j,m,k}\tau \ = \ S_{j,m,(k_1,\,\dots,\,k_j,\,k_{j+1}-m,\,k_{j+2},\,\dots,\,k_{n-1})}\tau\,, \qquad (2.44)$$

which amounts to a restatement of the final assertion of lemma 2.18. In particular the integral in the last line of (2.43) depends only on the class of r modulo m, as claimed. The integral converges in the strong distribution topology because the sum (2.42) does. Finally, shifting the variable of integration by $\frac{k_{j+1}}{m}$, allows us to replace $S_{j,m,(\dots,r,\dots)}\tau$ by its $\ell(h_j(\frac{r}{m}))$-translate. □

By construction $R_{j-1,m,k}\tau$ and $S_{j,m,k}\tau$ transform under the left action of $N_j(\mathbb{R})$ according to characters of that group – recall (2.19). Like any vector in $V_{\lambda,\delta}^{-\infty}$ they also transform under the right action of $B_-(\mathbb{R})$ according to the inducing character; cf. (2.1). Since $N_j(\mathbb{R})\cdot \operatorname{Im}\Phi_j \cdot B_-(\mathbb{R})$ is open in $G(\mathbb{R})$, we can legitimately restrict $R_{j-1,m,k}\tau$ and $S_{j,m,k}\tau$ to the image of Φ_j. In analogy to (1.2), pairs $(\nu,\eta)\in\mathbb{C}\times\mathbb{Z}/2\mathbb{Z}$ parameterize principal series representations of $SL(2,\mathbb{R})$:

$$W_{\nu,\eta} \ = \ \{\, f: SL(2,\mathbb{R})\to\mathbb{C} \,\big|\, f\big(g\left(\begin{smallmatrix} a & 0\\ c & a^{-1}\end{smallmatrix}\right)\big) = |a|^{1-\nu}(\operatorname{sgn} a)^{\eta} f(g)\,\}\,; \qquad (2.45)$$

$W_{\nu,\eta}^{-\infty}$ shall denote the space of distribution vectors. Under right translation by elements of $\operatorname{Im}\Phi_j\cap B_-(\mathbb{R})$, $R_{j-1,m,k}\tau$ and $S_{j,m,k}\tau$ transform according to the (restriction of) the inducing character. Thus

$$(R_{j-1,m,k}\tau)\circ\Phi_j\,,\ (S_{j,m,k}\tau)\circ\Phi_j \quad\in\quad W^{-\infty}_{\lambda_j-\lambda_{j+1}\,,\,\delta_j-\delta_{j+1}}\,. \qquad (2.46)$$

One can restrict $R_{j-1,m,k}\tau\circ\Phi_j$ and $S_{j,m,k}\tau\circ\Phi_j$ to the upper triangular unipotent subgroup of $SL(2,\mathbb{R})$, just as it is legitimate to restrict τ to $N(\mathbb{R})$. Thus we can define distributions of one variable $\rho_{j,m,k}$, $\sigma_{j,m,k}$ by the equations

$$\begin{aligned} \rho_{j,m,k}(x) \ &= \ \Big(\ell\big(h_{j+1}(-\tfrac{k_j}{m})\big)R_{j,m,k}\tau\Big) \circ \Phi_{j+1}\left(\begin{smallmatrix} 1 & -x \\ 0 & 1 \end{smallmatrix}\right), \\ \sigma_{j,m,k}(x) \ &= \ \Big(\ell\big(h_j(\tfrac{k_{j+1}}{m})\big)S_{j,m,k}\tau\Big) \circ \Phi_j\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right); \end{aligned} \tag{2.47}$$

in the exceptional cases (2.13) the integers k_0, k_n should be interpreted as zero.

At the extremes, $\rho_{0,1,k}$ and $\sigma_{n-1,1,k}$ are expressible in terms of the Fourier coefficients c_k of τ_{abelian} , and hence, via (2.9), also in terms of the a_k :

Lemma 2.48. $\rho_{0,1,k}(x) = \sum_{r\neq 0} c_{r,k_2,k_3,\dots,k_{n-1}}\, e(-rx)$ *and* $\sigma_{n-1,1,k}(x) = \sum_{r\neq 0} c_{k_1,\dots,k_{n-2},r}\, e(rx)$.

Proof. The integrations with respect to the y variables in (2.13) convert τ'' into τ', and the integrations with respect to the remaining variables turn the series (2.8) into a Fourier series in the first, respectively last, variable by fixing the remaining indices. □

The Voronoi formula amounts to a connection between the Fourier coefficients of the $\rho_{0,1,k}$ to those of the $\sigma_{n-1,1,k}$. Our proof establishes that connection by relating the $\rho_{j-1,m,k}$ to the $\sigma_{j,m,k}$ and the $\sigma_{j,m,k}$ to the $\rho_{j,m,k}$. Lemma 2.29 embodies a weak form of the former; we strengthen it to a useful version in section 3. Implicitly lemma 2.36 relates the $\sigma_{j,m,k}$ to the $\rho_{j,m,k}$. Our next proposition – which also collects some additional information – makes that quite explicit.

We normalize the Fourier transform $\mathcal{F}f = \widehat{f}$ of a Schwartz function $f \in \mathcal{S}(\mathbb{R})$ by the formula

$$\widehat{f}(x) \ = \ \int_{\mathbb{R}} f(y)\, e(-xy)\, dy\,. \tag{2.49}$$

The same formula expresses the Fourier transform of any $f \in \mathcal{S}'(\mathbb{R})$, the space of tempered distributions, to which the Fourier transform extends via the duality between functions and distributions. The proposition also involves the finite Fourier transform

$$\widehat{a}_k \ = \ \sum\nolimits_{\ell\in\mathbb{Z}/m\mathbb{Z}} e\big(\tfrac{k\ell}{m}\big)\, a_\ell \qquad (\, a = (a_k)_{k\in\mathbb{Z}/m\mathbb{Z}}\,)\,, \tag{2.50}$$

of functions on $\mathbb{Z}/m\mathbb{Z}$, normalized following a common convention.

Proposition 2.51. *The* $\rho_{j,m,k}$ *and* $\sigma_{j,m,k}$ *are tempered distributions. The* $\rho_{j,m,k}$ *do not depend on* k_{j+1} *and the* $\sigma_{j,m,k}$ *do not depend on* k_j. *For* $1 \le j \le n-2$ $\rho_{j,m,k}$ *depends on* k_j *only modulo* m, *and* $\sigma_{j,m,k}$ *depends on* k_{j+1} *only modulo* m. *Still for* $1 \le j \le n-2$, *and for any* $a = (a_k)_{k\in\mathbb{Z}/m\mathbb{Z}}$,

$$\begin{aligned} &\sum\nolimits_{\ell\in\mathbb{Z}/m\mathbb{Z}} a_\ell\, \rho_{j,m,(k_1,\dots,\,k_{j-1},\,\ell,\,k_{j+1},\dots,\,k_{n-1})}(x) \ = \\ &\qquad = \ \sum\nolimits_{\ell\in\mathbb{Z}/m\mathbb{Z}} \widehat{a}_\ell\, \widehat{\sigma}_{j,m,(k_1,\dots,\,k_j,\,\ell,\,k_{j+2},\dots,\,k_{n-1})}(mx)\,, \end{aligned}$$

or equivalently,

$$\rho_{j,m,k}(x) \ = \ \sum\nolimits_{\ell\in\mathbb{Z}/m\mathbb{Z}} e\left(\frac{k_j\ell}{m}\right) \widehat{\sigma}_{j,m,(k_1,\dots,k_j,\ell,k_{j+2},\dots,k_{n-1})}(mx)\,.$$

Proof. By construction the $\rho_{j,m,k}$ and $\sigma_{j,m,k}$ are restrictions, to the upper triangular unipotent group in $SL(2,\mathbb{R})$, of vectors in various $W_{\nu,\eta}^{-\infty}$, and such distribution vectors can be paired continuously against vectors in the dual representations $W_{-\nu,-\eta}^{\infty}$. On the other hand, any $f\in\mathcal{S}(\mathbb{R})$ naturally extends to a vector in $W_{-\nu,-\eta}^{\infty}$, and thus pairs naturally and continuously against restrictions to $\mathbb{R}$ of vectors in $W_{\nu,\eta}^{-\infty}$. This proves the temperedness of the $\rho_{j,m,k}$ and $\sigma_{j,m,k}$. The assertions about the dependence on the k_i follow directly from the corresponding statements in lemma 2.28.

In order to relate the $\rho_{j,m,k}$ to the $\sigma_{j,m,k}$, we evaluate both sides of the identity in lemma 2.36 on $h_{j+1}(-x)$. In view of (2.39), with $u=-t-r/m$, $v=-x$, and (2.38),

$$\begin{aligned}&\big(\ell(h_j(t))\,\ell\big(h_j(\tfrac{r}{m})\big)S_{j,m,(\dots,r,\dots)}\tau\big)\big(h_{j+1}(-x)\big) \;=\\ &\quad=\;e\big(x(r+mt)\big)\,\big(\ell\big(h_j(\tfrac{r}{m})\big)\ell(h_{j+1}(x))S_{j,m,(\dots,r,\dots)}\tau\big)\big(h_j(-t)\big)\\ &\quad=\;e(m\,t)\,\big(\ell\big(h_j(\tfrac{r}{m})\big)S_{j,m,(\dots,r,\dots)}\tau\big)\big(h_j(-t)\big)\,.\end{aligned}\tag{2.52}$$

The second step uses the identity $\ell(h_{j+1}(x))S_{j,m,(\dots,r,\dots)}\tau=e(-r\,x)S_{j,m,(\dots,r,\dots)}\tau$, which follows from (2.41). Coupled with lemma 2.36, (2.52) allow us to conclude

$$\begin{aligned}\rho_{j,m,k}(x)\;&=\;\ell\big(h_{j+1}(-\tfrac{k_j}{m})\big)\,R_{j,m,k}\tau\big(h_{j+1}(-x)\big)\;=\\ &=\;\sum\nolimits_{r\in\mathbb{Z}/m\mathbb{Z}}e\left(\tfrac{r\,k_j}{m}\right)\int_{\mathbb{R}}e(m\,x\,t)\,\sigma_{j,m,(k_1,\,\dots,\,k_j,r,k_{j+2},\,\dots,\,k_{n-1})}(-t)\,dt\\ &=\;\sum\nolimits_{r\in\mathbb{Z}/m\mathbb{Z}}e\left(\tfrac{r\,k_j}{m}\right)\,\widehat{\sigma}_{j,m,(k_1,\,\dots,\,k_j,r,k_{j+2},\,\dots,\,k_{n-1})}\,(m\,x)\,.\end{aligned}\tag{2.53}$$

That is the second, equivalent statement about the connection between the $\rho_{j,m,k}$ and the Fourier transforms of the $\sigma_{j,m,k}$. □

3. Vanishing to infinite order

Let $I\subset\mathbb{R}$ be an open interval, and x_0 a point in I. In [13] we introduced the notion of a distribution $\sigma\in C^{-\infty}(I)$ vanishing at x_0 to infinite order. When σ happens to be a C^∞ function, this coincides with the usual notion of vanishing to infinite order. We shall not repeat the details of the definition here; instead we summarize the features that are relevant for this paper. First of all,

$$\begin{aligned}&\text{if } \sigma_1\,,\,\sigma_2\in C^{-\infty}(I) \text{ both vanish to infinite order at } x_0\,,\text{ and}\\ &\qquad\text{if } \sigma_1 \text{ and } \sigma_2 \text{ agree on } I-\{x_0\}\,,\text{ then } \sigma_1=\sigma_2 \text{ on all of } I\,.\end{aligned}\tag{3.1}$$

Thus, if $\sigma_0\in C^{-\infty}(I-\{x_0\})$ can be extended to a distribution $\sigma\in C^{-\infty}(I)$ which vanishes to infinite order at x_0, that extension is uniquely determined; we then call σ the canonical extension of σ_0 across x_0. The terminology "canonical extension" can be justified:

$$\begin{aligned}&\text{the property of vanishing to infinite order at } x_0 \text{ is preserved by } C^\infty\\ &\text{coordinate changes, by differentiation, and by multiplication with } C^\infty\\ &\text{functions or with } |x-x_0|^\nu(\operatorname{sgn}(x-x_0))^\eta,\ \text{for any } (\nu,\eta)\in\mathbb{C}\times\mathbb{Z}/2\mathbb{Z}\,;\end{aligned}\tag{3.2}$$

consequently these operations commute with the process of canonical extension. Everything that has been said also applies to distributions defined on an open neighborhood I of ∞ in $\mathbb{RP}^1 \cong \mathbb{R}\cup\{\infty\}$, via the coordinate change $x \rightsquigarrow 1/x$. The Fourier transform provides a connection between vanishing to infinite order at the origin and canonical extension across infinity:

$$\begin{aligned} &\sigma \in \mathcal{S}'(\mathbb{R}) \text{ vanishes to infinite order at the origin} \\ &\qquad \Longrightarrow \qquad \widehat{\sigma} \text{ has a canonical extension across } \infty\,. \end{aligned} \tag{3.3}$$

In particular,

$$\begin{aligned} &\text{any periodic distribution } \sigma = \textstyle\sum_{r\neq 0} a_r\, e(rx) \text{ with zero} \\ &\qquad \text{constant term has a canonical extension across } \infty\,, \end{aligned} \tag{3.4}$$

as follows from (3.3) since $\widehat{\sigma} = \sum_{r\neq 0} a_r\,\delta_r(x)$, with $\delta_r(x)$ denoting the delta function at r, vanishes identically near the origin, and that is a much stronger condition than vanishing to infinite order.

For the statements of the next proposition, we fix $k\in\mathbb{Z}^{n-1}$, $m\in\mathbb{Z}_{\neq 0}$, and j, $1\leq j\leq n-2$, as before. We define

$$\mu_j \ :\ \mathbb{R}^* \longrightarrow \mathbb{C}^*\,, \qquad \mu_j(x) \ =\ |x|^{\lambda_j-\lambda_{j+1}-1}\,\mathrm{sgn}(x)^{\delta_j+\delta_{j+1}}\,, \tag{3.5}$$

and we again use the notational conventions of lemma 2.29: $m_j = \gcd(m,k_j)$ is the greatest common divisor of m and k_j, and $\overline{k_j}\in\mathbb{Z}/m\mathbb{Z}$ an integer such that $\overline{k_j}\,k_j \equiv m_j \pmod{m}$.

Proposition 3.6. *All the $\rho_{j,m,k}$ and $\sigma_{j,m,k}$ extend canonically to distributions on the compactified real line $\mathbb{R}\cup\{\infty\}$, and vanish to infinite order at the origin. Both $\sigma_{1,m,k}$ and $\rho_{n-2,m,k}$ even vanish to infinite order at every rational point. In terms of the notational conventions just introduced,*

$$\begin{aligned} \sigma_{1,m,k}(x) \ &=\ \mu_1\big(\tfrac{m\,x}{m_2}\big)\textstyle\sum_{\ell\neq 0} c_{\ell,m_2,k_3,\dots,k_{n-1}}\ e\Big(\tfrac{\ell m_2}{m}\big(\overline{k_2}-\tfrac{m_2}{m\,x}\big)\Big)\,, \\ \rho_{n-2,m,k}(x) \ &=\ \mu_{n-1}\big(\tfrac{m\,x}{m_{n-2}}\big)\textstyle\sum_{\ell\neq 0} c_{k_1,\dots,k_{n-3},m_{n-2},\ell}\ e\Big(\tfrac{\ell m_{n-2}}{m}\big(\tfrac{m_{n-2}}{m\,x}-\overline{k_{n-2}}\big)\Big)\,, \end{aligned}$$

and, for $2\leq j\leq n-2$,

$$\begin{aligned} \sigma_{j,m,k}(x) \ &=\ \mu_j\big(\tfrac{m\,x}{m_{j+1}}\big)\,\rho_{j-1,\widetilde{m},\,(k_1,\dots,\,k_{j-2},\,\widetilde{k}_{j-1}\,,\,k_j\,,\,m_{j+1}\,,\,k_{j+2},\dots,k_{n-1})}\Big(\tfrac{m_{j+1}^2}{m^2\,x}\Big) \\ &\qquad\qquad \textit{with } \ \widetilde{m} = \tfrac{m\,k_{j-1}}{m_{j+1}} \ \ \textit{and } \ \widetilde{k}_{j-1} = k_{j-1}\,\overline{k_{j+1}}\,. \end{aligned}$$

In all three cases these are identities of distributions on $\mathbb{R}\cup\{\infty\}$.

The proof will occupy the remainder of this section. We begin with a remark on the action of $SL(2,\mathbb{R})$ on $W_{\nu,\eta}^{-\infty}$. If $a,b,c,d\in\mathbb{R}$ and $ad-bc=1$,

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} \ =\ \begin{pmatrix} 1 & \frac{ax+b}{cx+d} \\ 0 & 1\end{pmatrix}\begin{pmatrix} \frac{1}{cx+d} & 0 \\ c & cx+d\end{pmatrix}\,. \tag{3.7}$$

Hence, for $\sigma \in W_{\nu,\eta}^{-\infty}$,

$$\left(\ell\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\sigma\right)\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;=\; |cx+d|^{\nu-1}\,(\operatorname{sgn}(cx+d))^{\eta}\;\sigma\begin{pmatrix} 1 & \frac{ax+b}{cx+d} \\ 0 & 1 \end{pmatrix}. \qquad (3.8)$$

Although this appears to be an equality of distributions on $\mathbb{R}-\{-\frac{d}{c}\}$, it can be given meaning on $\mathbb{R}\cup\{\infty\}$: if $\sigma_0 \in C^{-\infty}(\mathbb{R})$ is the restriction of some $\sigma \in W_{\nu,\eta}^{-\infty}$ to the upper triangular unipotent subgroup of $SL(2,\mathbb{R})$ – identified with $\mathbb{R}$ in the usual manner – then $(\operatorname{sgn} x)^{\eta}|x|^{\nu-1}\sigma(1/x)$ extends across the origin, and σ_0 along with this extension completely determines σ. The extension is not unique since one can add any finite linear combination of derivatives of the delta function; in other words, σ_0 does not determine σ completely. However, according to (3.2), if σ_0 has a canonical extension across infinity when viewed as a distribution in the usual sense, then σ_0 can also be canonically extended to a vector in $W_{\nu,\eta}^{-\infty}$.

When the three identities in lemma 2.29 are restricted to the images of, respectively, Φ_1, Φ_{n-1}, Φ_j, the equalities asserted by proposition 3.6 follow from (3.8), but initially only as equalities of distributions on $\mathbb{R}-\{0\}$. To complete the proof of the proposition, we shall show that the $R_{j,m,k}\tau\circ\Phi_{j+1}$ and $S_{j,m,k}\tau\circ\Phi_j$ vanish to infinite order at infinity: in that case, the renormalized quantities $\ell(h_j(-\frac{k_{j-1}}{m}))R_{j-1,m,k}\tau\circ\Phi_j$ and $\ell(h_j(\frac{k_{j+1}}{m}))S_{j,m,k}\tau\circ\Phi_j$ also vanish to infinite order at ∞. Since lemma 2.29 relates the behavior of one near ∞ to the behavior of the other at the origin, both vanish to infinite order also at the origin. We can then conclude that the $\rho_{j,k,m}$ and $\sigma_{j,m,k}$ have canonical extensions across infinity, that they vanish to infinite order at the origin if $1\le j\le n-2$, and that the equalities asserted by the proposition are valid even at $x=0$ and $x=\infty$, as asserted. The second of the three identities relates the behavior of $\rho_{n-2,m,k}$ near ∞ to that of the periodic distribution

$$\sigma_{n-1,1,(k_1,\dots,k_{n-3},m_{n-2},k_{n-1})}(x) \;=\; \sum\nolimits_{\ell\neq 0} c_{k_1,\dots,k_{n-3},m_{n-2},\ell}\; e(\ell x) \qquad (3.9)$$

near $x=-\frac{\overline{k_{n-2}}}{m/m_{n-2}}$. Since m and k_{n-2} are multiples of m_{n-2}, we may write them as $m=am_{n-2}$ and $k=bm_{n-2}$. In this parametrization, $\overline{k_{n-2}}$ represents the inverse $\bar{b}$ of b modulo a, and the previous fraction is equal to $-\bar{b}/a$. Thus, when one considers all pairs of integers $m\neq 0$ and k_{n-2} having greatest common divisor m_{n-2}, one obtains every rational number as such a fraction. It follows that both the Fourier series (3.9) and the $\rho_{n-2,m,k}$ vanish to infinite order also at every rational point. The same assertion about the $\sigma_{1,m,k}$ can be proved the same way, of course.

In short, to prove proposition 3.6 it suffices to show that all the $S_{j,m,k}\tau\circ\Phi_j$ and $R_{j,m,k}\tau\circ\Phi_{j+1}$ vanish to infinite order at ∞. The outer automorphism

$$g \;\mapsto\; w_{\text{long}}\,(g^{-1})^t\,w_{\text{long}}^{-1}\,, \quad \text{with} \quad w_{\text{long}} \;=\; \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}, \qquad (3.10)$$

of $G(\mathbb{R})$ relates the automorphic distribution τ to its own contragredient $\widetilde{\tau}$, and further relates the quantities $S_{j,m,k}\tau$ for τ to the $R_{j,m,k}\tau$ corresponding to $\widetilde{\tau}$. We therefore only need to treat the case of the $R_{j,m,k}\tau$.

The verification of the vanishing to infinite order requires global arguments – we need to regard τ and the various quantities related to it as sections of a line bundle on the (real) flag variety

$$X \;=\; G(\mathbb{R})/B_-(\mathbb{R})\,; \tag{3.11}$$

here $B_-(\mathbb{R})$ refers to the group of lower triangular matrices, as before. We let $\mathcal{L}_{\lambda,\delta} \to X$ denote the equivariant line bundle – i.e., line bundle on which $G(\mathbb{R})$ acts, compatibly with its action on X – on whose fiber at the identity coset $B_-(\mathbb{R})$ operates via the character

$$\begin{pmatrix} a_1 & 0 & 0 \\ & \ddots & \\ \star & \ddots & 0 \\ \star & \star & a_n \end{pmatrix} \;\mapsto\; \prod\nolimits_{1\le j\le n} \left(|a_j|^{\lambda_j + j - \frac{n+1}{2}}\,\mathrm{sgn}(a_j)^{\delta_j} \right). \tag{3.12}$$

The representation space in which τ lies coincides with the space of distribution sections of $\mathcal{L}_{\lambda,\delta}$,

$$V_{\lambda,\delta}^{-\infty} \;=\; C^{-\infty}(X,\mathcal{L}_{\lambda,\delta})\,, \tag{3.13}$$

on which $G(\mathbb{R})$ acts via left translation, as it does in the case of $V_{\lambda,\delta}^{\infty}$; cf. [14]. The analogous description applies to the representation space $W_{\nu,\eta}^{-\infty}$ of the group $SL(2,\mathbb{R})$, whose flag variety is equivariantly embedded in X via each of the Φ_j .

Let $o \in X$ denote the identity coset $e\,B_-(\mathbb{R})$. The upper unipotent group $N(\mathbb{R})$ acts freely on the $N(\mathbb{R})$-orbit through o , and

$$N(\mathbb{R}) \;\cong\; X_0 \;=_{\mathrm{def}}\; N(\mathbb{R})\cdot o \;\subset\; X \tag{3.14}$$

is the open Schubert cell. The isotropy subgroup $B_-(\mathbb{R})$ at o intersects $N(\mathbb{R})$ only in the identity, and that makes the line bundle $\mathcal{L}_{\lambda,\delta}$ canonically trivial over the open Schubert cell – another, equivalent way of identifying the restriction $\sigma|_{X_0}$ of any $\sigma \in V_{\lambda,\delta}^{-\infty}$ to the open Schubert cell with a scalar distribution. A simple computation in $SL(2,\mathbb{R})$ shows that, as $t\to\infty$, the curve $h_j(t)\,o$ converges to

$$\lim\nolimits_{t\to\infty} h_j(t)\,o \;=\; s_j\,o\,, \tag{3.15}$$

the translate of the base point o by

$$s_j \;=\; \Phi_j\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \qquad (\,1\le j\le n-1\,)\,, \tag{3.16}$$

which normalizes the diagonal subgroup of $G(\mathbb{R})$ and represents the j-th simple Weyl reflection. The $N(\mathbb{R})$-orbit through $s_j\,o$,

$$C_j \;=_{\mathrm{def}}\; N(\mathbb{R})\cdot s_j\,o \;\subset\; X \tag{3.17}$$

has codimension one. In fact, the C_j , $1\le j\le n-1$, are exactly the codimension one Schubert cells.

The codimension one subgroup $N_j(\mathbb{R})\subset N(\mathbb{R})$, defined in (2.17), acts freely at $s_j\,o$, hence

$$N_j(\mathbb{R}) \;\cong\; C_j \;=\; N_j(\mathbb{R})\cdot s_j\,o\,. \tag{3.18}$$

As a subgroup of $N(\mathbb{R})$, $N_j(\mathbb{R})$ acts freely also at the base point o, and the resulting orbit

$$N_j(\mathbb{R}) \;\cong\; \{x_j = 0\} \;=_{\text{def}}\; N_j(\mathbb{R}) \cdot o \tag{3.19}$$

lies in the open Schubert cell X_0 as a closed, codimension one submanifold. Since $s_j \in \Phi_j\big(SL(2,\mathbb{R})\big)$ normalizes $N_j(\mathbb{R})$, left translation by s_j relates the two orbits,

$$\ell(s_j) \;:\; \{x_j = 0\} \;\xrightarrow{\;\sim\;}\; C_j\,. \tag{3.20}$$

Note that s_j^2 lies in the diagonal subgroup of $G(\mathbb{R})$, and thus fixes the point o.

The group $\Phi_j\big(SL(2,\mathbb{R})\big)$ normalizes $N_j(\mathbb{R})$, and $N_j(\mathbb{R}) \times \mathbb{R} \cong N(\mathbb{R})$ via $(n,t) \mapsto n\,h_j(t)$. Consequently the $N_j(\mathbb{R})$-translates of

$$\Phi_j\big(SL(2,\mathbb{R})\big)\cdot o \;\cong\; SL(2,\mathbb{R})/\,\Phi_j^{-1}\big(B_-(\mathbb{R})\big) \;\cong\; \mathbb{RP}^1 \tag{3.21}$$

sweep out an $N_j(\mathbb{R})$-equivariant fibration

$$\begin{array}{ccccc} X_0 & \xrightarrow{\;\subset\;} & X_0 \cup C_j & \xleftarrow{\;\supset\;} & \Phi_j\big(SL(2,\mathbb{R})\big)\cdot o \\ \big\downarrow{\scriptstyle \mathbb{R}} & & \big\downarrow{\scriptstyle \mathbb{RP}^1} & & \big\downarrow{\scriptstyle \mathbb{RP}^1} \\ \{x_j = 0\} & = & \{x_j = 0\} & \xleftarrow{\;\supset\;} & \{o\}\;. \end{array} \tag{3.22}$$

Note that $X_0 \cup C_j$ is open in X since its complement, i.e., the union of all non-open Schubert cells other than C_j, is closed.

We shall need to consider $P_{j,n-j}(\mathbb{R})$, the standard upper parabolic subgroup of type $(j, n-j)$ – in other words, the parabolic subgroup of $GL(n,\mathbb{R})$ generated by $N(\mathbb{R})$ and the diagonally embedded copies of $GL(j,\mathbb{R})$, $GL(n-j,\mathbb{R})$ placed, respectively, into the top left $j \times j$ and bottom right $(n-j)\times(n-j)$ corners. Further notation: for any element w of the normalizer of the diagonal subgroup,

$$C_w \;=\; N(\mathbb{R})\cdot w\,o \tag{3.23}$$

is the Schubert cell containing the point $w\,o$; it depends only on the coset of w modulo the diagonal subgroup. In particular, $C_{s_j} = C_j$ as previously defined, and $C_e = X_0$.

Lemma 3.24. *The orbit $P_{j,n-j}(\mathbb{R})\cdot s_j\,o$ is a locally closed algebraic submanifold of X, of codimension one. It contains C_j as a Zariski open subset. Define, by downward induction,*

$$C_j^j \;=\; C_j\,, \quad C_j^i \;=\; s_i\,C_j^{i+1} \cup C_j^{i+1} \qquad \textit{for}\;\; 1 \le i \le j-1\,.$$

Then C_j^i is a Zariski open, $N(\mathbb{R})$-invariant subset of $P_{j,n-j}(\mathbb{R})\cdot s_j\,o \subset X$, which coincides with the union of the Schubert cells $C_{s_{j_1}s_{j_2}\dots s_{j_r}s_j}$, $i \le j_1 < \cdots < j_r < j$, $0 \le r \le j-i+1$. It is also the smallest $N(\mathbb{R})$-invariant subset of $P_{j,n-j}(\mathbb{R})\cdot s_j\,o$ containing $s_i\,C_j^{i+1}$.

Proof. We let $\{e_1, e_2, \dots, e_n\}$ denote the standard basis of $\mathbb{R}^n$ and $\{f_1, f_2, \dots, f_n\}$ the dual basis. The function

$$G(\mathbb{R}) \ni g \mapsto \langle f_1 \wedge f_2 \wedge \cdots \wedge f_j \,,\, g^{-1}(e_1 \wedge e_2 \wedge \cdots \wedge e_j) \rangle \qquad (3.25)$$

transforms according to a character under right translation by $B_-(\mathbb{R})$, is left invariant under the action of $N(\mathbb{R})$, and does not vanish at the identity. Its vanishing locus, which we can and shall regard as a subvariety of $X = G(\mathbb{R})/B_-(\mathbb{R})$, therefore consists of a union of lower dimensional $N(\mathbb{R})$-orbits. The action of s_j on the standard basis interchanges e_j and e_{j+1}, up to to sign, but leaves the other basis elements alone. The function (3.25) therefore vanishes at s_j, which implies that the vanishing locus contains the $N(\mathbb{R})$-orbit of $s_j\, o$, i.e., the codimension one Schubert cell $C_{s_j} = C_j$. Arguing similarly one sees that the vanishing locus does not contain any of the other codimension one Schubert cells C_i, $i \neq j$, and must therefore coincide with the closure of C_j. Since $P_{j,n-j}(\mathbb{R})$ preserves the line spanned by $e_1 \wedge e_2 \wedge \cdots \wedge e_j$, the vanishing locus contains $P_{j,n-j}(\mathbb{R}) \cdot s_j\, o$. But $N(\mathbb{R}) \subset P_{j,n-j}(\mathbb{R})$, so $C_j = N(\mathbb{R}) \cdot s_j\, o$ is contained, and necessarily Zariski open, in $P_{j,n-j}(\mathbb{R}) \cdot s_j\, o$. Since $P_{j,n-j}(\mathbb{R})$ acts transitively on $P_{j,n-j}(\mathbb{R}) \cdot s_j\, o$, which contains the locally Zariski closed submanifold C_j as a Zariski open subset, the orbit must also be a locally closed submanifold.

By construction, the C_j^i are finite unions of translates of C_j by elements of $P_{j,n-j}(\mathbb{R})$, and are therefore Zariski open in $P_{j,n-j}(\mathbb{R}) \cdot s_j\, o$. The $N(\mathbb{R})$-invariance of C_j^i is not immediately obvious, but is a consequence, of course, of its description as a union of Schubert cells. This description follows inductively from the following assertion: let w be an element of the normalizer of the diagonal subgroup, $1 \leq i \leq n-1$, and let $\mathfrak{n}$ denote the Lie algebra of $N(\mathbb{R})$; then

$$w^{-1} E_{i,i+1} w \in \mathfrak{n} \quad \Longrightarrow \quad s_i\, C_w \cup C_w \;=\; C_w \cup C_{s_i w}\,. \qquad (3.26)$$

To see this, recall that every $n \in N(\mathbb{R})$ can be expressed uniquely as $n = n_i h_i(t)$, with $n_i \in N_i(\mathbb{R})$, $t \in \mathbb{R}$. Since s_i normalizes $N_i(\mathbb{R})$,

$$\begin{aligned} C_w &= N(\mathbb{R}) \cdot w\, o \;=\; N_i(\mathbb{R}) \cdot \{ h_i(t)\, w\, o \mid t \in \mathbb{R} \}\,, \\ s_i\, C_w &= s_i\, N(\mathbb{R}) \cdot w\, o \;=\; N_i(\mathbb{R}) \cdot \{ s_i\, h_i(t)\, w\, o \mid t \in \mathbb{R} \}\,, \quad \text{and} \\ C_{s_i w} &= N(\mathbb{R}) \cdot s_i\, w o \;=\; N_i(\mathbb{R}) \cdot \{ h_i(t)\, s_i\, w\, o \mid t \in \mathbb{R} \}\,. \end{aligned} \qquad (3.27)$$

The hypothesis of (3.26) implies that the isotropy subgroup of $\Phi_i(SL(2,\mathbb{R}))$ at the point wo coincides with the Φ_i-image of the lower triangular subgroup of $SL(2,\mathbb{R})$. Thus (3.27) reduces (3.26) to the corresponding statement about the flag variety of $SL(2,\mathbb{R})$ – i.e., about $\mathbb{RP}^1$ – which is essentially obvious. As was remarked before, (3.26) implies the description of C_j^i as a union of Schubert cells, by downward induction on i. The final assertion of the lemma also follows: with w and i as in (3.26), $N(\mathbb{R})\, s_i\, C_w$ contains $N_i(\mathbb{R}) \cdot \{ h_i(t_1) s_i h_i(t_2) w o \mid t_1, t_2 \in \mathbb{R} \}$, hence both $C_{s_i w}$ and C_w. □

The notions of vanishing to infinite order and canonical extension can be defined for distributions on manifolds; the locus along which the vanishing or canonical extension takes place must be a locally closed submanifold [13]. When the submanifold has codimension one, as is the case for the C_j, this can be thought of as the one variable case with parameters. Recall that the $R_{j,m,k}\tau \circ \Phi_{j+1}$ can be regarded as a distribution section of a line bundle over the flag variety of $SL(2,\mathbb{R})$, i.e., over the compactified real line $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$.

Lemma 3.28. *For $2 \le j \le n-1$, $m \ne 0$, and $k \in \mathbb{Z}^{n-1}$, the following are equivalent:*
a) *The $R_{j-1,m,(k_1,\dots,k_{j-2},\ell,k_j,\dots,k_{n-1})}\tau\big|_{X_0}$, indexed by a set of representatives ℓ modulo m, have canonical extensions across C_j;*
b) *The $R_{j-1,m,(k_1,\dots,k_{j-2},\ell,k_j,\dots,k_{n-1})}\tau \circ \Phi_j\big|_{\mathbb{R}}$, indexed by a set of representatives ℓ modulo m, have canonical extensions across ∞;*
c) *$P_{j-1,m,k}\tau\big|_{X_0}$ has a canonical extension across C_j.*
The conditions analogous to a) - c)*, with "vanish to infinite order" in place of "have canonical extensions" are also equivalent to each other. Moreover,* a) *and* b) *are equivalent even when $j = 1$, as are the analogous conditions involving vanishing to infinite order. In all cases, these equivalences preserve uniformity in m and k, in the sense of* [13, definition 7.1].

Proof. The fibration (3.22) is $N_j(\mathbb{R})$-equivariant. We can therefore identify the total space $X_0 \cup C_j$ with $N_j(\mathbb{R}) \times \mathbb{RP}^1$. Since $N_j(\mathbb{R})$ acts on $R_{j-1,m,k}$ according to the character $\chi^R_{j-1,m,k}$, $R_{j-1,m,k}$ can be regarded as the product of its restriction to the fiber with the character:

$$R_{j-1,m,k}\tau\big|_{X_0 \cup C_j} \;=\; \chi^R_{j-1,m,k} \times R_{j-1,m,k}\tau \circ \Phi_j\,. \tag{3.29}$$

Thus b) implies a), both in the "canonical extension" and the "vanishing to infinite order" version. The converse is almost equally obvious. We can use the coordinates on $N_j(\mathbb{R})$, along with x_j, to establish vanishing to infinite order. The vector fields $\ell(E_{i_1,i_2})$, with $1 \le i_1 < i_2 \le n$, $(i_1,i_2) \ne (j,j+1)$, generate the linear differential operators on $N_j(\mathbb{R})$, and each of them acts on $R_{j,m,k}\tau$ as multiplication by a constant. Hence, when an expression verifying a) is written in terms of the $\ell(E_{i_1,i_2})$ and the partial derivative with respect to x_{j+1}, one can restrict it to the fiber $\mathbb{RP}^1$ in the product $N_j(\mathbb{R}) \times \mathbb{RP}^1$ and conclude b), and this implication also applies to both versions of a) and b). In view of lemma 2.28, increasing the index ℓ by m has the effect of translating $R_{j-1,m,(k_1,\dots,k_{j-2},\ell,k_j,\dots,k_{n-1})}\tau \circ \Phi_j$ by $h_j(1)$. Hence b) implies that the $R_{j-1,m,(k_1,\dots,k_{j-2},\ell,k_j,\dots,k_{n-1})}\tau \circ \Phi_j$, for all $\ell \in \mathbb{Z}$, have canonical extensions, or vanish to infinite order, uniformly in ℓ, in the sense of [13, definition 7.1]. An application of [13, Lemma 7.2] then implies that the sum of the terms in (3.29), for all $\ell \in \mathbb{Z}$, has a canonical extension across C_j, respectively vanishes to infinite order along C_j. But the sum is $P_{j-1,m,k}\tau$, since

$$P_{j-1,m,k}\tau \;=\; \sum\nolimits_{\ell \in \mathbb{Z}} R_{j-1,m,(k_1,\dots,k_{j-2},\ell,k_j,\,\dots,\,k_{n-1})}\tau\,, \tag{3.30}$$

in complete analogy to (2.42). Thus b) implies c), again in both versions. This last argument can be reversed because [13, Lemma 7.2] asserts an equivalence between the two relevant conditions. In these arguments, bounds embodying the uniformity in m and k are carried along, so uniformity is preserved by all of the equivalences. $\square$

In practice, we shall prove that a particular distribution vanishes to infinite order along a locally closed submanifold $C \subset X$ by first showing that its restriction to the complement of C has a canonical extension across C. The difference between the distribution and the canonical extension is a distribution[2] supported on C. That distribution is then shown to vanish by finding a contradiction between the support condition and other properties it is known to possess.

To make this concrete, let us consider a non-open Schubert cell C_w, as defined in (3.23), attached to an element $w \neq e$ of the normalizer of the diagonal subgroup. Since C_w is locally closed in X, we can choose a Zariski open subset $U_w \subset X$ which contains C_w as a Zariski closed subset. We suppose that a distribution section σ of $\mathcal{L}_{\lambda,\delta}$ is given, with support in C_w,

$$\sigma \in C^{-\infty}(U_w, \mathcal{L}_{\lambda,\delta}), \qquad \operatorname{supp}\sigma \subset C_w . \tag{3.31}$$

A particular example would be a distribution section on C_w, not of the line bundle $\mathcal{L}_{\lambda,\delta}$ itself, but of its tensor product with $\wedge^{\mathrm{top}} T_{C_w}X$, the top exterior power of the normal bundle of C_w in X. Distributions with values in $\mathcal{L}_{\lambda,\delta}$ are naturally dual to smooth measures with values in the dual line bundle $\mathcal{L}_{-\lambda,\delta}$; the shift by $\wedge^{\mathrm{top}} T_{C_w}X$ compensates for the discrepancy between the transformation under coordinate changes of smooth measures on U_w on the one hand, and smooth measures on C_w on the other. We should remark that the line bundle $\wedge^{\mathrm{top}} T_{C_w}X$ on C_w extends to a $G(\mathbb{R})$-equivariant line bundle on X, so the shift by $\wedge^{\mathrm{top}} T_{C_w}X$ merely amounts to a shift of the parameters λ and δ.

We shall call a distribution section σ of the particular type just discussed a distribution section of "normal degree zero". On general principle, one can express an $\mathcal{L}_{-\lambda,\delta}$-valued distribution with support on C_w as a linear combination of normal derivatives, applied to a $\mathcal{L}_{-\lambda,\delta}$-valued distribution of normal degree zero, though in general such expressions can be given only locally. In our applications σ will sometimes be invariant under a subgroup that acts on C_w with a compact fundamental domain. In that case such an expression exists globally, and there is a well defined "normal order" – i.e., the maximum number of normal derivatives that are required. Otherwise the normal order may be well defined only locally.

In the following, we let $\mathfrak{n}$, $\mathfrak{n}_-$, $\mathfrak{b}_-$, and $\mathfrak{a}$ denote the real Lie algebras of, respectively, $N(\mathbb{R})$, the lower triangular unipotent subgroup $N_-(\mathbb{R})$, the lower triangular Borel subgroup $B_-(\mathbb{R})$, and the diagonal subgroup. The unipotent group $N(\mathbb{R})$ is the pointwise product, in either order, of $N(\mathbb{R}) \cap wN(\mathbb{R})w^{-1}$ and $N(\mathbb{R}) \cap wN_-(\mathbb{R})w^{-1}$; the latter fixes the point $w\,o$, and the former acts freely at $w\,o$. Thus

$$N(\mathbb{R}) \cap wN(\mathbb{R})w^{-1} \;\cong\; C_w \quad \text{via} \quad N(\mathbb{R}) \ni n \;\mapsto\; n\,w\,o\,. \tag{3.32}$$

[2]Technically a distribution not on X, but rather on any open subset $U \subset X$ which contains C as a closed submanifold.

Since $N(\mathbb{R})\cap wN(\mathbb{R})w^{-1}$ is closed in $wN(\mathbb{R})w^{-1}$, we can let its orbit at the point wo play the role of U_w:

$$U_w \;=\; w\,X_0 \;=\; w\,N(\mathbb{R})\cdot o \;=\; wN(\mathbb{R})w^{-1}\text{- orbit at } w\,o\,. \tag{3.33}$$

The tangent space to U_w at $w\,o$ is naturally isomorphic to $w\mathfrak{n}w^{-1}$, and that of C_w is naturally isomorphic to $\mathfrak{n}\cap w\mathfrak{n}w^{-1}$; cf. (3.32). Conjugation by the group $N(\mathbb{R})\cap wN(\mathbb{R})w^{-1}$ preserves both $w\mathfrak{n}w^{-1}$ and $\mathfrak{n}\cap w\mathfrak{n}w^{-1}$, so these are isomorphisms not just at $w\,o$, but at any point $n\,w\,o$ with $n\in N(\mathbb{R})\cap wN(\mathbb{R})w^{-1}$. We conclude that

$$w\mathfrak{n}w^{-1}/(\mathfrak{n}\cap w\mathfrak{n}w^{-1}) \;\cong\; \text{normal space to } C_w \text{ at } n\,w\,o\,, \tag{3.34}$$

for any $n\in N(\mathbb{R})\cap wN(\mathbb{R})w^{-1}$. Let $Z_1\,,\,Z_2\,,\,\dots\,,\,Z_M$ be a basis of $\mathfrak{n}_-\cap w\mathfrak{n}w^{-1}$, and let $\ell(Z_j)$ denote the vector field generated by Z_j under infinitesimal left translation. Since $\mathfrak{n}_-\cap w\mathfrak{n}w^{-1}$ is a linear complement to $\mathfrak{n}\cap w\mathfrak{n}w^{-1}$ in $w\mathfrak{n}w^{-1}$,

$$\text{the } \ell(Z_j)\,,\ 1\leq j\leq M, \text{ generate the normal space to } C_w \text{ at each point.} \tag{3.35}$$

We conclude: an $\mathcal{L}_{\lambda,\delta}$-valued distribution σ on U_w with support on C_w, as in (3.31), can be expressed locally as

$$\sigma \;=\; \sum\nolimits_L \ell(Z^L)\,\sigma_L\,; \tag{3.36}$$

here L runs over all M-tuples $(\ell_1,\ell_2,\dots,\ell_M)$ of nonnegative integers of total length up to the normal degree of σ, $\ell(Z^L)$ is shorthand for the ordered product $\ell(Z_1)^{\ell_1}\ell(Z_2)^{\ell_2}\dots\ell(Z_N)^{\ell_N}$, and the σ_L are $\mathcal{L}_{\lambda,\delta}$-valued distributions supported on C_w, of normal degree zero.

The functions $f\in C^\infty(U_w)$ that vanish on C_w constitute an ideal I_w. One calls a vector field – always understood to have C^∞ coefficients – tangential to C_w if the one parameter group of diffeomorphisms it generates preserves C_w, or entirely equivalently,

$$\text{a vector field } V \text{ is tangential to } C_w \text{ if } VI_w\subset I_w\,. \tag{3.37}$$

As before, we suppose that $\sigma\in C^{-\infty}(U_w,\mathcal{L}_{\lambda,\delta})$ has support in C_w. Then

$$\begin{aligned}&\text{multiplication by any } f\in I_w \text{ reduces the normal degree of } \sigma \text{ by one, and}\\ &\text{multiplication by any } f\in C^\infty(U_w) \text{ does not increase the normal degree.}\end{aligned} \tag{3.38}$$

In particular, if σ has normal degree zero and $f\in I_w$, $f\,\sigma$ vanishes. This follows easily from the definitions.

By infinitesimal left translation, any element Z of the Lie algebra of $G(\mathbb{R})$ acts on sections of the $G(\mathbb{R})$-equivariant line bundle $\mathcal{L}_{\lambda,\delta}$. Arbitrary vector fields, on the other hand, do not obviously act on sections of $\mathcal{L}_{\lambda,\delta}$. Like any line bundle, $\mathcal{L}_{\lambda,\delta}$ can be locally trivialized, and any two local trivializations are related by multiplication with a C^∞ function without zeroes. In view of (3.38), multiplication by such a function does not affect the normal degree. Therefore, without loss of generality, we may as well suppose that σ is a scalar distribution. We note:

$$\text{if the vector field } V \text{ is tangential to } C_w\,, \text{ the normal degree of } V\sigma \text{ does not exceed the normal degree of } \sigma\,. \tag{3.39}$$

Like (3.38) this follows directly from the definitions. The Schubert cell C_w is not only an $N(\mathbb{R})$-orbit, but is also invariant under the diagonal subgroup, which fixes the point $w\,o$ and normalizes $N(\mathbb{R})$. Thus

$$\text{the vector field } \ell(Z), \text{ for any } Z \in \mathfrak{a}\oplus\mathfrak{n}\,, \text{ is tangential to } C_w\,, \tag{3.40}$$

since the one parameter group generated by $\ell(Z)$ preserves C_w. For lack of a better term, we shall call a vector field V *hypertangential* to C_w if

$$V I_w^k \ \subset\ I_w^{k+1} \quad \text{for all } k \geq 0\,; \tag{3.41}$$

this is not standard terminology, however.

Lemma 3.42. *A vector field V is hypertangential to C_w if and only if both V and its commutator $[V,W]$ with any other vector field W are tangential to C_w. If the vector field V is hypertangential to C_w, $V\sigma$ has strictly lower normal degree than σ; in particular, when σ has normal degree zero, then $V\sigma$ must vanish.*

Proof. We choose local coordinates $x_1, x_2, \dots, x_m, y_1, \dots, y_n$ on X so that C_w is the set of common zeroes of the y_j, $1 \leq j \leq n$. When we express V as

$$V \ = \ \sum\nolimits_{1\leq i\leq m} a_i \tfrac{\partial}{\partial x_i} \ + \ \sum\nolimits_{1\leq j\leq n} b_j \tfrac{\partial}{\partial y_j}\,, \tag{3.43}$$

tangentiality is characterized by the condition $b_j \in I_w$ for all j, whereas hypertangentiality translates into the conditions $a_i \in I_w$, $b_j \in I_w^2$, for all indices i and j. Using these characterizations, one obtains the alternative description of hypertangentiality by computing the commutators of V with the $\frac{\partial}{\partial y_\ell}$ and the $x_k \frac{\partial}{\partial y_\ell}$. Since

$$V \ = \ \sum\nolimits_{1\leq i\leq m} \big(\tfrac{\partial}{\partial x_i}\circ a_i - \tfrac{\partial a_i}{\partial x_i}\big) \ + \ \sum\nolimits_{1\leq j\leq n} \big(\tfrac{\partial}{\partial y_j}\circ b_j - \tfrac{\partial b_j}{\partial y_j}\big)\,, \tag{3.44}$$

and since $\frac{\partial}{\partial x_i} I_w \subset I_w$, a hypertangential vector field V can be expressed as a linear combination $\sum_\ell W_\ell \circ f_\ell + g$ with $f_\ell, g \in I_w$. At this point (3.38) implies the second assertion of the lemma. □

We had remarked earlier that any $n \in wN(\mathbb{R})w^{-1}$ can be expressed uniquely as a product $n = n_1\, n_2$, with $n_1 \in wN(\mathbb{R})w^{-1} \cap N(\mathbb{R})$, $n_2 \in wN(\mathbb{R})w^{-1} \cap N_-(\mathbb{R})$. Thus, for any Z in the Lie algebra of $G(\mathbb{R})$, we can define a vector field $m(Z)$ on $U_w = wN(\mathbb{R})\cdot o$ by the formula

$$\begin{gathered} \big(m(Z)f\big)(n_1\, n_2\, w\, o) \ = \ \tfrac{\partial}{\partial t} f(n_1 \exp(-t\,Z)\, n_2\, w\, o)\big|_{t=0} \ \text{ if } f \in C^\infty(U_w) \\ \text{and } n_1 \in wN(\mathbb{R})w^{-1} \cap N(\mathbb{R})\,,\ n_2 \in wN(\mathbb{R})w^{-1} \cap N_-(\mathbb{R})\,. \end{gathered} \tag{3.45}$$

By construction,

$$\text{the vector fields } m(Z) \text{ are } \big(wN(\mathbb{R})w^{-1}\cap N(\mathbb{R})\big)\text{-invariant.} \tag{3.46}$$

Since $\ell(Z)f(n_1n_2wo)$ is the derivative at $t=0$ of $f(\exp(-tZ)n_1n_2wo)$, we can describe the value $m(Z)|_{n_1n_2wo}$ of $m(Z)$ at the point n_1n_2wo as follows:

$$m(Z)\big|_{n_1n_2wo} \;=\; \ell(\widetilde{Z})\big|_{n_1n_2wo} \quad \text{with} \quad \widetilde{Z}=\operatorname{Ad}(n_1)Z\,; \tag{3.47}$$

here, as before, $n_1\in wN(\mathbb{R})w^{-1}\cap N(\mathbb{R})$, $n_2\in wN(\mathbb{R})w^{-1}\cap N_-(\mathbb{R})$.

Lemma 3.48. **a)** *$m(Z)$ is tangential to C_w if $Z\in\mathfrak{a}\oplus\mathfrak{n}$ or $Z\in w\,\mathfrak{n}_-\,w^{-1}$.*
b) $Z_1,Z_2\in w\,\mathfrak{n}\,w^{-1}\cap\mathfrak{n}_- \implies [m(Z_1),m(Z_2)]\;=\;m([Z_1,Z_2])$.
c) $Z_1,Z_2\in w\,\mathfrak{n}\,w^{-1}\cap\mathfrak{n} \implies [m(Z_1),m(Z_2)]\;=\;-\,m([Z_1,Z_2])$.
d) $Z_1\in w\,\mathfrak{n}\,w^{-1}\cap\mathfrak{n}_-$, $Z_2\in w\,\mathfrak{n}\,w^{-1}\cap\mathfrak{n} \implies [m(Z_1),m(Z_2)]\;=\;0$.
e) $Z_1\in w\,\mathfrak{n}_-\,w^{-1}\cap\mathfrak{n}$, $Z_2\in w\,\mathfrak{n}\,w^{-1}\cap\mathfrak{n}_- \implies$ *the vector field* $[m(Z_1),m(Z_2)]-m([Z_1,Z_2])$ *is tangential to* C_w.
f) $Z_1\in w\,\mathfrak{n}_-\,w^{-1}\cap\mathfrak{n}$, $[\,Z_1\,,\,w\,\mathfrak{n}w^{-1}\cap\mathfrak{n}_-\,]\;\subset\;\mathfrak{a}+\mathfrak{n}+w\,\mathfrak{n}_-\,w^{-1} \implies$ $m(Z_1)$ *is hypertangential to* C_w.

Proof. The one parameter group $\exp(-tZ)$, with $Z\in w\,\mathfrak{n}_-\,w^{-1}$, fixes the point $w\,o$. We conclude that the value $m(Z)|_{n_1wo}$ of the vector field $m(Z)$ at any point $n_1wo\in C_w$ – i.e., when $n_2=e$ – vanishes. That is an even stronger condition than tangentiality. When $Z\in\mathfrak{a}\oplus\mathfrak{n}$, on the other hand, the values $m(Z)|_{n_1wo}$ along C_w are tangential to C_w but possibly non-zero; in that case, too, $m(Z)$ is tangential to C_w. That implies a). For $Z_1\in w\,\mathfrak{n}\,w^{-1}\cap\mathfrak{n}_-$ and $n_2\in wN(\mathbb{R})w^{-1}\cap N_-(\mathbb{R})$, the product $\exp(-tZ_1)n_2$ also lies in $wN(\mathbb{R})w^{-1}\cap N_-(\mathbb{R})$. Hence, for any Z_2 in the Lie algebra of $G(\mathbb{R})$, any $n_1\in wN(\mathbb{R})w^{-1}\cap N(\mathbb{R})$ and any $f\in C^\infty(U_w)$,

$$\begin{aligned}\big(m(Z_1)m(Z_2)f\big)(n_1\,n_2\,w\,o) \;&=\; \tfrac{\partial}{\partial t}\,\big(m(Z_2)f\big)(n_1\exp(-t\,Z_1)\,n_2\,w\,o)\big|_{t=0}\\ &=\; \tfrac{\partial^2}{\partial t\partial s}\,f(n_1\exp(-sZ_2)\exp(-t\,Z_1)\,n_2\,w\,o)\big|_{s=t=0}\,.\end{aligned} \tag{3.49}$$

Similarly, with $Z_1\in w\,\mathfrak{n}\,w^{-1}\cap\mathfrak{n}$ and $n_1\in wN(\mathbb{R})w^{-1}\cap N(\mathbb{R})$, $n_1\exp(-tZ_1)$ lies in $wN(\mathbb{R})w^{-1}\cap N(\mathbb{R})$. Hence, for any Z_2 in the Lie algebra of $G(\mathbb{R})$, any $n_2\in wN(\mathbb{R})w^{-1}\cap N_-(\mathbb{R})$ and any $f\in C^\infty(U_w)$,

$$\begin{aligned}\big(m(Z_1)m(Z_2)f\big)(n_1\,n_2\,w\,o) \;&=\; \tfrac{\partial}{\partial t}\,\big(m(Z_2)f\big)(n_1\exp(-t\,Z_1)\,n_2\,w\,o)\big|_{t=0}\\ &=\; \tfrac{\partial^2}{\partial t\partial s}\,f(n_1\exp(-tZ_1)\exp(-s\,Z_2)\,n_2\,w\,o)\big|_{s=t=0}\,.\end{aligned} \tag{3.50}$$

These two identities imply b) - d).

For the proof of e), we fix $Z_1\in w\,\mathfrak{n}_-\,w^{-1}\cap\mathfrak{n}$ and $Z_2\in w\,\mathfrak{n}\,w^{-1}\cap\mathfrak{n}_-$. As Y_j runs over a basis of $w\,\mathfrak{n}\,w^{-1}$, the values of the vector fields $m(Y_j)$ at any point of U_w span the tangent space. We can therefore write

$$\begin{aligned}&m(Z_1)\;=\;\textstyle\sum_i a_i\,m(Y_i^+)\;+\;\sum_j b_j\,m(Y_j^-)\,,\\ &\quad\text{with}\quad Y_i^+\in w\,\mathfrak{n}\,w^{-1}\cap\mathfrak{n}\,,\quad Y_j^-\in w\,\mathfrak{n}\,w^{-1}\cap\mathfrak{n}_-\,,\quad a_i,b_j\in C^\infty(U_w)\,;\end{aligned} \tag{3.51}$$

this expression becomes unique when we assume, as we may, that the Y_i^+, Y_j^- are linearly independent. Then

$$\begin{aligned}[m(Z_1),m(Z_2)] \ &= \ \sum\nolimits_i \big(a_i\,[m(Y_i^+),m(Z_2)] - (m(Z_2)a_i)\,m(Y_i^+)\,\big) \\ &\quad + \ \sum\nolimits_j \big(b_j\,[m(Y_j^-),m(Z_2)] - (m(Z_2)b_j)\,m(Y_j^-)\,\big)\,. \end{aligned} \tag{3.52}$$

On the other hand, for $f\in C^\infty(U_w)$,

$$\begin{aligned} &(m([Z_1,Z_2])f)(w\,o) \ = \ \tfrac{\partial}{\partial t} f(\exp(-t[Z_1,Z_2])\,w\,o)\big|_{t=0} \\ &= \ \tfrac{\partial^2}{\partial t\partial s}\big(f(\exp(-sZ_2)\exp(-tZ_1)\,w\,o) - f(\exp(-tZ_1)\exp(-sZ_2)\,w\,o)\big)\big|_{s=t=0} \\ &= \ -\,\tfrac{\partial^2}{\partial t\partial s}\,f(\exp(-tZ_1)\exp(-sZ_2)\,w\,o)\big|_{s=t=0} \\ &= \ -\,\tfrac{\partial}{\partial s}\big(m(Z_1)f\big)(\exp(-s\,Z_2)\,w\,o)\big|_{s=0} \\ &= \ -\,\big(m(Z_2)m(Z_1)f\big)(w\,o)\,; \end{aligned} \tag{3.53}$$

at the second step we have used the identity

$$\exp(sZ_2)\exp(tZ_1) = \exp(sZ_2+tZ_1+\tfrac{1}{2}st[Z_2,Z_1]+s^2\cdots+t^2\cdots+\cdots)\,, \tag{3.54}$$

at the third, the fact that $\exp(-tZ_1)wo\equiv wo$, and at the last two steps the definition of $m(Z)$. We now substitute the expression (3.51) for $m(Z_1)$ and note that $m(Z_1)|_{wo}=0$, hence $a_i(wo)=0$ and $b_j(wo)=0$:

$$\begin{aligned}(m([Z_1,Z_2]f)(w\,o) \ &= \ -\sum\nolimits_i \big(m(Z_2)\,a_i\big)(w\,o)\big(m(Y_i^+)f\big)(w\,o) \\ &\quad -\sum\nolimits_j \big(m(Z_2)\,b_j\big)(w\,o)\big(m(Y_j^-)f\big)(w\,o)\,. \end{aligned} \tag{3.55}$$

Comparing this to (3.52), and using the vanishing of the a_i and b_j at wo, we see that the values at wo of the two vector fields $[m(Z_1),m(Z_2)]$ and $m([Z_1,Z_2])$ coincide. Both vector fields are invariant under $wN(\mathbb{R})w^{-1}\cap N(\mathbb{R})$, which acts transitively on C_w, so their difference vanishes along C_w. It is therefore tangential to C_w, as asserted by e).

We use the criterion in lemma 3.42 to verify f). By a) $m(Z_1)$ is tangential to C_w. It remains to be shown that the commutators of $m(Z_1)$ with $a\,m(Y^+)$ and $b\,m(Y^-)$ are tangential to C_w, for any $Y^+\in w\mathfrak{n}w^{-1}\cap\mathfrak{n}$, $Y^-\in w\mathfrak{n}w^{-1}\cap\mathfrak{n}_-$, and $a,b\in C^\infty(U_w)$. Both $m(Z_1)$ and $a\,m(Y^+)$ are tangential to C_w by a), hence so is their commutator. Because of a), e) and the hypotheses on Z_1, the commutator of $m(Z_1)$ and $m(Y^-)$ is also tangential to C_w. It remains to be shown that $\big(m(Z_1)b\big)m(Y^-)$ is tangential to C_w, or equivalently that $m(Z_1)b$ vanishes along C_w – or, in view of the $(wN(\mathbb{R})w^{-1}\cap N(\mathbb{R}))$-invariance, that $(m(Z_1)b)(wo)=0$. But this is clear: $Z_1\in w\mathfrak{n}_-w^{-1}=$ isotropy subalgebra at wo, so the value $m(Z_1)|_{wo}$ of the vector field $m(Z_1)$ at wo is zero. □

Proposition 3.56. *We consider a distribution section σ of $\mathcal{L}_{\lambda,\delta}$ with support in C_w, as in (3.23), and some $Z\in w\mathfrak{n}_-w^{-1}\cap\mathfrak{n}$, subject to the following hypotheses:*
i) $[\,Z\,,\,w\,\mathfrak{n}w^{-1}\cap\mathfrak{n}_-\,]\ \subset\ \mathfrak{a}+\mathfrak{n}+w\,\mathfrak{n}_-\,w^{-1}$;
ii) $\ell(Z)-m(Z)$ *annihilates* σ.
Then $\ell(Z)$ lowers the normal degree of σ by one.

The hypothesis ii) requires explanation. The meaning of the action of $\ell(Z)$ on σ is clear, not just for the specific Z in the proposition, but for any Z in the Lie algebra of $G(\mathbb{R})$. The formula (3.45), in contrast, only defines $m(Z)$ as a vector field on U_w. To attach meaning to $m(Z)\sigma$, we express $m(Z)$ as a linear combination $m(Z) = \sum_j a_j\,\ell(Z_j)$, with Z_j running over a basis of $w\mathfrak{n}w^{-1}$, and with coefficients $a_j \in C^\infty(U_w)$; we interpret $m(Z)\sigma$ as equivalent to $\sum_j a_j\,\ell(Z_j)\sigma$.

In our applications, σ has invariance properties that rule out a strictly lower normal degree for $\ell(Z)\sigma$, and that will allow us to conclude σ must vanish. As mentioned in the introduction, this is closely related to the main mechanism of proof in the paper [2] of Casselman-Hecht-Miličić.

Proof. We remarked earlier that we may suppose, without loss of generality, that σ is a scalar distribution. According to lemma 3.48 $m(Z)$ is hypertangential to C_w. Thus, by lemma 3.42, $\ell(Z)\,\sigma$ has strictly lower normal degree than σ. □

We return to our earlier notation, with $\tau \in C^{-\infty}(X, \mathcal{L}_{\lambda,\delta})^{G(\mathbb{Z})}$, and with the $R_{j,m,k}\tau$ and $S_{j,m,k}\tau$ as defined in section 2. Recall the definition of the codimension one Schubert cells $C_j = C_{s_j}$.

Lemma 3.57. *For $1 \leq j \leq n-2$ and all choices of m and k, $R_{j,m,k}\tau$ vanishes to infinite order along the codimension one Schubert cells $C_{\tilde{j}}$, provided $\tilde{j}$ equals neither j nor $j+1$. In the case of $j=0$, $R_{0,1,k}\tau$ vanishes to infinite order along all the $C_{\tilde{j}}$. In both cases, the vanishing to infinite order is uniform in m and the multi-index k, in the sense of* [13, definition 7.1].

Proof. We begin with an auxiliary result. In all cases covered by the statement of the lemma, with the single exception of $j=0$, $\tilde{j}=1$,

$$k_{\tilde{j}} \;=\; 0 \qquad \Longrightarrow \qquad R_{j,m,k}\tau \;=\; 0\,; \tag{3.58}$$

in the remaining exceptional case, $\ell(h_1(1))$ acts as the identity on $R_{0,1,k}\tau$, and

$$\int_{\mathbb{R}/\mathbb{Z}} \ell(h_1(t))\, R_{0,1,k}\tau\, dt \;=\; 0\,. \tag{3.59}$$

Indeed, by (2.12),

$$\begin{aligned} R_{j,m,k}\tau \;&=\; \int_{\left\{\substack{(x,y)\in(\mathbb{R}/\mathbb{Z})^{2n-3}\\ x_{j+1}=0}\right\}} e(k\cdot x + my_j)\;\ell(n_{x,y})\tau''\,dx\,dy \\ &=\; \int_{\left\{\substack{(x,y)\in(\mathbb{R}/\mathbb{Z})^{2n-3}\\ x_{j+1}=0}\right\}} \int_{N''(\mathbb{R})/N''(\mathbb{Z})} e(k\cdot x + my_j)\;\ell(n_{x,y}\,n'')\tau\,dn''\,dx\,dy\,. \end{aligned} \tag{3.60}$$

Let $U_{\tilde{j},n-\tilde{j}}(\mathbb{R}) \subset N(\mathbb{R})$ denote the unipotent radical of the standard upper parabolic of type $\tilde{j}\times(n-\tilde{j})$. The double integral in (3.60) can be combined into a single integral. When $(j,\tilde{j}) \neq (0,1)$, the resulting integration can be performed by first integrating over $U_{\tilde{j},n-\tilde{j}}(\mathbb{R})/U_{\tilde{j},n-\tilde{j}}(\mathbb{Z})$, then over the remaining variables. The hypotheses, including the assumption that $k_{\tilde{j}}=0$, ensure that the character

$e(k\cdot x+my_j)$ is identically equal to one on $U_{\tilde{j},n-\tilde{j}}(\mathbb{R})$. Hence the cuspidality of τ implies

$$\begin{aligned}\int_{U_{\tilde{j},n-\tilde{j}}(\mathbb{R})/U_{\tilde{j},n-\tilde{j}}(\mathbb{Z})} e(k\cdot x+my_j)\ \ell(n)\tau\, dn \ &= \\ = \int_{U_{\tilde{j},n-\tilde{j}}(\mathbb{R})/U_{\tilde{j},n-\tilde{j}}(\mathbb{Z})} \ell(n)\tau\, dn \ &= \ 0\,.\end{aligned} \tag{3.61}$$

and that, in turn, implies the vanishing of the integral (3.60). When $j=0$, $\tilde{j}=1$, the same argument applies, provided we first average $\ell(h_1(t))\,R_{0,1,k}\tau$ over $\mathbb{R}/\mathbb{Z}$.

As the first step towards the proof of vanishing to infinite order, we show that the restrictions of the $R_{j,m,k}\tau$ to the open Schubert cell X_0 have canonical extensions across the $C_{\tilde{j}}$, $\tilde{j}\neq j,\, j+1$, and in the case of $R_{0,1,k}\tau$ across all the $C_{\tilde{j}}$. The argument is slightly different, and also slightly simpler, for $R_{0,1,k}\tau$. We use (3.22) to identify X_0 with $N_{\tilde{j}}(\mathbb{R})\times\mathbb{R}$ and simultaneously, $X_0\cup C_{\tilde{j}}$ with $N_{\tilde{j}}(\mathbb{R})\times\mathbb{RP}^1$. In view of (2.7), (2.8), (2.11), and (2.13), $R_{0,1,k}\tau\big|_{X_0}$ can then be regarded as a Fourier series on $N_{\tilde{j}}(\mathbb{R})\times\mathbb{R}$ – which happens to be constant in the entries corresponding to $N'(\mathbb{R})$ – whose expansion in the variable $x_{\tilde{j}}$, corresponding to the factor $\mathbb{R}$, has zero constant term. Thus [13, proposition 2.19] applies directly: $R_{0,1,k}\tau\big|_{X_0}$ has a canonical extension across $C_{\tilde{j}}$, at least as a scalar distribution. However, the discrepancy between a distribution section of $\mathcal{L}_{\lambda,\delta}$ and a scalar distribution on the complement of $C_{\tilde{j}}$, in terms of coordinates valid along $C_{\tilde{j}}$, is a factor of the type $|x_{\tilde{j}}|^\nu\,(\operatorname{sgn} x_{\tilde{j}})^\eta$, which does not affect the notion of vanishing to infinite order; cf. (3.2). The uniformity in the sense of [13], finally, follows from the fact that the Fourier coefficients of the distribution τ_{abelian} – like those of any periodic distribution – are bounded by some polynomial in the length $\|k\|$ of the multi-index k.

We now suppose $1\leq j\leq n-2$, $1\leq\tilde{j}\leq n-1$, $\tilde{j}\neq j,\, j+1$. Under these conditions we want to show that $R_{j,m,k}\tau\big|_{X_0}$ has a canonical extension across $C_{\tilde{j}}$. Just as in the case of $R_{0,1,k}\tau$, we may as well regard $R_{j,m,k}\tau\big|_{X_0}$ as a scalar distribution. We consider the fibration (3.22) with j replaced by $j+1$, which then allows us to identify

$$R_{j,m,k}\tau\big|_{X_0} \ \cong\ \chi^R_{j,m,k}\times R_{j,m,k}\tau\circ\Phi_{j+1}\big|_{\mathbb{R}} \tag{3.62}$$

as in (3.29). Except for a translation and change of sign, $R_{j,m,k}\tau\circ\Phi_{j+1}\big|_{\mathbb{R}}$ coincides with $\rho_{j,m,k}$ (2.47), which is a tempered distribution by proposition 2.51. We can therefore express $R_{j,m,k}\tau\circ\Phi_{j+1}\big|_{\mathbb{R}}$ as a sufficiently high derivative of a continuous function of polynomial growth:

$$R_{j,m,k}\tau\circ\Phi_{j+1}\big|_{\mathbb{R}} \ =\ \tfrac{d^r}{dx^r}\,f \qquad \text{with} \quad |f(x)|\,=\,O(|x|^s) \quad \text{as} \quad |x|\to\infty\,, \tag{3.63}$$

for some $r,s\in\mathbb{N}$. The variable x in this identity is really x_{j+1}, the $(j+1,j+2)$ matrix entry of $n_{x,y}$ (2.10), and differentiation in this direction is the vector field $\ell(-E_{j+1,j+2})$, i.e., infinitesimal translation by $-E_{j+1,j+2}$ – the inverse in (2.3) accounts for the minus sign. The diffeomorphism

$$X_0 \ \cong\ N(\mathbb{R}) \ \cong N_{j+1}(\mathbb{R})\times\mathbb{R}\,, \tag{3.64}$$

which underlies the identification (3.62), involves left multiplication by the $N_{j+1}(\mathbb{R})$ factor. Hence (3.63) can be re-written as follows:

$$R_{j,m,k}\tau(n_{x,y}n'') \;=\; \ell(-\operatorname{Ad}(n_{x,y}n'')E_{j+1,j+2})^r \left(\chi^R_{j,m,k} \times f\right)(n_{x,y}n'')\,, \qquad (3.65)$$

for $(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-2}$ and $n'' \in N''(\mathbb{R})$. A simple calculation shows

$$\operatorname{Ad}(n_{x,y}n'')E_{j+1,j+2} \;\equiv\; E_{j+1,j+2} + x_j\, E_{j,j+2} \quad \text{modulo} \quad \operatorname{Ker}\chi^R_{j,m,k}\,. \qquad (3.66)$$

Since $E_{j+1,j+2}$ and $E_{j,j+2}$ commute, and since $\ell(E_{j,j+2})$ acts on $\chi^R_{j,m,k}$ as multiplication by $-2\pi i\, m$,

$$R_{j,m,k}\tau(n_{x,y}n'') \;=\; (2\pi i\, m\, x_j \,-\, \ell(E_{j+1,j+2}))^r \left(\chi^R_{j,m,k} \times f\right)(n_{x,y}n'')\,. \qquad (3.67)$$

Since $\tilde{j} \neq j$, the function x_j is smooth along $C_{\tilde{j}}$ – cf. (3.22), with $\tilde{j}$ in place of j. That makes $(2\pi i\, m\, x_j - \ell(E_{j+1,j+2}))$ a linear differential operator with C^∞ coefficients on some neighborhood[3] of $C_{\tilde{j}}$. The action of such a differential operator does not affect vanishing to infinite order along $C_{\tilde{j}}$. Hence, to show that $R_{j,m,k}\tau$, has a canonical extension across $C_{\tilde{j}}$, it suffices to establish the same fact for $\chi^R_{j,m,k} \times f$.

We may suppose $k_{\tilde{j}} \neq 0$ by (3.58), and the vector field $\ell(E_{\tilde{j},\tilde{j}+1})$ acts on $\chi^R_{j,m,k}$ as multiplication by the factor $-2\pi i\, k_{\tilde{j}}$. Thus, for any $r \in \mathbb{N}$,

$$\chi^R_{j,m,k} \times f \;=\; (-2\pi i\, k_{\tilde{j}})^{-r}\, \ell(E_{\tilde{j},\tilde{j}+1})^r\, \chi^R_{j,m,k} \times f\,. \qquad (3.68)$$

We use (3.22), with $\tilde{j}$ in place of j, to identify $X_0 \cup C_{\tilde{j}} \cong N_{\tilde{j}} \times \mathbb{RP}^1$. Then $x_{\tilde{j}}$ becomes a coordinate on the fiber, which takes the value ∞ exactly along $C_{\tilde{j}}$. Arguing as we did between (3.63) and (3.65), we find

$$\frac{\partial}{\partial x_{\tilde{j}}} \;=\; \ell(-\operatorname{Ad}(n_{x,y}n'')E_{\tilde{j},\tilde{j}+1}) \qquad (\, n'' \in N''(\mathbb{R})\,). \qquad (3.69)$$

Since $\tilde{j} \neq j\,, j{+}1$, $Ad(n_{x,y}n'')E_{\tilde{j},\tilde{j}+1} \equiv E_{\tilde{j},\tilde{j}+1}$ modulo the kernel of $\chi^R_{j,m,k}$. That makes (3.68) equivalent to

$$\chi^R_{j,m,k} \times f \;=\; (2\pi i\, k_{\tilde{j}})^{-r}\, \frac{\partial^r}{\partial x^r_{\tilde{j}}} \left(\chi^R_{j,m,k} \times f\right)\,. \qquad (3.70)$$

According to (3.63) the function $\chi^R_{j,m,k} \times f$ is locally bounded along $C_{\tilde{j}}$. Since $\frac{\partial}{\partial x_{\tilde{j}}}$ vanishes to second order[4] on $\{x_{\tilde{j}} = \infty\} = C_{\tilde{j}}$, the criterion of [13, definition 2.4] applies: $R_{j,m,k}\tau$ has a canonical extension across $C_{\tilde{j}}$, as asserted.

To establish the uniformity in the sense of [13], we only need to argue that the bound implicit in (3.63) holds uniformly in m and k, with a bounding constant which depends polynomially on m and $\|k\|$. But this is a consequence of the fact that the $\chi^R_{j,m,k}$ are the Fourier coefficients of a globally defined distribution.

[3] On the complement of the closure of C_j in X, in fact.

[4] In terms of the coordinate change $t = 1/x_{\tilde{j}}$, $\frac{\partial}{\partial x_{\tilde{j}}} = -t^2\, \frac{\partial}{\partial t}$.

To finish the proof we must show that $R_{j,m,k}\tau$ vanishes to infinite order on $C_{\tilde{j}}$, or equivalently, that the difference

$$s_{j,m,k,\tilde{j}} \;=\; R_{j,m,k}\tau \;-\; \text{canonical extension of } R_{j,m,k}\tau \text{ across } C_{\tilde{j}} \tag{3.71}$$

vanishes. We note that $s_{j,m,k,\tilde{j}}$ is a $\mathcal{L}_{\lambda,\delta}$-valued distribution, defined on a neighborhood of $C_{\tilde{j}}$, specifically on $X_0 \cup C_{\tilde{j}}$. By definition,

$$s_{j,m,k,\tilde{j}} \text{ is supported on } C_{\tilde{j}}\,. \tag{3.72}$$

Because of the canonical nature[5] of the canonical extension, $s_{j,m,k,\tilde{j}}$ inherits invariance properties from $R_{j,m,k}\tau$. In particular,

$$\ell(n)\, s_{j,m,k,\tilde{j}} \;=\; \chi^R_{j,m,k}(n^{-1})\, s_{j,m,k,\tilde{j}} \quad \text{for } \; n\in N_{j+1}(\mathbb{R})\,; \tag{3.73}$$

because $R_{j,m,k}\tau$ satisfies the corresponding equation; cf. (2.19). On the infinitesimal level, this implies

$$\ell(E_{\tilde{j},\tilde{j}+1})\, s_{j,m,k,\tilde{j}} \;=\; -2\,\pi\, i\, k_{\tilde{j}}\, s_{j,m,k,\tilde{j}}\,, \tag{3.74}$$

except when $j=0$, $\tilde{j}=1$; this exceptional case will be treated afterwards. We shall show that (3.74) forces $s_{j,m,k,\tilde{j}}=0$, by applying proposition 3.56 with $w=s_{\tilde{j}}$ and $Z=E_{\tilde{j},\tilde{j}+1}$. In this situation $w\mathfrak{n}_-w^{-1}\cap\mathfrak{n}$ is spanned by $E_{\tilde{j},\tilde{j}+1}$ and $w\mathfrak{n}w^{-1}\cap\mathfrak{n}_-$ by $E_{\tilde{j}+1,\tilde{j}}$. Since $[E_{\tilde{j},\tilde{j}+1},E_{\tilde{j}+1,\tilde{j}}]\in\mathfrak{a}$, hypothesis i) of the proposition is satisfied. In view of (3.47), to verify ii), we must show that $(\operatorname{Ad} n_1 - 1)E_{\tilde{j},\tilde{j}+1}$ annihilates $s_{j,m,k,\tilde{j}}$, for every $n_1\in N_{\tilde{j}}(\mathbb{R})$; note that $N_{\tilde{j}}(\mathbb{R})$ plays the role of $wN(\mathbb{R})w^{-1}\cap N(\mathbb{R})$ in the present context. But

$$\big(\operatorname{Ad} N_{\tilde{j}}(\mathbb{R})-1\big)E_{\tilde{j},\tilde{j}+1} \;\subset\; \mathbb{R}\,E_{\tilde{j}-1,\tilde{j}+1} \,\oplus\, \mathbb{R}\,E_{\tilde{j},\tilde{j}+2} \,\oplus\, \mathfrak{n}''\,; \tag{3.75}$$

here $\mathfrak{n}''$ denotes the Lie algebra of $N''(\mathbb{R})$, the second derived subgroup of $N(\mathbb{R})$. Since $\tilde{j}\neq j,j{+}1$, $\big(\operatorname{Ad} N_{\tilde{j}}(\mathbb{R}){-}1\big)E_{\tilde{j},\tilde{j}+1}$ annihilates $\chi^R_{j,m,k}$, and hence, by (3.74), $s_{j,m,k,\tilde{j}}$ – that is the hypothesis ii). Thus $\ell(E_{\tilde{j},\tilde{j}})$ lowers the normal degree of $s_{j,m,k,\tilde{j}}$, contradicting (3.74) unless $s_{j,m,k,\tilde{j}}=0$.

We now turn to the case $j=0$, $\tilde{j}=1$. At the beginning of this proof, we pointed out that the action of the one parameter group $h_1(t)$ – which is generated by $E_{1,2}$ – on $R_{0,1,k}\tau$ drops to an action of $\mathbb{R}/\mathbb{Z}$, and the average of $R_{0,1,k}\tau$ over $\mathbb{R}/\mathbb{Z}$ vanishes (3.59). In view of [13, proposition 7.20], these properties are inherited by $s_{0,1,k,1}$:

$$\ell(h_1(1))\, s_{0,1,k,1} \;=\; s_{0,1,k,1}\,, \qquad \text{and} \qquad \int_{\mathbb{R}/\mathbb{Z}} \ell(h_1(t))\, s_{0,1,k,1}\, dt \;=\; 0\,. \tag{3.76}$$

We now apply proposition 3.56 to the Fourier coefficients of this $(\mathbb{R}/\mathbb{Z})$-action. Since $h_1(t)$ preserves C_1, normalizes $N_1(\mathbb{R})$, and acts on $E_{1,2}$ via a character, not only does $s_{0,1,k,1}$ satisfy the hypothesis ii) of proposition 3.56 with $Z=E_{1,2}$, but its Fourier coefficients also do. We can then argue exactly as before, and conclude that all the Fourier coefficients vanish, except possibly the one corresponding to the trivial character. But (3.76) asserts the vanishing of the "constant" Fourier coefficient, so $s_{0,1,k,1}=0$ in the case $j=0$, $\tilde{j}=1$, as well. □

[5]Specifically (3.2).

Lemma 3.77. *For $0 \leq j \leq n-2$ and all choices of m as well as k, the $R_{j,m,k}\tau$ vanish to infinite order along C_{j+1}. Moreover, this is the case uniformly in m and k, in the sense of* [13].

Proof. We argue by induction on j. Lemma 3.57 already contains this assertion for $j = 0$. Let us suppose then that $j \geq 1$, and that the assertion is correct at the previous step. By induction and lemma 3.28,

$$R_{j-1,m,k}\tau \circ \Phi_j \quad \text{vanishes to infinite order at } \infty\,, \tag{3.78}$$

uniformly in m and k. Both for $j = 1$ and $j > 1$, lemma 2.29 relates the behavior of the $R_{j-1,m,k}\tau$ at $C_j \cong \{x_j = \infty\}$ to that of the $\ell(h_j(k_{j+1}/m))S_{j,m,k}\tau$ at $\{x_j = 0\}$, though with different choices of m and k on the two sides; these choices can be bounded by polynomials in m and $\|k\|$, so uniformity in m and k is not affected. Composing the identities with Φ_j, we conclude that

$$\ell\big(h_j(\tfrac{k_{j+1}}{m})\big)S_{j,m,k}\tau \circ \Phi_j \quad \text{vanishes to infinite order at the origin,} \tag{3.79}$$

uniformly in m and k. Hence by (2.47),

$$\sigma_{j,m,k} \quad \text{vanishes to infinite order at the origin,} \tag{3.80}$$

again uniformly in m and k. Combining this with proposition 2.51, with (3.3), and with the relationship (2.47) between the $\rho_{j,m,k}$ and the $R_{j,m,k}\tau \circ \Phi_{j+1}$, we find that every $R_{j,m,k}\tau \circ \Phi_{j+1}\big|_{\mathbb{R}}$ has a canonical extension across ∞. That is also true in the uniform sense, since (3.3) preserves uniformity; see [13, lemma 7.15]. Hence, by lemma 3.28,

$$R_{j,m,k}\tau \quad \text{has a canonical extension across } C_{j+1}\,, \tag{3.81}$$

uniformly in m and k. It remains to be shown that all the $R_{j,m,k}\tau$ vanish to infinite order along C_{j+1}. We argue by contradiction, and suppose

$$R_{j,m,k}\tau \ \text{ does not vanish to infinite order along } C_{j+1}\,, \tag{3.82}$$

for at least one choice of m and k. We must derive a contradiction from (3.81) and (3.82).

Recall the definition of the locally closed submanifolds C_j^i in lemma 3.24. By downward induction on i, for $0 \leq i \leq j$, we shall show that

$$\begin{aligned} &\text{the } R_{i,m,k}\tau \text{ have canonical extensions across } C_{j+1}^{i+1}, \text{ uniformly in } m,\, k\,, \\ &\text{but for some } m,\, k\,,\ R_{i,m,k}\tau \text{ does not vanish to infinite order on } C_{j+1}^{i+1}\,. \end{aligned} \tag{3.83}$$

For $i = j$, this is the case by (3.81–3.82). Suppose than that $0 \leq i \leq j-1$ and that (3.83) is satisfied for all larger values of i. The actions of the one parameter groups $h_{i+1}(t)$, $h_{i+2}(t)$ on X preserve C_{j+1}^{i+2}, and drop to actions of the compact group $\mathbb{R}/\mathbb{Z}$ on $N(\mathbb{Z})$-invariant objects, such as $P_{i+1,m,k}\tau$. According to (2.40–2.41), the $R_{i+1,m,k}\tau$ and $S_{i+1,m,k}\tau$ are the Fourier coefficients of $P_{i+1,m,k}\tau$ with respect to the two actions. Thus, by [13, lemma 7.2], the fact that the $R_{i+1,m,k}\tau$

have canonical extensions, uniformly in m and k, implies the corresponding assertion about the $P_{i+1,m,k}\tau$, and then [13, Proposition 7.20] allows us to draw the analogous conclusion about the $S_{i+1,m,k}\tau$. Differences between these distributions and their canonical extensions inherit the invariance properties of their "parents". Since the various components of a Fourier expansion are necessarily linearly independent, if some $R_{i+1,m,k}\tau$ fails to vanish to infinite order, that must also be the case for some $S_{i+1,m,k}\tau$. We conclude:

$$\begin{aligned}&\text{the } S_{i+1,m,k}\tau \text{ have canonical extensions across } C^{i+2}_{j+1}, \text{ uniformly in } m,\, k\,,\\ &\text{but for some } m,\, k\,,\ S_{i+1,m,k}\tau \text{ does not vanish to infinite order on } C^{i+2}_{j+1}\,.\end{aligned} \tag{3.84}$$

Translation by elements of $N(\mathbb{R})$ – or, for that matter, by diagonal matrices – does not affect vanishing to infinite order along Schubert cells. Thus (3.84) remains correct for the renormalized quantities $\ell(h_{i+1}(k_{i+2}/m))S_{i+1,m,k}\tau$. In case $i>0$, lemma 2.29 relates these renormalized quantities to the $\ell(h_{i+1}(-k_i/m))R_{i,m,k}\tau$, though with different choices of m and k, via translation by

$$\Phi_{i+1}\begin{pmatrix}0 & -m_{i+2}/m\\ m/m_{i+2} & 0\end{pmatrix} \;=\; \Phi_{i+1}\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\Phi_{i+1}\begin{pmatrix}-m/m_{i+2} & 0\\ 0 & -m_{i+2}/m\end{pmatrix}. \tag{3.85}$$

The preceding statement remains correct even for $i=0$, provided we replace $\ell(h_{i+1}(-k_i/m))R_{i,m,k}\tau$ by $\ell(h_1(-\bar{k}_2 m_2/m))R_{0,m,k}\tau$; to see this, note that

$$\begin{pmatrix}0 & m_2/m\\ -m/m_2 & \bar{k}_2\end{pmatrix}^{-1} \;=\; \begin{pmatrix}1 & -\bar{k}_2 m_2/m\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}0 & -m_2/m\\ m/m_2 & 0\end{pmatrix}. \tag{3.86}$$

The first matrix on the right side of (3.85) is s_{i+1}, as defined in (3.16), and the second lies in the diagonal subgroup. We had just remarked that translation by elements of $N(\mathbb{R})$ or the diagonal subgroup does not affect vanishing to infinite order along Schubert cells. The uniformity is not affected by the different choices of m and k on the two sides, as was remarked earlier, nor by translation by a diagonal matrix with entries that are bounded by multiples of $\|k\|$, and translation by $h_{i+1}(k_{i+2}/m)$ or $h_{i+1}(-k_i/m)$. This establishes (3.83), though so far only with $s_{i+1}C^{i+2}_{j+1}$ in place of C^{i+1}_{j+1}.

If $i>0$, $s_{i+1}C^{i+2}_{j+1}$ is invariant under the one parameter group $h_i(t)$ – not invariant under $h_{i+1}(t)$, however – so for $i>0$ we can argue as we did in the case of the $R_{i+1,m,k}\tau$ and conclude that the $P_{i,m,k}\tau$ have canonical extensions across $s_{i+1}C^{i+2}_{j+1}$, uniformly in m and k as always. But each $P_{i,m,k}\tau$ is $N(\mathbb{Z})$-invariant, and therefore has a canonical extension across all $N(\mathbb{Z})$-translates of $s_{i+1}C^{i+2}_{j+1}$. According to lemma 3.24, the $N(\mathbb{R})$-translates constitute a Zariski open cover of C^{i+1}_{j+1}. We claim that even the $N(\mathbb{Z})$-translates cover C^{i+1}_{j+1}. If that were not true, some non-empty Zariski closed subset of $s_{i+1}C^{i+2}_{j+1}$ would have to be $N(\mathbb{Z})$-invariant and, in view of the Zariski density of $N(\mathbb{Z})$ in $N(\mathbb{R})$, even $N(\mathbb{R})$-invariant. But then the $N(\mathbb{R})$-translates of $s_{i+1}C^{i+2}_{j+1}$ could not cover C^{i+1}_{j+1} – contradiction! Having a canonical extension, whether in a uniform sense or not, is a local property. The $P_{i,m,k}\tau$ therefore have canonical extensions across $C^{i+1}_{j+1}=\cup_{n\in N(\mathbb{Z})}\, n\, s_{i+1}C^{i+2}_{j+1}$, uniformly in m and k. We now can conclude (3.83) by taking Fourier components with respect to the action of $h_i(t)$. If $i = 0$ we need to modify the argument slightly. The $R_{0,1,k}\tau$ themselves are $N(\mathbb{Z})$-invariant, so we can argue as before and conclude that the $R_{0,1,k}\tau$ have canonical extensions across not only $s_1 C^2_{j+1}$, but all of C^1_{j+1}, uniformly in k. That completes the verification of (3.83) for all i, $1 \le i \le j$.

We now apply (3.83) with $i = 0$: there exists some k such that $R_{0,1,k}\tau$ has a canonical extension across C^1_{j+1} but does not vanish there to infinite order. The same must then be true for at least one of the Fourier coefficients of $R_{0,1,k}\tau$ with respect to the action of $\mathbb{R}/\mathbb{Z}$ via $h_1(t)$. In view of (3.59) the non-zero Fourier coefficients are Whittaker distributions, i.e., $N(\mathbb{Z})$-equivariant extensions of characters

$$N(\mathbb{R}) \ni n \;\; \mapsto \;\; e(k\,x) \;\text{ if }\; n = n_{x,y} n'' \;\text{ with }\; n'' \in N''(\mathbb{R}) \quad (\, k \in \mathbb{Z}^{n-1}_{\neq 0}\,) \tag{3.87}$$

to distribution vectors in $C^{-\infty}(X, \mathcal{L}_{\lambda,\delta})$. It is known, of course, that these extend uniquely in an $N(\mathbb{Z})$-equivariant manner [2], as follows readily also from proposition 3.56. We have arrived at the required contradiction to (3.81–3.82). □

Together, lemma 3.28 and lemma 3.77 imply that all the $R_{j,m,k}\tau \circ \Phi_{j+1}$ vanish to infinite order at ∞. Proposition 3.6 follows, as was pointed out earlier.

4. Classical proof of the formula

We begin by recalling some analytic ingredients from our paper [13] that were used in [14], in particular some results from [13, §6] that apply directly to our context as well. Let $\mathcal{S}_{\mathrm{sis}}(\mathbb{R})$ denote the space of all finite linear combinations of the products $(\operatorname{sgn} x)^\eta |x|^\alpha (\log|x|)^j \phi(x)$, where $\eta \in \mathbb{Z}/2\mathbb{Z}$, $\alpha \in \mathbb{C}$, $j \in \mathbb{Z}_{\ge 0}$, and ϕ is an element of the Schwartz space $\mathcal{S}(\mathbb{R})$. By hypothesis, the test function f in theorem 1.10 is an element of this space, as its transform F is also asserted to be by (1.8).

Propositions 2.51 and 3.6 display relations involving Fourier transforms and $x \mapsto x^{-1}$ amongst some of the Fourier components of the automorphic distribution. These relations, chained together, ultimately lead to a distributional identity which is equivalent to the statement of the summation formula in theorem 1.10. For this it is convenient to use the operators

$$T_{\alpha,\eta} \;\; = \;\; \mathcal{F}\left(x \mapsto \phi(x^{-1}) \operatorname{sgn}(x)^\eta |x|^{-\alpha-1}\right), \quad \alpha \in \mathbb{C} \text{ and } \eta \in \mathbb{Z}/2\mathbb{Z}\,, \tag{4.1}$$

which are the subject of [13, §6]. This Fourier integral converges uniformly for Re α sufficiently large when $\phi \in \mathcal{S}(\mathbb{R})$, and extends to an operator which maps $\mathcal{S}_{\mathrm{sis}}(\mathbb{R})$ to itself by [13, Theorem 6.6]. Its adjoint operator given by

$$T^*_{\alpha,\eta}\sigma(x) \;\; = \;\; \operatorname{sgn}(x)^\eta\, |x|^{\alpha-1}\, \widehat{\sigma}(\tfrac{1}{x}) \tag{4.2}$$

is shown there to extend to and preserve the space of tempered distributions which vanish to infinite order at the origin. We also require the slightly more general operators

$$\begin{aligned}\mathcal{T}_{j,a,b} &= \mathcal{F}\left(x \mapsto f(\tfrac{b}{ax})\,\mathrm{sgn}(bx)^{\delta_j+\delta_{j+1}}|a|^{-1}|b|^{\lambda_j-\lambda_{j+1}}|x|^{-\lambda_j+\lambda_{j+1}-1}\right), \\ \mathcal{T}^*_{j,a,b}\sigma(x) &= \mathrm{sgn}(ax)^{\delta_j+\delta_{j+1}}\,|ax|^{\lambda_j-\lambda_{j+1}-1}\,\widehat{\sigma}(\tfrac{b}{ax}) \\ &= \mu_j(ax)\,\widehat{\sigma}(\tfrac{b}{ax})\end{aligned} \tag{4.3}$$

for $j=1,\ldots,n-1$, where μ_j is defined in (3.5). Their analytic properties of course follow from those of $T_{\alpha,\eta}$ and its adjoint by simple rescalings. In particular, $\mathcal{T}_{j,a,b}$ maps $\mathcal{S}_{\mathrm{sis}}(\mathbb{R})$ to itself, and $\mathcal{T}^*_{j,a,b}$ preserves the space of tempered distributions which vanish to infinite order at the origin. Using the relation

$$M_\eta\widehat{\phi}(s) \;=\; (-1)^\eta\,G_\eta(s)\,M_\eta\phi(1-s) \tag{4.4}$$

for an arbitrary element $\phi\in\mathcal{S}(\mathbb{R})$ ([13, (4.58)]), the proof of [13, Lemma 6.19] is trivially modified to show that

$$\begin{aligned}M_\eta(\mathcal{T}_{j,a,b}\phi)(s) &= (-1)^\eta\,\mathrm{sgn}(a)^{\eta+\delta_j+\delta_{j+1}}\,|a|^{s+\lambda_j-\lambda_{j+1}-1}\,\times \\ &\quad\times\,\mathrm{sgn}(b)^\eta\,|b|^{-s}\,G_\eta(s)\,(M_{\eta+\delta_j+\delta_{j+1}}\phi)(s+\lambda_j-\lambda_{j+1}),\end{aligned} \tag{4.5}$$

and consequently

$$\begin{aligned}M_\eta(\mathcal{T}_{1,a_1,b_1}\mathcal{T}_{2,a_2,b_2}\cdots\mathcal{T}_{n-1,a_{n-1},b_{n-1}}\phi)(s) &= (-1)^{(n-1)\eta+(n-1)\delta_1+\delta_n}\,\times \\ \times\left(\prod_{j=1}^{n-1}\mathrm{sgn}(a_j)^{\eta+\delta_1+\delta_{j+1}}|a_j|^{s+\lambda_1-\lambda_{j+1}-1}G_{\eta+\delta_1+\delta_j}(s+\lambda_1-\lambda_j)\right)&\times \\ \times\left(\prod_{j=1}^{n-1}\mathrm{sgn}(b_j)^{\eta+\delta_1+\delta_j}|b|^{-s-\lambda_1+\lambda_j}\right)\,M_{\eta+\delta_1+\delta_n}\phi(s+\lambda_1-\lambda_n)&.\end{aligned} \tag{4.6}$$

To complement the distributions $\sigma_{j,m,k}$ and $\rho_{j,m,k}$ from the previous section, we now introduce some auxiliary distributions related to them. The distribution

$$\tau_R(t) \;=\; \sum_{r\neq 0} c_{c_{n-2},\cdots,c_1,r}\,e(r(t-\tfrac{a}{q}))\,. \tag{4.7}$$

depends on the parameters $c_1,\cdots,c_{n-2}$ and $\frac{a}{q}$, which are fixed in the statement of theorem 1.10; however, the analogous distributions

$$\Delta_{L;k_2,\ldots,k_{n-1},\theta} \;=\; \sum_{r\neq 0} c_{r,k_2,\ldots,k_{n-1}}\,e(r\,\theta)\,\delta_r(t) \tag{4.8}$$

and

$$\tau_{L;k_2,\ldots,k_{n-1},\theta}(t) \;=\; \widehat{\Delta}_{L;k_2,\ldots,k_{n-1},\theta}(t) \;=\; \sum_{r\neq 0} c_{r,k_2,\ldots,k_{n-1}}\,e(r(\theta-t)) \tag{4.9}$$

do depend on different but similar parameters, which are therefore indicated in the notation. The first formula in proposition 3.6 can be restated in terms of this notation as

$$\sigma_{1,m,k} \;=\; \mathcal{T}^*_{1,\frac{m}{m_2},\frac{m_2}{m}}\,\Delta_{L;m_2,k_3,\ldots,k_{n-1},\frac{\overline{k_2}}{m/m_2}}\,. \tag{4.10}$$

Likewise, the second formula in proposition 3.6 relates τ_R to σ_{n-2} by

$$\tau_R \quad = \sum_{\ell \pmod{c_1 q}} e(\tfrac{\ell\,\bar a}{q})\ \mathcal{T}^*_{n-1,q,c_1}\ \sigma_{n-2,qc_1,(c_{n-2},\dots,c_2,0,\ell)}\,, \tag{4.11}$$

as can be seen by relating it to $\rho_{n-2,c_1q,(c_{n-2},\dots,c_2,c_1\bar a,0)}$ and applying proposition 2.51. In this last formula as well as elsewhere in this section, we set the indices k_j of $\sigma_{j,m,k}$ and k_{j+1} of $\rho_{j,m,k}$ to zero, as we may. Finally, the third formula in proposition 3.6 can also be restated in a way similar to (4.10):

$$\begin{aligned}\sigma_{j,m,k} \quad &= \sum_{\ell \pmod{\frac{mk_{j-1}}{m_{j+1}}}} e(\tfrac{\ell\,\overline{k_{j+1}}\,m_{j+1}}{m})\ \times \\ &\qquad \times\ \mathcal{T}^*_{j,\frac{m}{m_{j+1}},k_{j-1}}\ \sigma_{j-1,\frac{mk_{j-1}}{m_{j+1}},(k_1,\dots,k_{j-2},0,\ell,m_{j+1},k_{j+2},\dots,k_{n-1})}\,.\end{aligned} \tag{4.12}$$

Here we use that the $\sigma_{j,m,k}$ all vanish to infinite order at the origin; see proposition 3.6. The distributions $\Delta_{L;\dots}$ obey a stronger property: they vanish identically on the interval $(-1,1)$. Chaining together formulas (4.10-4.12) now allows us to calculate τ_R in terms of the action of the $\mathcal{T}^*_{j,a,b}$ on (4.8). We next parametrize ℓ in (4.11) as $d_1\ell_1$, where d_1 is ranges over divisors of c_1q and ℓ_1 ranges over $(\mathbb{Z}/\frac{qc_1}{d_1}\mathbb{Z})^*$, so that

$$\tau_R \quad = \sum_{d_1|qc_1}\ \sum_{\ell_1\in(\mathbb{Z}/\frac{qc_1}{d_1}\mathbb{Z})^*} e(\tfrac{d_1\ell_1\bar a}{q})\ \mathcal{T}^*_{n-1,q,c_1}\ \sigma_{n-2,qc_1,(c_{n-2},\dots,c_2,0,d_1\ell_1)}. \tag{4.13}$$

Now consider equation (4.12) with $j = n-2$. With this parametrization of ℓ, the quantity m_{n-1} – the GCD of qc_1 and $d_1\ell_1$ – is equal to d_1, and $\overline{k_{n-1}}$ can be taken to be $\overline{\ell_1}$, the modular inverse of ℓ_1 in $(\mathbb{Z}/\frac{qc_1}{d_1}\mathbb{Z})^*$. Using (4.12) for $2 \le h \le n-2$, with $d_{h-1}|\frac{qc_1\cdots c_{h-1}}{d_1\cdots d_{h-2}}$ and $\ell_{h-1} \in (\mathbb{Z}/\frac{qc_1\cdots c_{h-1}}{d_1\cdots d_{h-1}}\mathbb{Z})^*$, we obtain successive relations

$$\begin{aligned}&\sigma_{n-h,\frac{qc_1\cdots c_{h-1}}{d_1\cdots d_{h-2}},(c_{n-2},\dots,c_h,0,d_{h-1}\ell_{h-1},d_{h-2},\dots,d_1)} \quad = \\ &\sum_{d_h|\frac{qc_1\cdots c_h}{d_1\cdots d_{h-1}}}\ \sum_{\ell_h\in(\mathbb{Z}/\frac{qc_1\cdots c_h}{d_1\cdots d_h}\mathbb{Z})^*} e\left(\frac{d_h\ell_h\overline{\ell_{h-1}}}{\frac{qc_1\cdots c_{h-1}}{d_1\cdots d_{h-1}}}\right) \times \\ &\qquad \times\ \mathcal{T}^*_{n-h,\frac{qc_1\cdots c_{h-1}}{d_1\cdots d_{h-1}},c_h}\sigma_{n-h-1,\frac{qc_1\cdots c_h}{d_1\cdots d_{h-1}},(c_{n-2},\dots,c_{h+1},0,d_h\ell_h,d_{h-1},\dots,d_1)}\,.\end{aligned} \tag{4.14}$$

When $h = 2$ the parameters on the left hand side match those on the right hand side of (4.13). Thus τ_R equals both of the following expressions:

$$\begin{aligned}&\sum_{r\neq 0} c_{c_{n-2},\cdots,c_1,r}\, e(-r\tfrac{a}{q})\, e(rt) \quad = \\ &\underbrace{\sum_{d_h|\frac{qc_1\cdots c_h}{d_1\cdots d_{h-1}}}\ \sum_{\ell_h\in(\mathbb{Z}/\frac{qc_1\cdots c_h}{d_1\cdots d_h}\mathbb{Z})^*}}_{\text{for all } h\le n-2} e\left(\frac{d_1\ell_1\bar a}{q} + \sum_{h=2}^{n-2}\frac{d_h\ell_h\overline{\ell_{h-1}}}{\frac{qc_1\cdots c_{h-1}}{d_1\cdots d_{h-1}}}\right) \times \\ &\times\ \mathcal{T}^*_{n-1,q,c_1}\mathcal{T}^*_{n-2,\frac{qc_1}{d_1},c_2}\cdots\mathcal{T}^*_{2,\frac{qc_1\cdots c_{n-3}}{d_1\cdots d_{n-3}},c_{n-2}}\sigma_{1,\frac{qc_1\cdots c_{n-2}}{d_1\cdots d_{n-3}},(0,d_{n-2}\ell_{n-2},d_{n-3},\dots,d_1)}\,,\end{aligned} \tag{4.15}$$

in which both sums involve all d_h and ℓ_h for $1 \le h \le n-2$. By (4.10) the σ_1 term in the last line of this formula is

$$\mathcal{T}^*_{1,\frac{qc_1\cdots c_{n-2}}{d_1\cdots d_{n-2}},\frac{d_1\cdots d_{n-2}}{qc_1\cdots c_{n-2}}}\ \Delta_{L;d_{n-2},d_{n-3},\ldots,d_1,\frac{\overline{\ell_{n-2}}}{\frac{qc_1\cdots c_{n-2}}{d_1\cdots d_{n-2}}}}\,. \tag{4.16}$$

Thus (4.15) remains equal after the following modifications are performed: a sum over $r \in \mathbb{Z}$ is inserted; $\frac{\overline{\ell_{n-2}}r}{\frac{qc_1\cdots c_{n-2}}{d_1\cdots d_{n-2}}}$ is added to the argument of the exponential; $\mathcal{T}^*_{1,\frac{qc_1\cdots c_{n-2}}{d_1\cdots d_{n-2}},\frac{d_1\cdots d_{n-2}}{qc_1\cdots c_{n-2}}}$ is added to the end of the chain of $\mathcal{T}^*$ operators; and the σ_1 term is replaced by $c_{r,d_{n-2},d_{n-3},\ldots,d_1}\delta_r(t)$.

We will now show that equation (4.15) is the distributional equivalent of the summation formula in theorem 1.10, by integrating both sides against the test function $\mathcal{N}\cdot g_1(t)$, where $\mathcal{N}$ is the normalizing factor

$$\mathcal{N} \;=\; \prod_{j\,\le\, n-2} \operatorname{sgn}(c_{n-1-j})^{\delta_1+\cdots\delta_j}|c_{n-1-j}|^{\lambda_1+\cdots+\lambda_j} \tag{4.17}$$

and $g_1 = \mathcal{F}(f(x)|x|^{-\lambda_n}\operatorname{sgn}(x)^{\delta_n})$, in terms of the function f in theorem 1.10. In particular,

$$M_\delta g_1(s) \;=\; (-1)^{\delta}\, G_\delta(s)\,(M_{\delta+\delta_n}f)(1-s-\lambda_n)\,, \tag{4.18}$$

because of (4.4). By our hypothesis that $f \in |x|^{\lambda_n}\operatorname{sgn}(x)^{\delta_n}\mathcal{S}(\mathbb{R})$, g_1 is an arbitrary Schwartz function, and so may be integrated against the periodic – and hence tempered – distribution on the left hand side of (4.15). Taking into account the normalization (2.9), this gives precisely the left hand side of the formula in theorem 1.10.

Now recall the description of the right hand side of (4.15) given after (4.16). The variables ℓ_h occur only in the argument of the exponential, and yield exactly the hyperkloosterman sum $S(r,\bar{a};q,c,d)$. The integration of the right hand side of (4.15) is now equal to

$$\mathcal{N}\ \cdot \sum_{\substack{d_h|\frac{qc_1\cdots c_h}{d_1\cdots d_{h-1}}\\ \text{for all } h\le n-2}}\ \sum_{r\neq 0} S(r,\bar{a};\,q,c,d)\ c_{r,d_{n-2},d_{n-3},\ldots,d_1}\,g_2(r)\,, \tag{4.19}$$

where now

$$g_2 \;=\; \mathcal{T}_{1,\frac{qc_1\cdots c_{n-2}}{d_1\cdots d_{n-2}},\frac{d_1\cdots d_{n-2}}{qc_1\cdots c_{n-2}}}\mathcal{T}_{2,\frac{qc_1\cdots c_{n-3}}{d_1\cdots d_{n-3}},c_{n-2}}\cdots\mathcal{T}_{n-2,\frac{qc_1}{d_1},c_2}\mathcal{T}_{n-1,q,c_1}\,g_1. \tag{4.20}$$

Introducing the quantity $c_{n-1}=1$ for convenience, one can use (4.6) to express the Mellin transform of g_2 as

$$\begin{aligned} M_\delta g_2(s) \;&=\; (-1)^{(n-1)\delta+(n-1)\delta_1+\delta_n}\,(M_{\delta+\delta_1+\delta_n}g_1)(s+\lambda_1-\lambda_n)\,\times \\ &\times\ \operatorname{sgn}(\tfrac{qc_1\cdots c_{n-2}}{d_1\cdots d_{n-2}})^{\delta}\left|\tfrac{qc_1\cdots c_{n-2}}{d_1\cdots d_{n-2}}\right|^{s}\Bigg(\prod_{j=1}^{n-1}\operatorname{sgn}(c_j)^{\delta+\delta_1+\delta_{n-j}}|c_j|^{-s-\lambda_1+\lambda_{n-j}}\,\times \\ &\times\ \operatorname{sgn}(\tfrac{qc_1\cdots c_{j-1}}{d_1\cdots d_{j-1}})^{\delta+\delta_1+\delta_{n-j+1}}\left|\tfrac{qc_1\cdots c_{j-1}}{d_1\cdots d_{j-1}}\right|^{s+\lambda_1-\lambda_{n-j+1}-1}G_{\delta+\delta_1+\delta_j}(s+\lambda_1-\lambda_j)\Bigg)\,. \end{aligned} \tag{4.21}$$

Letting

$$\widetilde{\mathcal{N}} \quad = \quad \prod_{j=1}^{n-2} (\operatorname{sgn} d_{n-1-j})^{\delta_1+\cdots+\delta_{j+1}} \, |d_{n-1-j}|^{\lambda_1+\cdots+\lambda_{j+1}} \,, \tag{4.22}$$

the equality of (4.19) with the right hand side of the formula in theorem 1.10 reduces to the identity

$$\mathcal{N} \cdot g_2(r) \quad = \quad \operatorname{sgn}(r)^{\delta_1} \, |r|^{\lambda_1} \, \widetilde{\mathcal{N}} \, \left| \frac{q}{r d_1 \cdots d_{n-2}} \right| \, F\!\left(\tfrac{r \, d_{n-2}^2 \, d_{n-3}^3 \cdots d_1^{n-1}}{q^n \, c_{n-2} \, c_{n-3}^2 \cdots c_1^{n-2}} \right), \tag{4.23}$$

or the following equivalent relation between Mellin transforms:

$$\mathcal{N} \cdot \frac{|d_1 \cdots d_{n-2}|}{|q|} \, M_\delta g_2(s) \quad = \quad \widetilde{\mathcal{N}} \cdot \operatorname{sgn}\!\left(\tfrac{d_{n-2}^2 \, d_{n-3}^3 \cdots d_1^{n-1}}{q^n \, c_{n-2} \, c_{n-3}^2 \cdots c_1^{n-2}} \right)^{\delta+\delta_1} \left| \tfrac{d_{n-2}^2 \, d_{n-3}^3 \cdots d_1^{n-1}}{q^n \, c_{n-2} \, c_{n-3}^2 \cdots c_1^{n-2}} \right|^{1-s-\lambda_1} M_{\delta+\delta_1} F(s+\lambda_1-1). \tag{4.24}$$

After substituting (4.18) into (4.21), and then into the left hand side of the previous equation, while substituting (1.5) into its right hand side, both sides have identical occurrences of $M_{\delta+\delta_1} f(1-s-\lambda_1)$ and the product $\prod_{j=1}^{n} G_{\delta+\delta_1+\delta_j}(s+\lambda_1-\lambda_j)$. A short computation using (2.2) then verifies that the remaining terms – powers of (-1), $|q|$, $\operatorname{sgn}(q)$, $|c_j|$, $\operatorname{sgn}(c_j)$, $|d_j|$, and $\operatorname{sgn}(d_j)$ – on both sides agree.

5. Adelic proof of the formula

In this section, we give a second, self-contained derivation of the Voronoi formula using adelic automorphic distributions. Since it is a second proof, we will describe only the details of the calculation, and not its rigorous justification (which can be handled using the techniques of the previous sections anyhow).

To begin, we will describe the adelization of the classical automorphic distributions from section 2, considering a $GL(n,\mathbb{Z})$-invariant automorphic distribution that comes from a cuspidal automorphic representation of $GL(n)$ over $\mathbb{A}$, the adele group of $\mathbb{Q}$. This process is formally identical to the usual adelization of classical automorphic functions using strong approximation, though we shall present it via Fourier expansions because of the key role they play later.

Let us first review Whittaker functions for automorphic representations. We use $\psi = \prod \psi_p$ to denote the standard additive character on $\mathbb{Q}\backslash\mathbb{A}$, whose restriction to $\mathbb{R}$ coincides with $e(\cdot)$. It can be used to form the standard character $\psi_N = \prod \psi_{N,p}$ of $N(\mathbb{Q})\backslash N(\mathbb{A})$, by composing ψ with the sum of the entries just above the diagonal. A famous formula of Piatetski-Shapiro and Shalika shows that the smooth vectors can be represented as sums of left-translates of adelic Whittaker functions $W = \prod W_p$, each of which transforms on the left under $N(\mathbb{Q}_p)$ by the character $\psi_{N,p}$. By convention, the Fourier coefficient a_k of such a vector is the renormalized value of the finite part of the adelic Whittaker function on the matrix

$\Delta_k = \mathrm{diag}(k_1 \cdots k_{n-1}, k_2 \cdots k_{n-1}, \ldots, k_{n-1}, 1) \in GL(n,\mathbb{Q})$:

$$W_f(\Delta_k) \;=\; \prod_{p<\infty} W_p(\Delta_k) \;=\; \frac{a_{k_1,\ldots,k_{n-1}}}{\prod_{j=1}^{n-1} |k_j|^{j(n-j)/2}}\,. \tag{5.1}$$

We shall define adelic automorphic distributions by replacing $W = W_\infty W_f$ with a "boundary Whittaker distribution" $B = B_\infty W_f$ according to the following procedure. The distribution $B_\infty \in V_{\lambda,\delta}^{-\infty}$, like the Whittaker function W_∞ it shall replace, will also transform on the left under $N(\mathbb{R})$ according to this character. Up to scaling, it must therefore be equal to this character on $N(\mathbb{R})$, and is completely described as such on the open Schubert cell. We define B_∞ to be its unique extension to an $N(\mathbb{R})$-equivariant distribution in $V_{\lambda,\delta}^{-\infty}$, which was proven to exist in [2]. The motivation for this definition is as follows. Consider the relation (2.9) between the Fourier coefficients of an automorphic form associated to τ, and the Fourier coefficients of (2.8). The latter were just interpreted in terms of $W_f(\Delta_k)$. The product in (2.9) is the reciprocal of the *unnormalized* inducing character (i.e. without including ρ) from (2.1) on the diagonal matrix Δ_k; the presence of ρ accounts for the product in (5.1). Hence $B(\Delta_k) = c_{k_1,\ldots,k_{n-1}}$.

The adelic automorphic distribution $\tau_{\mathbb{A}}$, in analogy to the Piatetski-Shapiro-Shalika Fourier expansion of cusp forms in terms of Whittaker functions, is defined as the sum of $B\left(\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right) g\right)$, where γ runs over all cosets of $N^{(n-1)}(\mathbb{Q})\backslash GL(n-1,\mathbb{Q})$, $N^{(n-1)}$ being the subgroup of unit upper triangular matrices in $GL(n-1)$. When τ is restricted to the factor $GL(n,\mathbb{R}) \hookrightarrow GL(n,\mathbb{A})$, it corresponds to the automorphic distribution from section 2 that embeds elements of $V_{-\lambda,\delta}^{\infty}$, the smooth vectors in the dual principal series representation, to smooth vectors of the automorphic representation. Our assumption of $GL(n,\mathbb{Z})$ invariance forces the p-adic Whittaker functions W_p, for p finite, to be right invariant under the maximal compact subgroup $GL(n,\mathbb{Z}_p)$ of $GL(n,\mathbb{Q}_p)$. For congruence subgroups, one must alter the p-adic Whittaker functions W_p for p dividing the level (this must be done even classically, and corresponds to vector valued automorphic forms or distributions which transform under $GL(n,\mathbb{Z})$ by a matrix action). We shall not pursue this here, except to note that this is a computational obstacle to deriving the Voronoi formula for arbitrary congruence subgroups that in principal can be solved with enough information about ramified Whittaker functions. For notational convenience, we drop the subscript $\mathbb{A}$ from $\tau_{\mathbb{A}}$, since only this object and not τ itself will be used for the remained of this section.

Our proof is based on two different formulas for the following period of the adelic automorphic distribution:

$$V(g) \;=\; \int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})} \tau(ng)\,\overline{\psi_N(n)}\,dn\,, \tag{5.2}$$

where N_1 is the unipotent radical of the standard $(2,1,1,\ldots,1)$ parabolic of $GL(n)$. One easily sees that it has the Fourier expansion

$$V(g) \;=\; \sum_{r\,\in\,\mathbb{Q}^*} B\left(\left(\begin{smallmatrix} r & \\ & I_{n-1}\end{smallmatrix}\right) g\right). \tag{5.3}$$

Just like automorphic forms, the automorphic distribution τ has a contragredient dual automorphic distribution $\widetilde{\tau}$ defined through the map (3.10) and the convention that

$$\widetilde{\tau}(g) \quad = \quad \tau(\tilde{g})\,. \tag{5.4}$$

This convention also serves to define dual Whittaker functions and distributions $\widetilde{W_p}$ and $\widetilde{B}_\infty$ which transform on the left according to the complex conjugate character $\overline{\psi}$ of N, and $\widetilde{\tau}$ has a similar Fourier expansion in terms of $\widetilde{B} = \widetilde{B}_\infty \widetilde{W_p}$. Since the finite Whittaker functions W_p are assumed to be right invariant under $GL(n, \mathbb{Z}_p)$, and B_∞ is a distribution vector for a principal series representation (2.1),

$$\tau,\ \widetilde{\tau},\ V,\ B, \text{ and } \ \widetilde{B} \ \text{ are all right invariant under } N_-(\mathbb{R}) \times K_f\,, \tag{5.5}$$

$K_f = \prod_{p<\infty} GL(n, \mathbb{Z}_p)$ denoting the standard maximal compact subgroup of $GL(n, \mathbb{A}_f)$. We also use the customary notation $\widehat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Z}_p$ and $\widehat{\mathbb{Z}}^* = \prod_{p<\infty} \mathbb{Z}_p^*$.

Proposition 5.6. *Let $\widetilde{V}(g)$ denote the analogous period of $\widetilde{\tau}$, but with ψ instead of ψ^{-1} in the integral (5.2). Then*

$$\widetilde{V}(g) \quad = \quad \int_{\mathbb{A}^{n-2}} V\left(\begin{pmatrix} 0 & 0 & 1 \\ I_{n-2} & 0 & x \\ 0 & 1 & 0 \end{pmatrix} \tilde{g}\right) dx\,.$$

This proposition is formally equivalent to a well-known result in the Rankin-Selberg theory that unfortunately does not seem to be in the literature. For that reason we have chosen to give a proof of it in the appendix.

Our alternate derivation of the formula in theorem 1.10 uses the formula in proposition 5.6 with

$$\begin{aligned} g \quad &= \quad \begin{pmatrix} 1 & -b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \begin{pmatrix} c_1\cdots c_{n-2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1\cdots c_{n-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_2\cdots c_{n-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3\cdots c_{n-2} & 0 & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ \text{and} \quad \tilde{g} \quad &= \quad \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1\cdots c_{n-2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1\cdots c_{n-3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1\cdots c_{n-4} & 0 & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} z^{-1}\,, \end{aligned} \tag{5.7}$$

where $z = c_1 \cdots c_{n-2} I$ is in the center of $GL(n, \mathbb{A})$ (which, in our full level situation, we may tacitly assume τ and $\widetilde{\tau}$ are invariant under). Here the c_j are elements of $\mathbb{Z}_{\neq 0}$, regarded as a subset of the diagonally-embedded $\mathbb{Q}^* \subset \mathbb{A}$, and $b = t + \frac{\alpha}{q} \in \mathbb{A}$, where $t \in \mathbb{R}$ and $\alpha \in \widehat{\mathbb{Z}}^*$ is equal to the integer a modulo q. Note that α is not simply equal to the diagonal embedding of the integer a, but is altered at its prime factors so that it is a unit at each place. Both sides of the expression in the proposition are distributions in $t \in \mathbb{R}$. In our calculation we will restrict $t \neq 0$, though this is only for formal convenience as the distributions can indeed be shown to vanish to infinite order at $t=0$. These distributions are in fact identical those in (4.15), but are instead packaged in a way connected to the Jacquet-Piatetski-Shapiro-Shalika derivation of the standard L-function on $GL(n)$.

Recalling (5.3) and that the definition of $\widetilde{V}$ involves ψ^{-1} instead of ψ, the left hand side is equal to

$$\sum_{r\,\in\,\mathbb{Q}^*} e(rt - r\tfrac{a}{q})\,\widetilde{B}\begin{pmatrix} rc_1\cdots c_{n-2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1\cdots c_{n-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_2\cdots c_{n-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3\cdots c_{n-2} & 0 & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.8}$$

The presence of the minus sign for the finite factor $\frac{a}{q}$ comes from the contribution of the finite places of b, and is due to the fact that ψ is additively-invariant under the diagonal embedding of $\mathbb{Q}$ inside $\mathbb{A}$. By definition, the value of $\widetilde{B}$ on diagonal matrices factors into the contragredient Fourier coefficient (2.9) from section 2 times the value of the unnormalized inducing character. The Fourier coefficient vanishes unless $r\in\mathbb{Z}$, so the above equals

$$\sum_{r\neq 0} e(rt - r\tfrac{a}{q})\,a_{c_{n-2},\ldots,c_1,r}\,|r|^{\lambda_n}\,\mathrm{sgn}(r)^{\delta_n}\prod_{j=1}^{n-2}|c_j|^{\sum_{i\geq n-j}\lambda_i}\,\mathrm{sgn}(c_j)^{\sum_{i\geq n-j}\delta_i}. \tag{5.9}$$

Integrating in $t\in\mathbb{R}$ against the test function

$$\prod_{j=1}^{n-2}|c_j|^{-\sum_{i\geq n-j}\lambda_i}\,\mathrm{sgn}(c_j)^{\sum_{i\geq n-j}\delta_i}\cdot\mathcal{F}\left(f(u)|u|^{-\lambda_n}\,\mathrm{sgn}(u)^{\delta_n}\right)(t) \tag{5.10}$$

gives the left hand side of the formula in theorem 1.10. This is the exact same integration that was performed in the previous section to obtain the left hand side of the Voronoi formula, though in adelic terminology.

Now we examine the right hand side of the formula in proposition 5.6, with the purpose of integrating it against (5.10). Unlike the calculation in the previous section, this does not directly involve the distributions $\sigma_{j,m,k}$ that played a prominent role there, though the mechanics are fundamentally related. The product of the two matrices inside of $V(\cdot)$ and the first factor of $\tilde{g}$ in (5.7) equals

$$\begin{pmatrix} 0 & 0 & 1 \\ I_{n-2} & 0 & x \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ I_{n-2} & 0 & x \\ 0 & 1 & b \end{pmatrix}. \tag{5.11}$$

Let D_c equal to the matrix $\mathrm{diag}(c_1\cdots c_{n-2}, c_1\cdots c_{n-3},\cdots,c_1)\in GL(n-2,\mathbb{Q})$, and x equal the column vector $(x_{n-2},\ldots,x_1)$. Using the fact $b^{-1}=t^{-1}+q\alpha^{-1}$ is in $\mathbb{R}^*\times\widehat{\mathbb{Z}}$, (5.5) implies the integrand in proposition 5.6 equals

$$\begin{aligned} V\left(\begin{pmatrix} 0 & 0 & 1 \\ I_{n-2} & 0 & x \\ 0 & 1 & b \end{pmatrix}\begin{pmatrix} D_c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -b^{-1} & 1 \end{pmatrix}\right) &= V\begin{pmatrix} 0 & -b^{-1} & 1 \\ D_c & -b^{-1}x & x \\ 0 & 0 & b \end{pmatrix} \\ &= V\left(\begin{pmatrix} 1 & 0 & b^{-1} \\ 0 & I_{n-1} & b^{-1}x \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -b^{-1} & 0 \\ D_c & -b^{-1}x & 0 \\ 0 & 0 & b \end{pmatrix}\right) \qquad (5.12) \\ &= \psi(b^{-1}x_1)\,V\begin{pmatrix} 0 & -b^{-1} & 0 \\ D_c & -b^{-1}x & 0 \\ 0 & 0 & b \end{pmatrix}. \end{aligned}$$

We change variables $x \mapsto bD_c x = (t + \frac{\alpha}{q})D_c x$ in that integral, which has the overall effect of changing the measure by $|tq|^{n-2}$ because D_c is rational (the valuation here always refers to the archimedean one unless specified otherwise). Thus the integral equals

$$|tq|^{n-2} \int_{\mathbb{A}^{n-2}} \psi(c_1 x_1)\, V\left(\begin{pmatrix} b^{-1} & 0 & 0 \\ 0 & D_c & 0 \\ 0 & 0 & b \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ I_{n-2} & -x & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) dx\,. \tag{5.13}$$

In effect, this last matrix step is the only tool needed to compute the value of V and, as we now demonstrate, also reduces the range of integration.

Proposition 5.14. *Let $\mathcal{X}$ denote the subset of $(x_1, \ldots, x_{n-2}) \in \mathbb{A}^{n-2}$ with successively nondecreasing p-adic valuations at each prime, that is*

$$\mathcal{X} \ : \quad |x_{n-2}|_p \quad \geq \quad |x_{n-3}|_p \quad \geq \quad \cdots \quad \geq |x_2|_p \quad \geq \quad |x_1|_p \quad \geq \quad 1\,. \tag{5.15}$$

Then (5.13) is equal to

$$|tq|^{n-2} \int_{\mathcal{X}} \sum_{r \in \mathbb{Q}^*} \psi\left(c_1 x_1 + \sum_{j=2}^{n-2} \tfrac{c_j x_j}{x_{j-1}} + \tfrac{r}{b x_{n-2}} \right) B(\Delta)\, dx\,, \tag{5.16}$$

where Δ is the diagonal matrix

$$\Delta \quad = \quad \operatorname{diag}(-r\tfrac{c_1 \cdots c_{n-2}}{b x_{n-2}}, -\tfrac{c_1 \cdots c_{n-2} x_{n-2}}{x_{n-3}}, -\tfrac{c_1 \cdots c_{n-3} x_{n-3}}{x_{n-4}}, \ldots, -\tfrac{c_1 c_2 x_2}{x_1}, -c_1 x_1, b). \tag{5.17}$$

Proof. By (5.5), V is invariant under right translation by the matrix $\left(\begin{smallmatrix} I_{n-2} & v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ for $v \in \widehat{\mathbb{Z}}^{n-2}$, which has the effect of translating x to $x - v$. Thus the integration over $\mathbb{A}^{n-2}$ reduces to the quotient $(\mathbb{A}/\widehat{\mathbb{Z}})^{n-2}$. By ignoring the set where all $(x_i)_{\mathbb{R}}$ vanish and appealing to strong approximation, we may chose coset representatives $x = (x_{n-2}, \ldots, x_1)$ such that each $x_i^{-1} \in \mathbb{R}^* \times \widehat{\mathbb{Z}}$. This condition for x_1 along with the matrix identity

$$\begin{pmatrix} & & & -1 \\ 1 & & & -x_{n-2} \\ & \ddots & & \vdots \\ & & 1 & -x_2 \\ & & 1 & -x_1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & \frac{1}{x_1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & -1 \\ & \ddots & & -x_{n-2} \\ & & 1 & \vdots \\ & & 1 & -x_2 \\ & & & -x_1 \end{pmatrix} \cdot \begin{pmatrix} & & & -\frac{1}{x_1} \\ 1 & & & -\frac{x_{n-2}}{x_1} \\ & \ddots & & \vdots \\ & & 1 & -\frac{x_2}{x_1} \\ & & & 1 \end{pmatrix} \tag{5.18}$$

allows us to replace the last matrix in (5.13) by these last two, regarded as embedded into the upper left $(n-1) \times (n-1)$ block of $GL(n)$. The first of these matrices may be passed through to the left, and causes V to transform by $\psi(\frac{c_2 x_2}{x_1})$. The last matrix now has the same form as the original one, but of size one dimension smaller. Right translation by an integer matrix as before, along with the structure of the character, shows that this integral is unchanged if $x_2 \mapsto x_2 + z x_1$, for any $z \in \widehat{\mathbb{Z}}$. Thus we may again assume that $\frac{x_1}{x_2}$ is a p-adic unit at each place. Continuing this way, we reduce the integration to the domain $\mathcal{X}$. To summarize, we have thus factored $\begin{pmatrix} & -1 \\ I_{n-2} & x \end{pmatrix}$ as

$$AU = \begin{pmatrix} -\frac{1}{x_{n-2}} & -\frac{1}{x_{n-3}} & -\frac{1}{x_{n-4}} & \cdots & -\frac{1}{x_1} & -1 \\ & -\frac{x_{n-2}}{x_{n-3}} & -\frac{x_{n-2}}{x_{n-4}} & \cdots & -\frac{x_{n-2}}{x_1} & -x_{n-2} \\ & & -\frac{x_{n-3}}{x_{n-4}} & \cdots & -\frac{x_{n-3}}{x_1} & -x_{n-3} \\ & & & \ddots & \vdots & \vdots \\ & & & & -\frac{x_2}{x_1} & -x_2 \\ & & & & & -x_1 \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ -\frac{x_{n-3}}{x_{n-2}} & 1 & & & & \\ & -\frac{x_{n-4}}{x_{n-3}} & \ddots & & & \\ & & \ddots & & & \\ & & & -\frac{x_1}{x_2} & 1 & \\ & & & & -\frac{1}{x_1} & 1 \end{pmatrix}. \tag{5.19}$$

The matrix U lies in $N_-(\mathbb{R}) \times GL(n-1,\widehat{\mathbb{Z}})$ in the region $\mathcal{X}$, and in this situation the last matrix in (5.13) can be replaced by simply $\left(\begin{smallmatrix} A & \\ & 1 \end{smallmatrix}\right)$, for V is right-invariant under such a matrix U. Therefore (5.13) can be written as

$$|tq|^{n-2} \int_{\mathcal{X}} \psi(c_1x_1)\, V\left(\begin{pmatrix} b^{-1} & 0 & 0 \\ 0 & D_c & 0 \\ 0 & 0 & b \end{pmatrix} \begin{pmatrix} A & \\ & 1 \end{pmatrix}\right) dx\,, \tag{5.20}$$

which equals

$$|tq|^{n-2} \int_{\mathcal{X}} \psi(c_1x_1) \sum_{r\,\in\,\mathbb{Q}^*} B\left(\begin{pmatrix} rb^{-1} & 0 & 0 \\ 0 & D_c & 0 \\ 0 & 0 & b \end{pmatrix} \begin{pmatrix} A & \\ & 1 \end{pmatrix}\right) \tag{5.21}$$

by (5.3). To obtain the expression (5.16) we change the index of summation r to $rc_1\cdots c_{n-2}$, and commute the upper triangular matrix A across to the left so that it transforms out according the character ψ of N on the left of B. □

The value of $B(\Delta)$ factors into an archimedean factor (which is the value of the inducing character on this matrix), times local Whittaker factors. Since these local Whittaker functions are unramified, their value is determined by the p-adic valuations on the simple roots, and vanish unless these are all ≤ 1:

$$|\tfrac{qrx_{n-3}}{x_{n-2}^2}|_p\,,\ |\tfrac{c_{n-2}x_{n-2}x_{n-4}}{x_{n-3}^2}|_p\,,\ |\tfrac{c_{n-3}x_{n-3}x_{n-5}}{x_{n-4}^2}|_p\,,\ldots,\ |\tfrac{c_3x_3x_1}{x_2^2}|_p\,,\ |\tfrac{c_2x_2}{x_1^2}|_p\,,\ |c_1x_1q|_p \leq 1. \tag{5.22}$$

Let us first consider the last of these inequalities, recalling that $|x_1|_p \leq 1$. It constrains d_1x_1 to be a p-adic unit at all places, for some divisor d_1 of $qc_1 \in \mathbb{Z}$. Since $d_1 \mapsto \frac{qc_1}{d_1}$ is a bijection of such divisors, we may instead write $|x_1|_p = |\frac{d_1}{qc_1}|_p$, for $d_1|qc_1$. Likewise, the constraints $|x_1|_p \leq |x_2|_p \leq |\frac{x_1^2}{c_2}|_p$ mean that $1 \leq |\frac{x_2}{x_1}|_p \leq |\frac{x_1}{c_2}|_p = |\frac{d_1}{qc_1c_2}|_p$. Thus, $|\frac{x_2}{x_1}|_p$ equals $|\frac{d_2}{qc_1c_2/d_1}|_p$ for some $d_2|\frac{qc_1c_2}{d_1}$, i.e. $|x_2|_p = |\frac{d_1^2d_2}{q^2c_1^2c_2}|_p$. Continuing, we see that the range of integration $\mathcal{X}$ in (5.16) can be broken up as the disjoint union over

$$\begin{array}{c} d_1\,|\,qc_1 \\ d_2\,|\,\frac{qc_1c_2}{d_1} \\ d_3\,|\,\frac{qc_1c_2c_3}{d_1d_2} \\ \vdots \\ d_{n-2}\,|\,\frac{qc_1\cdots c_{n-2}}{d_1\cdots d_{n-3}} \end{array} \tag{5.23}$$

of

$$\begin{aligned}\Big\{\, |x_1|_p \;=\; |\tfrac{d_1}{qc_1}|_p\ ,\ \ |\tfrac{x_j}{x_{j-1}}|_p \;=\; |\tfrac{d_1\cdots d_j}{qc_1\cdots c_j}|_p \quad \text{for}\ \ j\,\geq\,2\Big\} \;&= \\ =\;\Big\{\, |x_1|_p \;=\; |\tfrac{d_1}{qc_1}|_p\ ,\ \ |x_j|_p \;=\; |\tfrac{d_1^j d_2^{j-1}\cdots d_j}{q^j c_1^j c_2^{j-1}\cdots c_j}|_p \quad \text{for}\ \ j\,\geq\,2\Big\}&. \end{aligned} \tag{5.24}$$

The divisors in (5.23) are precisely the ones occurring on the right hand side of the formula in theorem 1.10, so we are reduced to showing that (5.16) – when integrated over (5.24) instead of $\mathcal{X}$ – corresponds to the rest of the right hand side of that formula. The first constraint in (5.22) governs the integrality of r, namely that on the piece (5.24) one has that r times $\frac{q^n c_1^{n-1} c_2^{n-2}\cdots c_{n-2}^2}{d_1^{n-1} d_2^{n-2}\cdots d_{n-2}^2}$ is an integer. When r is divided by that quantity, the sum over it in (5.16) becomes a sum over nonzero integers, corresponding to the one on the right hand side in theorem 1.10. We simultaneously change variables

$$x_1 \;\mapsto\; \tfrac{d_1}{qc_1}\,x_1\ ,\quad x_j \;\mapsto\; \tfrac{d_1^j d_2^{j-1}\cdots d_j}{q^j c_1^j c_2^{j-1}\cdots c_j}\,x_j \quad \text{for}\ \ j\,\geq\,2\,, \tag{5.25}$$

which incurs no change of measure factor because these ratios are rational numbers. This converts the domain (5.24) to $\{x_j \in \mathbb{R}\times\widehat{\mathbb{Z}}^*\}$, i.e. adeles which are p-adic units at each finite place. Then the relevant contribution of (5.16) is

$$\begin{aligned}&|tq|^{n-2}\int_{(\mathbb{R}\times\widehat{\mathbb{Z}}^*)^{n-2}} \psi\Big(\tfrac{x_1 d_1}{q} + \sum_{j\,=\,2}^{n-2} \tfrac{d_j x_j x_{j-1}^{-1}}{\frac{qc_1\cdots c_{j-1}}{d_1\cdots d_{j-1}}} + \tfrac{r}{b}\,\tfrac{d_1 d_2\cdots d_{n-2}}{q^2 c_1 c_2\cdots c_{n-2}}\,x_{n-2}^{-1}\Big)\times \\ &B(\mathrm{diag}(-\tfrac{r}{b}\,\tfrac{d_1 d_2\cdots d_{n-2}}{q^2 x_{n-2}}, -\tfrac{d_1\cdots d_{n-2}}{q}\,\tfrac{x_{n-2}}{x_{n-3}}, -\tfrac{d_1\cdots d_{n-3}}{q}\,\tfrac{x_{n-3}}{x_{n-4}},\ldots, -\tfrac{d_1 d_2 x_2}{q x_1}, -\tfrac{d_1 x_1}{q}, b)))dx. \end{aligned} \tag{5.26}$$

We next modify the signs of r and the x_j so that the signs of x_1, $\frac{x_2}{x_3},\ldots,\frac{x_{n-2}}{x_{n-3}}$, and $\frac{r}{x_{n-2}}$ are all flipped. Thus we replace (5.26) with

$$\begin{aligned}&|tq|^{n-2}\int_{(\mathbb{R}\times\widehat{\mathbb{Z}}^*)^{n-2}} \psi\Big(-\tfrac{x_1 d_1}{q} - \sum_{j\,=\,2}^{n-2} \tfrac{d_j x_j x_{j-1}^{-1}}{\frac{qc_1\cdots c_{j-1}}{d_1\cdots d_{j-1}}} - \tfrac{r}{b}\,\tfrac{d_1 d_2\cdots d_{n-2}}{q^2 c_1 c_2\cdots c_{n-2}}\,x_{n-2}^{-1}\Big)\times \\ &\quad\times\ B(\mathrm{diag}(\tfrac{r}{b}\,\tfrac{d_1 d_2\cdots d_{n-2}}{q^2 x_{n-2}}, \tfrac{d_1\cdots d_{n-2}}{q}\,\tfrac{x_{n-2}}{x_{n-3}}, \tfrac{d_1\cdots d_{n-3}}{q}\,\tfrac{x_{n-3}}{x_{n-4}},\cdots, \tfrac{d_1 d_2 x_2}{q x_1}, \tfrac{d_1 x_1}{q}, b))). \end{aligned} \tag{5.27}$$

We now observe that both ψ and B in (5.27) split into an archimedean factor (which is a distribution in $t \in \mathbb{R}$, the archimedean part of b), and a nonarchimedean factor (in which we similarly replace b by $\frac{\alpha}{q}$). In the latter, the value of B is precisely equal to

$$|r|^{-(n-1)/2}\,\left(\prod_{j\,=\,1}^{n-2} |d_j|^{-j(n-j)/2}\right)\ a_{r,d_{n-2},\ldots,d_{n-1}}\,, \tag{5.28}$$

because α and the x_i are all p-adic units.

An arbitrary element of $\mathbb{A}_f$ has the form $x_f + z$, where x_f is the finite part of a rational number x and $z \in \widehat{\mathbb{Z}}$; the value of ψ on such an adele is equal to $e(-x)$.

Thus if an element of $\frac{qc_1\cdots c_j}{d_1\cdots d_j}\widehat{\mathbb{Z}}$ is added to x_j, the character ψ is unchanged, and the $(\widehat{\mathbb{Z}}^*)^{n-2}$ part of the integral in (5.27) breaks up into a sum over cosets: it equals the hyperkloosterman sum in theorem 1.10, divided by

$$\left|\prod_{j\le n-2}\frac{qc_1\cdots c_j}{d_1\cdots d_j}\right| \;=\; \left|\frac{q^{n-2}c_1^{n-2}c_2^{n-3}\cdots c_{n-2}}{d_1^{n-2}d_2^{n-3}\cdots d_{n-2}}\right|, \tag{5.29}$$

which is the adelic measure (i.e. index) of this set that it is trivial on within $\widehat{\mathbb{Z}}^{n-2}$.

At this point the calculation has shifted from adeles to reals, and essentially repeats the final steps of the calculation in the previous section. The archimedean part (5.27), including the factor of $|tq|^{n-2}$, is a distribution in $t\in\mathbb{R}$ which is integrated against the function (5.10). In order to symmetrize the expression which follows, we now relabel t, the archimedean part of b, as x_{n-1}, and the variable u from (5.10) as x_n. This contribution is equal to the product of powers of $|c_j|$ and $\operatorname{sgn}(c_j)$ there times

$$\begin{aligned}
&\left(\prod_{j=1}^{n-2}|d_j|^{\sum_{i\le n-j}(\frac{n+1}{2}-i-\lambda_i)}\operatorname{sgn}(d_j)^{\sum_{i\le n-j}\delta_i}\right)\times\\
&\times\int_{\mathbb{R}^n} e\Big(-\tfrac{x_1d_1}{q}-\sum_{j=2}^{n-2}\frac{d_jx_jx_{j-1}^{-1}}{\frac{qc_1\cdots c_{j-1}}{d_1\cdots d_{j-1}}}-\frac{r}{x_{n-1}}\frac{d_1d_2\cdots d_{n-2}}{q^2c_1c_2\cdots c_{n-2}}x_{n-2}^{-1}-x_{n-1}x_n\Big)\times\\
&\times f(x_n)\,|x_n|^{-\lambda_n}\operatorname{sgn}(x_n)^{\delta_n}\,|r|^{\frac{n-1}{2}-\lambda_1}\operatorname{sgn}(r)^{\delta_1}\,|qx_{n-1}|^{\lambda_1-\lambda_n-1}\times\\
&\times\operatorname{sgn}(qx_{n-1})^{\delta_1+\delta_n}\left(\prod_{j=1}^{n-2}|x_j|^{\lambda_{n-j-1}-\lambda_{n-j}-1}\operatorname{sgn}(x_j)^{\delta_{n-j-1}+\delta_{n-j}}\right)dx_1\cdots dx_n\,.
\end{aligned}\tag{5.30}$$

To symmetrize the argument of the exponential, we change $x_{n-1}\mapsto x_{n-1}^{-1}$; this changes its occurrence outside the exponential to $|x_{n-1}|^{-\lambda_1+\lambda_n-1}\operatorname{sgn}(x_{n-1})^{\delta_1+\delta_n}$. Each term in the exponential, aside from the first, is now a ratio of successive variables $\frac{x_j}{x_{j-1}}$. Changing variables $x_j\mapsto x_1\cdots x_j$ converts these ratios each to x_j, and the expression (5.30) as a whole to

$$\begin{aligned}
&|q|^{\lambda_1-\lambda_n-1}\operatorname{sgn}(q)^{\delta_1+\delta_n}\left(\prod_{j=1}^{n-2}|d_j|^{\sum_{i\le n-j}\frac{n+1}{2}-i-\lambda_i}\operatorname{sgn}(d_j)^{\sum_{i\le n-j}\delta_i}\right)|r|^{\frac{n-1}{2}-\lambda_1}\times\\
&\times\operatorname{sgn}(r)^{\delta_1}\int_{\mathbb{R}^n}e\Big(-\tfrac{x_1d_1}{q}-\sum_{j=2}^{n-2}\frac{d_jx_j}{\frac{qc_1\cdots c_{j-1}}{d_1\cdots d_{j-1}}}-r\frac{d_1d_2\cdots d_{n-2}}{q^2c_1c_2\cdots c_{n-2}}x_{n-1}-x_n\Big)\times\\
&\qquad\times f(x_1\cdots x_n)\,|x_n|^{-\lambda_n}\operatorname{sgn}(x_n)^{\delta_n}\,|x_{n-1}|^{-\lambda_1}\operatorname{sgn}(x_{n-1})^{\delta_1}\times\\
&\qquad\qquad\times\left(\prod_{j=1}^{n-2}|x_j|^{-\lambda_{n-j}}\operatorname{sgn}(x_j)^{\delta_{n-j}}\right)dx_1\cdots dx_n\,.
\end{aligned}\tag{5.31}$$

We now come to the final change of variables, which maps $x_j \mapsto \frac{qc_1\cdots c_{j-1}}{d_1\cdots d_j}x_j$ for $j \leq n-2$, and $x_{n-1} \mapsto \frac{q^2c_1\cdots c_{n-2}}{rd_1\cdots d_{n-2}}x_{n-1}$. The argument of f is divided by the product of these ratios, $\frac{rd_1^{n-1}d_2^{n-2}\cdots d_{n-2}^2}{q^nc_1^{n-2}c_2^{n-3}\cdots c_{n-2}}$, and so this integral is therefore by (1.3) equal to the same instance of $F(\frac{rd_1^{n-1}d_2^{n-2}\cdots d_{n-2}^2}{q^nc_1^{n-2}c_2^{n-3}\cdots c_{n-2}})$ occuring in theorem 1.10, times the change of variables factor

$$\left(\prod_{j=1}^{n-2} \left|\tfrac{qc_1\cdots c_{j-1}}{d_1\cdots d_j}\right|^{1-\lambda_{n-j}} \operatorname{sgn}(\tfrac{qc_1\cdots c_{j-1}}{d_1\cdots d_j})^{\delta_{n-j}}\right) \left|\tfrac{q^2c_1\cdots c_{n-2}}{rd_1\cdots d_{n-2}}\right|^{1-\lambda_1} \operatorname{sgn}(\tfrac{q^2c_1\cdots c_{n-2}}{rd_1\cdots d_{n-2}})^{\delta_1}. \tag{5.32}$$

To finish, we multiply the products of $|q|$, $\operatorname{sgn}(q)$, $|r|$, $\operatorname{sgn}(r)$, $|c_j|$, $\operatorname{sgn}(c_j)$, $|d_j|$, and $\operatorname{sgn}(d_j)$ from (5.10), (5.28), (5.30), and (5.32), and divide by (5.29). Using (2.2), this indeed results in $\frac{|q|}{|rd_1\cdots d_{n-2}|}$, verifying the right hand side of the formula in theorem 1.10.

A. Appendix: Proof of Proposition 5.6

In this appendix we prove proposition 5.6, and more generally a crucial but unpublished ingredient in the work of Jacquet, Piatetski-Shapiro, and Shalika on the tensor product L-functions on $GL(n_1) \times GL(n_2)$ for $|n_1 - n_2| > 1$. Despite not appearing in their papers, it is both the mechanism by which their functional equation is established, and the source for the local integrals which they study in detail. We therefore felt it worthwhile to present a proof here.

The more general statement concerns a smooth integrable function ϕ on $GL(n,\mathbb{Q})\backslash GL(n,\mathbb{A})$, and thus does not involve a level assumption. By convention, we define the dual form $\tilde{\phi}$ by the formula $\widetilde{\phi}(g) = \phi(\tilde{g})$ (see (5.4)).

Proposition A.1. *Fix $1 < m < n$ and let N° denote the unipotent radical of the $(m,1,1,\ldots,1)$ standard parabolic subgroup of $GL(n)$. Define periods*

$$\begin{aligned} V(g) &= \int_{N^\circ(\mathbb{Q})\backslash N^\circ(\mathbb{A})} \phi(ng)\,\overline{\psi_N(n)}\,dn \\ \textit{and}\quad \widetilde{V}(g) &= \int_{N^\circ(\mathbb{Q})\backslash N^\circ(\mathbb{A})} \tilde{\phi}(ng)\,\psi_N(n)\,dn \\ &= \int_{\widetilde{N^\circ}(\mathbb{Q})\backslash \widetilde{N^\circ}(\mathbb{A})} \phi(n\tilde{g})\,\overline{\psi_N(n)}\,dn\,, \end{aligned}$$

where $\widetilde{N^\circ}$ is the unipotent radical of the $(1,1,\ldots,1,m)$ standard parabolic subgroup of $GL(n)$. Then

$$\widetilde{V}(g) = \int_{M_{n-m,m-1}(\mathbb{A})} V\left(\begin{pmatrix} I_{m-1} & & \\ X & I_{n-m} & \\ & & 1 \end{pmatrix} w\,\tilde{g}\right) dX\,,$$

in which the integral converges absolutely, $M_{k,\ell}$ denotes $k \times \ell$ matrices, and w denotes the permutation matrix $\begin{pmatrix} & I_{m-1} \\ I_{n-m+1} & \end{pmatrix}$.

Proposition 5.6 corresponds to the case $m=2$, but for distributions instead of smooth forms. However, the distributional version stated there is equivalent to the one stated for all cusp forms here, because automorphic distributions are equivalently linear functionals which control their embeddings into spaces of smooth functions. The periods V defined here have Fourier expansions as sums of Whittaker functions, left translated by elements of $GL(m-1,\mathbb{Q})$ embedded into the upper left corner of $GL(n,\mathbb{A})$. Together with the relation above, this implies the unfolding of Jacquet-Piatetski-Shapiro-Shalika's integral representation, as quoted in [3, Theorem 2.1], for example.

Before giving the proof, it is helpful to describe some relevant aspects of integration over quotients of nilpotent groups. Suppose that $G = G_1 \ltimes G_2$ is the semidirect product of locally compact Hausdorff groups such that G_2 is abelian, and that the continuous action ρ of G_1 on G_2 preserves its Haar measure:

$$(g_1,g_2)\cdot(h_1,h_2) \quad = \quad (g_1h_1,\, g_2+\rho(g_1)h_2)\,. \qquad \text{(A.2)}$$

Under this assumption, the left Haar measure on G is the product measure of the left Haar measure on G_1 with the Haar measure on G_2. The group G is furthermore unimodular if G_1 is.

We further suppose each G_i has a discrete subgroup Γ_i, with the semidirect product $\Gamma = \Gamma_1 \ltimes \Gamma_2$ itself discrete in G. In particular, for each $\gamma_1 \in \Gamma_1$, $\rho(\gamma_1)$ acts bijectively on Γ_2. Recall that a fundamental domain for a discrete subgroup Δ of a topological group H is an open set $S \subset H$ whose left Δ-translates are disjoint yet dense in H. For any $h \in H$, Sh is also a fundamental domain for Δ, but hS is instead a fundamental domain for $h\Delta h^{-1}$. If S_2 is a fundamental domain for $\Gamma_2 \subset G_2$, $\rho(\gamma_1)S_2$ is also a fundamental domain for each $\gamma_1 \in \Gamma_1$, because Γ_1 normalizes Γ_2. It follows from this that if S_1 is a fundamental domain for $\Gamma_1 \subset G_1$, then $S = S_1 \times S_2$ is a fundamental domain for $\Gamma \subset G$. More generally, if $f : S_1 \to G_2$ is continuous, then $\{(s_1, f(s_1)+\rho(s_1)s_2) \mid s_1 \in S_1,\, s_2 \in S_2\}$ is also a fundamental domain for $\Gamma\backslash G$.

We now apply the above considerations to an iterated semidirect product. Namely, suppose that U is any unipotent radical of a parabolic subgroup of $GL(n)$, having dimension d. Its adelic points $U(\mathbb{A})$ can be identified, setwise, with d-tuples $(u_1,\ldots,u_d) \in \mathbb{A}^d$, and its Haar measure is the product of Haar measures $du_1 \cdots du_d$ from each copy. A fundamental domain for $U(\mathbb{Q})\backslash U(\mathbb{A})$ is given by any product of fundamental domains for each $u_i \in \mathbb{Q}\backslash\mathbb{A}$, one for each copy. We always normalize our Haar measures to give volume 1 to $(\mathbb{Q}\backslash\mathbb{A})^d$ under this identification. Because U is an iterated semidirect product of abelian groups, the remark at the end of the previous paragraph implies the following by induction:

Proposition A.3. *Let U be the unipotent radical of a parabolic subgroup of $GL(n)$. Then left translation by elements of $U(\mathbb{A})$ maps any fundamental domain for $U(\mathbb{Q})\backslash U(\mathbb{A})$ into another.*

Proof of proposition A.1. Let N' denote the subgroup $w\widetilde{N^\circ}w^{-1}$, so that

$$\widetilde{V}(g) \quad = \quad \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} \phi(nw\tilde{g})\,\overline{\psi_N(n)}\,dn\,. \qquad \text{(A.4)}$$

This reduces the proposition to showing

$$\begin{aligned}\int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} \phi(ng)\,\overline{\psi_N(n)}\,dn \quad &= \\ \int_{M_{n-m,m-1}(\mathbb{A})} \int_{N^\circ(\mathbb{Q})\backslash N^\circ(\mathbb{A})} &\phi\left(n\begin{pmatrix} I_{m-1} & & \\ X & I_{n-m} & \\ & & 1\end{pmatrix} g\right)\overline{\psi_N(n)}\,dn\,dX \end{aligned} \tag{A.5}$$

for an arbitrary $g \in GL(n,\mathbb{A})$. Let U denote the subgroup of unit upper triangular matrices in $GL(n-m+1)$. The character ψ_N for $GL(n-m+1)$ makes sense on $U(\mathbb{A})$, and agrees with its $GL(n)$ variant when U is embedded into the lower right corner; as no confusion will arise, we shall use the notation ψ_N for either sized matrix. Using the notation $M_{k,\ell}(\mathbb{Q}\backslash\mathbb{A})$ as shorthand for a fundamental domain for the quotient $M_{k,\ell}(\mathbb{Q})\backslash M_{k,\ell}(\mathbb{A})$, we define the following integrals for $0 \le j \le n-m$:

$$\begin{aligned} I_j \quad = \quad \int_{\substack{y\,\in\, M_{m-1,j}(\mathbb{Q}\backslash\mathbb{A}) \\ X_1\,\in\, M_{n-m-j,m-1}(\mathbb{Q}\backslash\mathbb{A}) \\ X_2\,\in\, M_{j,m-1}(\mathbb{A}) \\ u\,\in\, U(\mathbb{Q})\backslash U(\mathbb{A})}} &\phi\left(\begin{pmatrix} I_{m-1} & & y \\ & I_{n+1-m-j} & \\ & & I_j\end{pmatrix}\begin{pmatrix} I_{m-1} & \\ X_1 & \\ X_2 & u \\ 0 & \end{pmatrix} g\right) \times \\ &\times\ \overline{\psi_N(u)}\,du\,dX_2\,dX_1\,dy\,. \end{aligned} \tag{A.6}$$

The y integration is clearly well defined on the quotient. We now argue that the X_1 and u integrations are as well. To simplify notation, let us temporarily write $Y = [0\ y] \in M_{m-1,n+1-m}$ and $X = \left[\begin{smallmatrix} X_1 \\ X_2 \\ 0\end{smallmatrix}\right] \in M_{n+1-m,m-1}$, so that the argument of ϕ is $\left(\begin{smallmatrix} I_{m-1} & Y \\ & I_{n+1-m}\end{smallmatrix}\right)\left(\begin{smallmatrix} I_{m-1} & \\ X & u\end{smallmatrix}\right) g$. Suppose first that u is replaced by γu, for $\gamma \in U(\mathbb{Q})$. Since ϕ is left invariant under $\left(\begin{smallmatrix} I_{m-1} & \\ & \gamma^{-1}\end{smallmatrix}\right)$, the argument of ϕ can be replaced by $\left(\begin{smallmatrix} I_{m-1} & Y\gamma \\ & I_{n+1-m}\end{smallmatrix}\right)\left(\begin{smallmatrix} I_{m-1} & \\ \gamma^{-1}X & u\end{smallmatrix}\right)$. Since γ is unipotent upper triangular, a column of $Y\gamma$ is formed its counterpart in Y by adding multiples of preceding columns to it. The reverse change of variables $y \mapsto y\gamma^{-1}$ preserves the subgroup $M_{m-1,j}(\mathbb{Q})$ and the measure dy. Likewise, the variables X_1 and X_2 can be changed to convert the expression back to I_j. Thus the u integration is well defined.

To show that the X_1 integration is well defined, suppose $Q \in M_{n-m+1,m-1}(\mathbb{Q})$ has zero entries in its bottom $j+1$ rows, so that its nonzero entries correspond to positions in X_1. Adding Q to X likewise has the effect of replacing the matrix $\left(\begin{smallmatrix} I_{m-1} & Y \\ & I_{n+1-m}\end{smallmatrix}\right)$ in the argument of ϕ by

$$\begin{aligned} \begin{pmatrix} I_{m-1} & \\ -Q & I_{n-m+1}\end{pmatrix}\begin{pmatrix} I_{m-1} & Y \\ & I_{n+1-m}\end{pmatrix}\begin{pmatrix} I_{m-1} & \\ Q & I_{n-m+1}\end{pmatrix} &= \begin{pmatrix} I_{m-1} & \\ -Q & I_{n-m+1}\end{pmatrix}\begin{pmatrix} I_{m-1} & Y \\ Q & I_{n-m+1}\end{pmatrix} \\ = \begin{pmatrix} I_{m-1} & Y \\ & I_{n-m+1}-QY\end{pmatrix} &= \begin{pmatrix} I_{m-1} & Y \\ & I_{n-m+1}\end{pmatrix}\begin{pmatrix} I_{m-1} & \\ & I_{n-m+1}-QY\end{pmatrix}, \end{aligned} \tag{A.7}$$

since $YQ = 0$. With $u' = I_{n-m+1}-QY$, one has $\left(\begin{smallmatrix} I_{m-1} & \\ & u'\end{smallmatrix}\right)\left(\begin{smallmatrix} I_{m-1} & \\ X & u\end{smallmatrix}\right) = \left(\begin{smallmatrix} I_{m-1} & \\ u'X & u'u\end{smallmatrix}\right)$. As before, the change of variables $X \mapsto u'X$ can be undone without changing the measure. By proposition A.3, it furthermore maps one fundamental domain for X_1 to another. Another application of that proposition shows that the change of variables $u \mapsto (u')^{-1}u$ maps one fundamental domain for $u \in U(\mathbb{Q})\backslash U(\mathbb{A})$ to another. It also preserves the character $\psi_N(u)$, because u' – an member of the derived subgroup $[U(\mathbb{A}),U(\mathbb{A})]$ – lies in the kernel of ψ_N. Thus the integrands in each I_j are well-defined on their regions of integration.

The group N' is the semidirect product of the embedding of U into the lower right corner of $GL(n)$, and the embedding of $M_{n-m,m-1}$ into $GL(n)$ given by $X_1 \mapsto \left(\begin{smallmatrix} I_{m-1} & & \\ X_1 & I_{n-m} & \\ & & 1 \end{smallmatrix}\right)$. Hence the product of fundamental domains for $U(\mathbb{Q})\backslash U(\mathbb{A})$ and $M_{n-m,m-1}(\mathbb{Q})\backslash M_{n-m,m-1}(\mathbb{A})$ serves as a fundamental domain for $N'(\mathbb{Q})\backslash N'(\mathbb{A})$, and the product of their Haar measures is likewise the Haar measure on N' which gives this quotient volume 1. Therefore the integral I_0 reduces to the left hand side of (A.5). When $j = n-m$,

$$I_{n-m} \;=\; \int_{\substack{y \in M_{m-1,n-m}(\mathbb{Q}\backslash\mathbb{A}) \\ X_2 \in M_{n-m,m-1}(\mathbb{A}) \\ u \in U(\mathbb{Q})\backslash U(\mathbb{A})}} \phi\left(\begin{pmatrix} I_{m-1} & & y \\ & 1 & \\ & & I_{n-m} \end{pmatrix}\begin{pmatrix} I_{m-1} & \\ X_2 & u \\ 0 & \end{pmatrix} g\right) \times \tag{A.8}$$
$$\times\; \overline{\psi_N(u)}\, du\, dX_2\, dy\,.$$

Let us factor the last matrix in this expression as $\left(\begin{smallmatrix} I_{m-1} & \\ & u \end{smallmatrix}\right)\left(\begin{smallmatrix} I_{m-1} & \\ X_2' & I_{n-m+1} \\ 0 & \end{smallmatrix}\right)$, in which $u\left[\begin{smallmatrix} X_2' \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} X_2 \\ 0 \end{smallmatrix}\right]$. Both the change of variables $X_2 \mapsto X_2'$, as well as its inverse, involve adding multiples of lower rows to higher ones, which does not alter the measure dX_2. Applying the same considerations about product fundamental domains and measures for N° as observed for N' at the beginning of this paragraph, we see that (A.8) is hence equal to the right hand side of (A.5).

To complete the proof we will show that $I_j = I_{j+1}$, and afterwards argue the absolute convergence. Starting with (A.6), we enlarge y by adding a column to its left, denoted by y_1. Let $Y = (0\ y)$ be as above, but with this newly enlarged y. Let Q now denote the $(n-m+1)\times(m-1)$ matrix which has all zeroes except for its $n-m-j$-th row, which equals the row vector $q \in \mathbb{Q}^{m-1}$. We have that $YQ = 0$ and QY is strictly upper triangular, with its $(n-m-j, n-m-j+1)$-st entry equal to qy_1. Using ϕ's left invariance under $\left(\begin{smallmatrix} I_{m-1} & \\ Q & I_{n-m+1} \end{smallmatrix}\right)$, we may rewrite I_j in terms of its Fourier series expansion at $y_1 = 0$:

$$\sum_{q \in \mathbb{Q}^{m-1}} \int_{\substack{y \in M_{m-1,j+1}(\mathbb{Q}\backslash\mathbb{A}) \\ X_1 \in M_{n-m-j,m-1}(\mathbb{Q}\backslash\mathbb{A}) \\ X_2 \in M_{j,m-1}(\mathbb{A}) \\ u \in U(\mathbb{Q})\backslash U(\mathbb{A})}} \phi\left(\begin{pmatrix} I_{m-1} & \\ Q & I_{n-m+1} \end{pmatrix}\begin{pmatrix} I_{m-1} & Y \\ & I_{n-m+1} \end{pmatrix}\begin{pmatrix} I_{m-1} & \\ X_1 & \\ X_2 & u \\ 0 & \end{pmatrix} g\right) \times$$
$$\overline{\psi_N(u)}\,\overline{\psi(qy_1)}\, du\, dX_2\, dX_1\, dy\,. \tag{A.9}$$

We calculate

$$\begin{pmatrix} I_{m-1} & \\ Q & I_{n-m+1} \end{pmatrix}\begin{pmatrix} I_{m-1} & Y \\ & I_{n-m+1} \end{pmatrix} \;=\; \begin{pmatrix} I_{m-1} & Y \\ & I_{n-m+1} \end{pmatrix}\begin{pmatrix} I_{m-1} & \\ Q & I_{n-m+1} \end{pmatrix}\begin{pmatrix} I_{m-1} & \\ & u' \end{pmatrix}, \tag{A.10}$$

where $u' = I_{n-m+1} + QY \in U$. Therefore the argument of ϕ in (A.9) is equal to

$$\begin{aligned}
&\begin{pmatrix} I_{m-1} & Y \\ & I_{n-m+1} \end{pmatrix} \begin{pmatrix} I_{m-1} & \\ Q & I_{n-m+1} \end{pmatrix} \begin{pmatrix} I_{m-1} & \\ & u' \end{pmatrix} \begin{pmatrix} I_{m-1} & \\ X_1 & \\ X_2 & u \\ 0 & \end{pmatrix} g \\
&= \begin{pmatrix} I_{m-1} & Y \\ & I_{n-m+1} \end{pmatrix} \begin{pmatrix} I_{m-1} & \\ Q & I_{n-m+1} \end{pmatrix} \begin{pmatrix} I_{m-1} & \\ u' \begin{bmatrix} X_1 \\ X_2 \\ 0 \end{bmatrix} & I_{n-m+1} \end{pmatrix} \begin{pmatrix} I_{m-1} & \\ & u'u \end{pmatrix} g\,. \qquad \text{(A.11)}
\end{aligned}$$

Again, changing variables $u \mapsto (u')^{-1}u$ maps any fundamental domain for $U(\mathbb{Q})\backslash U(\mathbb{A})$ into another, and changes the character $\psi_N(u)$ to $\psi_N(u)\psi(qy_1)^{-1}$ – cf. proposition A.3. The multiplication of the unipotent matrix u' on $\begin{bmatrix} X_1 \\ X_2 \\ 0 \end{bmatrix}$ serves to add multiples of lower rows to a higher rows. Again, since X_1 and X_2 are integrated over abelian groups, this can be reversed by a change of variables which does not destroy the fundamental domain for X_1 or change either Haar measure. After combining the sum over q in (A.9) with the integration over the bottom row of X_1, so that it becomes an integration over $\mathbb{A}^{m-1}$ instead of $(\mathbb{Q}\backslash\mathbb{A})^{m-1}$, we have thus converted the expression for I_j into that for I_{j+1}.

Finally, we conclude by proving the absolute convergence. The above manipulations show that the absolute value of I_j is bounded by the analogous expression to (A.9), but with absolute values around the integrand. That expression is in turn bounded above by the expression for I_{j+1} – but again with absolute values around the integrand – because it is formed by combining a union of fundamental domains together. Thus each such expression is absolutely convergent, provided the last one – I_{n-m}, or equivalently the integral in proposition A.1 – is (see [10, §6.4] for this argument, in the context of the exterior square L-function). The absolute convergence of the I_{n-m} integral itself follows from the gauge estimates in [9, §5]. Indeed, although the estimates there are for the analogous local integrals instead, the mechanism of bounding the unipotent integration by a rapidly decaying function of the unipotent variable applies equally to the periods V, because of the expression for them as a sum of Whittaker functions mentioned just after the statement of this proposition. □

References

[1] W. Casselman, *Jacquet modules for real reductive groups*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), 1980, pp. 557–563.

[2] William Casselman, Henryk Hecht, and Dragan Miličić, *Bruhat filtrations and Whittaker vectors for real groups*, The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998), 2000, pp. 151–190.

[3] James W. Cogdell, *L-functions and converse theorems for* GL_n, Automorphic forms and applications, IAS/Park City Math. Ser., vol. 12, Amer. Math. Soc., Providence, RI, 2007, pp. 97–177. MR2331345 (2008e:11060)

[4] W. Duke and H. Iwaniec, *Estimates for coefficients of L-functions. I*, Automorphic forms and analytic number theory (Montreal, PQ, 1989), Univ. Montréal, Montreal, QC, 1990, pp. 43–47. MR1111010 (92f:11068)

[5] Dorian Goldfeld and Xiaoqing Li, *Voronoi formulas on* $\mathrm{GL}(n)$, Int. Math. Res. Not. (2006), Art. ID 86295, 25. MR2233713 (2007f:11052)

[6] ———, *The Voronoi formula for* $\mathrm{GL}(n,\mathbb{R})$, Int. Math. Res. Not. (2008).

[7] ———, *Addendum to: The Voronoi formula for* $\mathrm{GL}(n,\mathbb{R})$, Int. Math. Res. Not. (to appear).
[8] Aleksandar Ivić, *On the ternary additive divisor problem and the sixth moment of the zeta-function*, theory (Cardiff, 1995), London Math. Soc. Lecture Note Ser., vol. 237, Cambridge Univ. Press, Cambridge, 1997, pp. 205–243. MR1635762 (99k:11129)
[9] Hervé Jacquet and Joseph Shalika, *Rankin-Selberg convolutions: Archimedean theory*, sixtieth birthday, Part I (Ramat Aviv, 1989), Israel Math. Conf. Proc., vol. 2, Weizmann, Jerusalem, 1990, pp. 125–207. MR1159102 (93d:22022)
[10] ———, *Exterior square L-functions*, Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988), 1990, pp. 143–226.
[11] Xiaoqing Li, *Bounds for* $GL(3)\times GL(2)$ *L-functions and GL(3) L-functions* (2008). preprint.
[12] Stephen D. Miller, *Cancellation in additively twisted sums on* $\mathrm{GL}(n)$, Amer. J. Math. **128** (2006), no. 3, 699–729. MR2230922 (2007k:11078)
[13] Stephen D. Miller and Wilfried Schmid, *Distributions and analytic continuation of Dirichlet series*, J. Funct. Anal. **214** (2004), no. 1, 155–220.
[14] ———, *Automorphic distributions, L-functions, and Voronoi summation for* $\mathrm{GL}(3)$, Ann. of Math. (2) **164** (2006), no. 2, 423–488. MR2247965

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