

Math 99r: Complex Dynamics and the Mandelbrot Set

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1 Overview

Complex dynamics studies the action of a rational function $f \in \mathbb{C}(z)$ on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by repeated iteration:

$$z, f(z), f(f(z)), f(f(f(z))), \dots$$

Even in the case of a quadratic polynomial, the dynamics of f may be marvelously rich. For each map f , the Riemann sphere splits up into a set $F(f)$ called the *Fatou set* where the dynamics of f are tame and a set $J(f)$ called the *Julia set* where f is chaotic. A central question is to understand the structure of the Fatou and Julia sets and the action of f upon them.

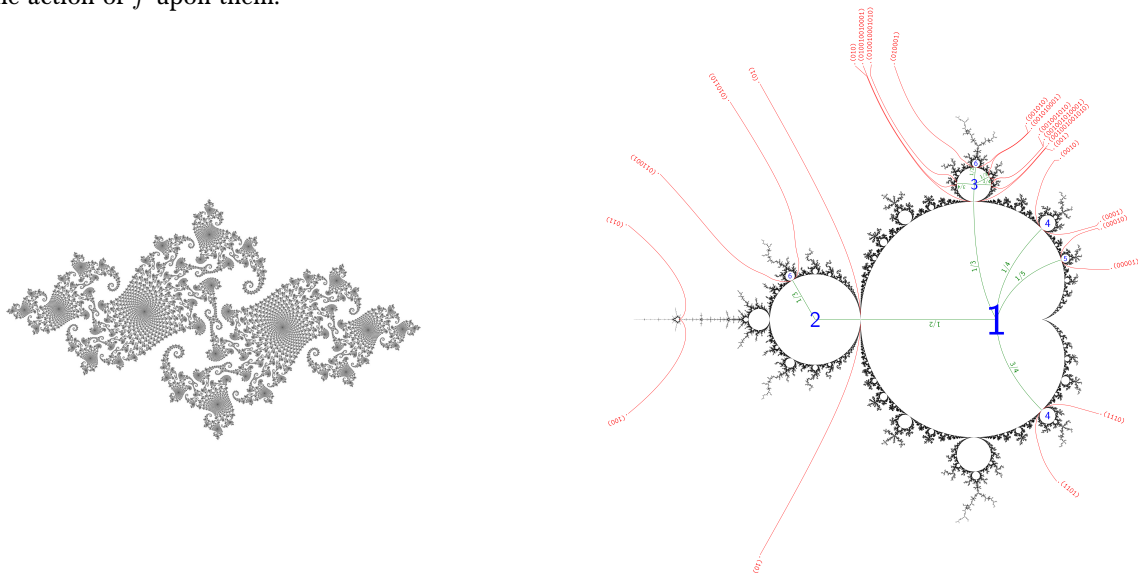


Figure 1: Left: The Julia set of $f(z) = z^2 + (-0.74543 + 0.11301i)$. Right: Structure in the Mandelbrot set.

In the first portion of the course, we will study general results about the dynamics of rational maps: topological properties of the Julia set, the local dynamics around periodic points, and the classification of Fatou components. Throughout this tour, we will encounter many concrete examples of rational maps with interesting behavior.

In the second part of the course we will consider the famed *Mandelbrot set* $\mathcal{M} \subset \mathbb{C}$. The Mandelbrot set is the set of parameters $c \in \mathbb{C}$ for which the quadratic polynomial $f_c(z) = z^2 + c$ has a connected Julia set. We will learn how the structure of $\mathcal{M}$ reflects the dynamics of quadratic polynomials and cover some of the known results about the geometry of $\mathcal{M}$. By the end of the course, we will have a combinatorial description of the Mandelbrot set—conditional upon the long-standing *MLC* conjecture. We will discuss this conjecture and other open problems surrounding $\mathcal{M}$.

2 Prerequisites

The only prerequisite for this course is an introductory course in complex analysis.

Students are encouraged to take Prof. McMullen's course on Advanced Complex Analysis (231a).

3 Course Outline

3.1 Introduction to Complex Dynamics

For this section I will closely follow the exposition in Milnor's excellent book [Mil06]. For a good exposition of hyperbolic geometry, see [Hub06].

- Introduction to the course
- Crash-course in complex analysis and Riemann surfaces
- Julia and Fatou sets
- Dynamics on hyperbolic surfaces
- Attracting and repelling fixed points
- Superattracting fixed points
- Parabolic fixed points
- Irrationally indifferent fixed points
- Fixed points in the Julia set
- Classification of Fatou components
- Local connectivity and combinatorial models
- Hyperbolic and subhyperbolic rational maps
- External rays

3.2 The Mandelbrot Set

I plan to primarily follow books by Hubbard [Hub16] and Lyubich [Lyu]. I will also refer to McMullen's book [McM94] and papers of Thurston [Thu09] and Milnor [Mil00].

- Stability and the Mandelbrot set
- Structure of the Mandelbrot set
- Combinatorial model for the Mandelbrot set

3.3 Student Presentations

Topics for final papers and presentations include the following:

- Irrationally indifferent fixed points and Diophantine approximation
- Sullivan's No-Wandering-Domains theorem
- Equilibrium measure and Brolin's theorem
- Thurston's algorithm for constructing postcritically finite real quadratic maps
- Elliptic curves and Lattès maps
- Newton's method and stable families

- Yoccoz's theorem on locally connected Julia sets
- Period-doubling and the logistic map
- Universality of the Mandelbrot set
- Tan Lei's theorem on asymptotic self-similarity
- Mating of Julia sets
- The Teichmüller space of a rational map
- DeMarco-Koch-McMullen's theorem on constructing postcritically finite maps
- Construction of Herman rings
- Dynamics of transcendental functions
- Arithmetic and nonarchimedean dynamics
- Sullivan's dictionary with hyperbolic 3-manifolds
- Complex dynamics in higher dimensions

4 Resources

There are a number of computer programs for drawing Julia sets and the Mandelbrot set. These include Mathematica, It [Man], Mandel [Wol], and FractalStream. Please let me know if you use any others. I am interested in seeing the results of your experiments and explorations.

5 Assignments

A problem set will be assigned every $N \geq 2$ weeks. There will be a choice of problems, some of which will be computational. Collaboration is encouraged. Late homework will not be accepted. Towards the end of the tutorial, you will write a final paper and give a presentation on an approved topic. The final grade will be a function of the homework and project grades.

6 Office Hours

I will hold office hours from 11:00-12:00am on Fridays in the common room.

References

- [Hub06] John Hamal Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1.* Matrix Editions, Ithaca, NY, 2006.
- [Hub16] John Hamal Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 2: Surface homeomorphisms and rational functions.* Matrix Editions, Ithaca, NY, 2016.
- [Lyu] Mikhail Lyubich. Conformal geometry and dynamics of quadratic polynomials, vol i-ii. <https://www.math.stonybrook.edu/~mlyubich/book.pdf>.
- [Man] Mannes Technology. It. <https://mannes-tech.com/It/index.html>. v3.1.
- [McM94] Curtis T. McMullen. *Complex dynamics and renormalization*, volume 135 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1994.
- [Mil00] John Milnor. Periodic orbits, external rays and the Mandelbrot set: an expository account. Number 261, pages xiii, 277–333. 2000. *Géométrie complexe et systèmes dynamiques* (Orsay, 1995).

- [Mil06] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.
- [Thu09] William P. Thurston. On the geometry and dynamics of iterated rational maps. In *Complex dynamics*, pages 3–137. A K Peters, Wellesley, MA, 2009. Edited by Dierk Schleicher and Nikita Selinger and with an appendix by Schleicher.
- [Wol] Wolf Jung. Mandel. <http://www.mndynamics.com/indexp.html>. v5.18.