Generalizations of Hodge-de-Rham degeneration for Fukaya categories

Semon Rezchikov

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1 Introduction

Let A be a dg-algebra over a field k. Then we have the famous

Theorem 1 (Kaledin [20], [19])**.** *Suppose A is smooth and proper, and char $k = 0$. Then the noncommutative Hodge-to-deRham spectral sequence degenerates, so $HP_\bullet(A) = HH_\bullet(A)((u))$.*

Recently, Efimov [10] disproved the following two potential generalizations of Kaledin's theorem by constructing explicit counterexamples:

Conjecture 1 (Kontsevich)**.** *Suppose A is proper, but not necessarily smooth. Then the composition*

$$(HH_\bullet(A) \otimes HC_\bullet(A^{op}))[1] \xrightarrow{id \otimes \delta} HH_\bullet(A) \otimes HH_\bullet(A^{op}) \to k,$$

where the last arrow is the pairing introduced by Shklyarov [27], is zero.

Conjecture 2 (Kontsevich)**.** *Suppose A is smooth, but not necessarily proper. Then the composition*

$$K_0(A \otimes A^{op}) \xrightarrow{ch} (HH_\bullet(A) \otimes HH_\bullet(A^{op}))_0 \xrightarrow{id \otimes \delta} (HH_\bullet(A) \otimes HC^-_\bullet(A^{op}))_{-1} \tag{1}$$

vanishes on the class of the diagonal $[\Delta] \in K_0(A \otimes A^{op})$. Here K_0 denotes the K-theory of the triangulated category of perfect A-modules.

Above, δ denotes the Connes operator, reviewed in Section 2.2.1, and ch is the noncommutative Chern character, reviewed in Section 3.2.2.

Symplectic geometry is a source of particularly interesting A_∞ categories. Moreover, categorical notions like *smoothness* and *properness* have correspond naturally to geometric conditions on symplectic manifolds. In this paper show that Fukaya categories are very special among A_∞ categories; in particular, they satisfy the conjectures of Kontsevich for geometric reasons.

Theorem 2. *Assume k has characteristic 2.*

Let M be Liouville domain, and let $Fuk(M)$ be the Fukaya category of closed exact Lagrangians in M (see e.g. Seidel [25]) with k coefficients. Then the composition

$$(HH_\bullet(Fuk(M)) \otimes HC_\bullet(Fuk(M)^{op})[1] \xrightarrow{id \otimes \delta} HH_\bullet(Fuk(M)) \otimes HH_\bullet(Fuk(M)^{op}) \to k, \tag{2}$$

where the last arrow is the pairing introduced by Shklyarov, is zero.

One geometric incarnation of *smoothness* in symplectic geometry is encapsulated in the following

Definition 1. We say that a Liouville domain M is *strongly nondegenerate* if the diagonal bimodule is generated (Section 3.2.1) by tensor products of Yoneda modules associated to Lagrangians in M and M^-, and moreover M is nondegenerate in the sense of [1].

Theorem 3. *Assume k has characteristic 2.*

Let M be a Liouville domain that is strongly nondegenerate*. Let $WF(M)$ denote the wrapped Fukaya category of M with k coefficients; due to the nondegeneracy assumption, $WF(M)$ is smooth. Let $WF(M){-}mod{-}WF(M)$ denote the A_∞ category of $(WF(M), WF(M))$-bimodules. Then the composition*

$$K_0(WF(M){-}mod{-}WF(M)) \xrightarrow{K\circ ch} (HH_\bullet(WF(M))\otimes HH_\bullet(WF(M)^{op}))_0 \xrightarrow{id\otimes\delta} (HH_\bullet(WF(M))\otimes HC^-_\bullet(WF(M)^{op}))_{-1} \tag{3}$$

vanishes on the class of the diagonal $[\Delta] \in K_0(WF(M) - mod - WF(M))$.

Here, the left-most term is the K-theory of perfect $WF(M) - WF(M)$-bimodules, ch is the *noncomutative Chern character*, and K is the Kunneth map on Hochschild homology.

Remark 1. Efimov's examples can be made Calabi-Yau (Efimov, personal communication), so these results do not follow algebraically from previous results on the existence of Calabi-Yau structures for Fukaya categories.

It is a fundamental theorem tat

Proposition 1. *[18], [7] If M is Weinstein then it is strongly nondegenerate.*

See the discussion around Lemma 11 for an explanation of what is needed to replace the strong nondegeneracy assumption above with the usual nondegeneracy condition of the Abouzaid generation criterion [1].

Remark 2. In Theorem 3, we use $K_0(WF(M) - mod - WF(M))$ as the domain of the map ch instead of "$WF(M) \otimes WF(M)^{op}$". This is because $WF(M)$ is naturally an A_∞-category, and the natural replacement for the tensor product of *dg* categories $A\otimes B$ is the category of bimodules $A - mod - B$. It is clear that Kontsevich's conjectures are invariant under replacing A by a quasi-equivalent *dg*-category; Theorem 3 implies Conjecture 2 for A any dga quasi-equivalent to $WF(M)$.

For the proof of Theorem 2, we take advantage of the degenerations of the moduli space of annuli with one distingushed marked point on each boundary component. This moduli space is closely related to the moduli space used in Abouzaid's proof [1] of the generation criterion: in our case, the distinguished marked points on each side of the annulus have a relative angle that is *allowed to vary*, while in Abouzaid's case the relative angle is *fixed*. For a visual summary of the degeneration argument, we direct the reader to Figure 2.

For the proof of Theorem 3, we use Ganatra's cyclic open-closed map [17] to rephrase the statement in terms of symplectic cohomology, and then perform a sequence of bordisms to the resulting moduli space of curves. At the level of TQFT, the argument is described schematically in Figure 1 below. In fact, it is possible to give an alternative proof of Theorem 2 that is *dual* to our proof of Theorem 3 simply by reading Figure 1 backwards, although we do not pursue this route in the paper. Thus, the conjectures of Kontsevich are dual to one another, and in the setting of Fukaya categories the admit *geometrically dual proofs*.

From the perspective of the cobordism hypothesis (see [14] for an introduction), natural algebraic conditions on an ∞-category should be the same as conditions on the TQFT that is equivalent to the ∞-category under the cobordism hypothesis. For example, smooth and proper *dg*-categories are exactly the fully dualizable *dg*-categories, which by the original form of the cobordism hypothesis correspond to fully extended $2d$ TQFTs (folkore, but see [28]). Similarly, making the TQFT an *oriented* TQFT instead of a *framed* TQFT is related to the Calabi-Yau condition on an A_∞ category [9]. We speculate that the schematic degenerations of Figure 1 should be propoted to axiomatic conditions on a TQFT, which should then correspond to a "good" algebraic condition on a *dg*-category. Historically, the identification of natural categorical properties, such as the notion of a smooth, proper, or Calabi-Yau category, has been helpful for researchers in many different fields, including symplectic topology. Based on the phenomena observed in this paper, we hope that the natural analytic behavior of pseudoholomorphic curves can be a guide for researchers in algebra, showing the way towards a better definition of a "good" noncommutative space.

2 Proper Fukaya Categories

To approach Theorem 2, we will borrow an explicit version of the conclusion in the proposition from Efimov. Please look at Appendix 6 for a discussion of the conventions on A_∞ algebras that we use.

Lemma 1 (Efimov [10] (Prop 2.2))**.** *Let A_1, A_2 be strictly unital A_∞ algebra over a field $\Bbbk$, and let M be a finite dimensional strictly unital A_∞ (A_1, A_2)-bimodule. Assume that $dimM < \infty$.*

$$\psi : HH_\bullet(A_1) \otimes HH_\bullet(A_2^{op}) \to HH_\bullet(A_1) \otimes HH_\bullet(A_2^{op}) \to HH_\bullet(End(V)) \to \Bbbk \tag{4}$$

is given by

$$\psi((a_0, \dots, a_n) \otimes (b_0, \dots, b_m)) \mapsto str_M(m \mapsto \sum_{0 \le i \le n, 0 \le j \le m} (-1)^{\maltese} \mu_{n+1,m+1}(a_i, \dots, a_n, \dots a_{i-1}, m, b_j, \dots, b_0, b_m, \dots, b_{j+1})). \tag{5}$$

Here, the str_M denotes the supertrace of an endomorphism of a graded vector space, the $\mu_{i,j}$ are the bimodule operations, and $(-1)^{\maltese}$ is a sign defined as follows. Let $b = (b_0, \dots, b_m)$, and $a = (a_0, \dots, a_m)$. Given an element x of an A_∞ algebra, let $|x|$ denote its degree and let $\|x\| = |x| - 1$ denote its reduced degree. Given a tuple $f = (f_0, \dots, f_r)$ of elements of a graded vector space and $0 \le i \le j \le r$, let

$$l_i^j(f) = \sum_{k=i}^{j} \|f_k\|.$$

Then

$$\maltese = |m| l_0^m(b) + l_0^n(a) + l_0^{i-1}(a) l_i^n(a) + \sum_{0 \le p < q \le j} \|b_p\| \|b_q\| + \sum_{j+1 \le p \le q \le m} \|b_p\| \|b_q\|.$$

Remark 3. The lemma is stated for finite dimensional A_∞ modules M, but the same computation proves the lemma for A_1, A_2 strictly unital A_∞ categories, with $M(X, Y)$, a finite dimensional k-vector space for any pair X, Y of objects in A_1, A_2, respectively. Likewise, the computations make sense for A_∞ modules which are only $\mathbb{Z}/2\mathbb{Z}$-graded.

Remark 4. If M is the diagonal bimodule then $\mu_{i,j} = \mu_{i+j+1}$.

The derivation of the above equation uses an expression for the Connes operator on the cyclic bar complex of a strictly unital A_∞ algebra. The Fukaya category $Fuk(M)$ of closed exact Lagrangians is not strictly unital, it is *c-unital*, namely, the homology category $H(Fuk(M))$ is unital ([25][I.2a]).

2.1 Units

The Fukaya category is not strictly unital; in this section we recall several chain-level notions of units in A_∞ categories and state a convenient algebraic lemma comparing two different such notions.

Definition 2. A *homology* unit for a c-unital A_∞ category A to be a choice $e_L \in Hom_A(L, L)$ of lift of the unit in $H(A)$ for every object L of A.

Given an object Δ in a c-unital A_∞ category A, we say that $e_L \in Hom_A(L, L)$ is a *homology unit for L* if it lifts the unit in $Hom_{H(A)}(L, L)$.

Given an A_∞ algebra A with a homology unit $e \in A$, there is the following useful notion introduced by FOOO [15]:

Definition 3. (see [16]) A homotopy unit for A is an A_∞ algebra structure on $A' := A \oplus kf[1] \oplus ke^+$, where

- The A_∞ operations preserve $A \subset A'$ and agree with the A_∞ operations on A;
- The element e^+ is a strict unit, and
- $\mu^1(f) = e - e^+$.

Remark 5. The definition implies that $A \to A'$ is a quasi-equivalence.

Remark 6. The definition immediately generalizes to arbitrary A_∞ categories: given an A_∞ category A with a homology unit $\{e_L\}_{L\in Ob(A)}$, one enlarges A to A' by adding a two-dimensional graded vector space $kf_L[1] \oplus ke_L^+$ to $End_A(L)$ for all objects of A, and defines a homotopy unit for A to be an A_∞ structure on A' extending that of A such that the $\{e_L^+\}$ are strict units and $\mu^1(f_L) = e_L - e_L^+$ for every L.

A geometric construction of homotopy units for a single compact lagrangian have been constructed by [15] and in a significantly more complex setting, for a certain model of the Wrapped Fukaya category of the product of a pair of Liouville domains, by [16]. For our purposes, these geometric constructions are not important, and we can simply invoke the following general algebraic lemma proven in Appendix 5:

Proposition 2. *Any c-unital A_∞ category with a choice of homology unit $\{e_L\}$ admits a homotopy unit.*

2.2 Variants of the Hochschild complex

There are several canonical complexes that compute Hochschild homology that may be associated to an A_∞ algebra, depending on whether it has a strict unit or not and what algebraic structure is required. The most basic, which makes sense for any A_∞ category, is the *bar complex*, which, as a graded vector space, is

$$C_\bullet(A) := \bigoplus_{k\geq 0} \bigoplus_{X_1,\ldots,X_k\in Ob(A)} Hom^*(X_0,X_1) \otimes Hom^*(X_1,X_2)[1] \otimes \ldots \otimes Hom^*(X_k,X_0)[1].$$

If A is strictly unital, then there is the *reduced bar complex*, which is a quotient of the bar complex in which any cyclically composable collection of morphisms may have at most the first element equal to the a strict unit. We denote the reduced bar complex by $C_\bullet^{red}(A)$; it is called $C_\bullet(A)$ by Efimov (see [10] Section 2). The quotient map $C_\bullet(A) \to C_\bullet^{red}(A)$ is an equivalence.

For any A there is also a *non-unital Hochschild complex* which, as a graded vector space is

$$C_\bullet^{nu}(A) = C_\bullet(A) \oplus C_\bullet(A)[1]$$

described, for example, in ([17] Section 3) and the inclusion $C_\bullet(A) \to C_\bullet^{nu}(A)$ is an equivalence whenever A is c-unital.

2.2.1 A review of S^1-actions on chain complexes

We briefly review the various S^1 actions on different variants of the Hochschild complexes; see ([17] Sections 2,3) for more details. Equipping $S^1 = \mathbb{R}/\mathbb{Z}$ with the CW structure with one 0-cell and one 1-cell, the graded group

$$C_{-\bullet}(S^1) = k[\Lambda]/\Lambda^2, |\Lambda| = -1$$

has the structure of a dg ring, and the inclusion into singular chains on S^1 is an equivalence of A_∞ algebras where the latter is equipped with the algebra structure induced by the group structure on S^1. A *dg*-module over $C_{-\bullet}(S^1)$ is called a *mixed complex*, and an A_∞-module over $C_{-\bullet}(S^1)$ is called an ∞-*mixed complex.* A *map* if ∞-mixed complexes just just a map of A_∞-modules. If A is unital then $C_\bullet(A)$, $C_\bullet^{red}(A)$ both have the structure of a mixed complex defined by Connes, and the quotient map is a map of mixed complexes. For any A the complex $C^{nu}_\bullet(A)$ is a mixed complex and if A is unital then $C_\bullet(A) \to C^{nu}_\bullet(A)$ is a equivalence of mixed complexes.

Given a chain complex M, the data of an A_∞-module structure on M over $C_{-\bullet}(S^1)$ is exactly the same as the data of a degree 1 map

$$\delta = M \to M[[u]] := M \widehat{\otimes_k k[[u]]} \tag{6}$$

where u is a formal variable of degree 2, the hat denotes u-adic completion; where the the u^0 term of δ is the usual differential on M, and the $k[[u]]$-linear extension f δ to an endomorphism of $M[[u]]$ is a differential. We will denote the components of this map by

$$\begin{aligned} \delta_k : M \to M[1-2k] \\ \delta = \sum_k \delta_k u^k. \end{aligned} \tag{7}$$

Clearly one can specify the structure of an ∞-mixed complex on M by specifying maps δ_k as above satisfying a necessary identity. One calls the resulting complex $M[[u]]$ the *negative cyclic complex*, the $k((u))$-linear complex $M((u))$ that one obtains by inverting u the *periodic cyclic complex*, and the quotient $M((u))/uM[[u]]$ the *cyclic complex.* Their homologies are the *negative cyclic homology*, *periodic cyclic homology*, and *cyclic homology*, respectively.

2.3 Reducing to the Hochschild complex of $Fuk(M)$

In this section, we explain how to apply Efimov's formula 5 to the setting of the (non-unital!) Fukaya category, even though it was derived using a unital model. Furthermore, we elaborate on why checking Theorem 2 on the image of Hochschild cycles in cyclic cycles suffices to prove Theorem 2.

Let $Fuk(M) \to Fuk(M)'$ be a choice of homotopy unit for $Fuk(M)$. There is an induced diagram of mixed complexes

$$\begin{array}{ccccc} C^{nu}_\bullet(Fuk(M)) & \longrightarrow & C^{nu}_\bullet(Fuk(M)') & & \\ & & \uparrow & & \\ C_\bullet(Fuk(M)) & \longrightarrow & C_\bullet(Fuk(M)') & \longrightarrow & C^{red}_\bullet(Fuk(M)') \end{array} \tag{8}$$

in which the left horizontal arrows are inclusions and all maps are equivalences. This zig-zag of equivalences of mixed complexes shows that understand the map in Equation 2, it suffices to compute the image of the map

$$(HH_\bullet(Fuk(M)') \otimes HC_\bullet((Fuk(M)')^{op})[1] \xrightarrow{id\otimes\delta} HH_\bullet(Fuk(M)') \otimes HH_\bullet((Fuk(M)')^{op}) \to k. \tag{9}$$

where the Hochschild and cyclic homologies are computed using the mixed complex structure on $C^{red}_\bullet(Fuk(M)')$. We now prove the lemma

Lemma 2. *Theorem 2 holds for $Fuk(M)$ if for all Hochschild cycles*

$$\sum_l k_l(a^l_0, \dots, a^l_{n_l}) \otimes (b^l_0, \dots, b^l_{m_l}) \in C_\bullet(Fuk(M)) \tag{10}$$

one has that

$$\sum_l k_l str_M\left(m \mapsto \sum_{0\le i\le n, 0\le j\le m} \mu_{n+m+3}(a^l_i, \dots, a^l_{n_l}, \dots a^l_{i-1}, m, b^l_j, \dots, b^l_0, b^l_{m_l}, \dots, b^l_{j+1})\right) = 0. \tag{11}$$

Proof. For an ∞-mixed complex N, the Connes map $\delta : HC_\bullet(N) \to HH_\bullet(N)$ is the boundary map of the long exact sequence of complexes (see [22, Theorem 2.2.1])

$$M = uM[[u]]/u^2M[[u]] \to M((u))/u^2M[[u]] \to M((u))/uM[[u]]. \tag{12}$$

where the identification $M = uM[[u]]/u^2M[[u]]$ shifts degree by two. Let $M = C^{red}_\bullet(Fuk(M)')$, and write $M = M' \oplus M''$, where M' is the subcomplex spanned by cycles of composable morphisms for which the first term in the tensor product is the strict unit, and M'' is the image of $C_\bullet(Fuk(M))$ in $C^{red}_\bullet(Fuk(M)')$, i.e. the subcomplex spanned by cycles of composable morphisms where the first term in the tensor product is *not* a strict unit. Let $y \in M((u))/uM[[u]]$ be a cocycle in the cyclic homology complex. Any cyclic chain y can be uniquely written as

$$y = y^{nu} + y^u + u^{-1}y^{neg}, \tag{13}$$

where

$$u^{-1}y^{neg} \in u^{-1}M((u))/uM[[u]];$$
$$y^{nu} \in u^0M'' \subset M((u))/uM[[u]];$$
$$y^u \in u^0M' \subset M((u))/uM[[u]].$$

In the mixed complex structure on $C^{red}_\bullet(Fuk(M)')$, all terms containing a unit are annihilated by the Connes B operator, and all terms in the image of B contain a unit in the first position (See Efimov [10], Equation (2.1) or [17]). Denote the Hochschild differential by b; then the differencial on the negative cyclic complex is $b + uB$. Given $z \in M((u))/uM[[u]]$, we introduce the notation

$$z = \sum_{i\leq 0}[z]_iu^{-i}, \text{ where}[z]_i \in M.$$

Suppose that y is a cycle, i.e. $(d + uB)y = 0$. Plugging in (13), one sees that

$$b(y^{nu} + y^u) = -[By^{neg}]_0 + -u^{-1}[By^{neg}]_{-1}u^{-1} + \ldots$$

Since $b(y^{nu})$ has no units in first position, but every term on the right hand side must have units in the first position, we must have that $b(y^{nu}) = 0$, and thus that $(b + uB)(y^{nu}) = 0$, so y^{nu} is a cyclic cycle. Therefore, since $By^u = 0$, we have that

$$B(y) \in By^{nu} + u^{-1}M((u))/M[[u]]$$

and in particular, by the definition of δ (12), one has that

$$\delta y = \delta y^{nu}.$$

But every y^{nu} with $by^{nu} = 0$ is the image of a Hochschild cycle in $C_\bullet(Fuk(M))$, and the map $HH_*(Fuk(M)) \to HH_*(Fuk(M)')$ is a quasi-isomorphism. This proves the Lemma. □

2.4 Fukaya-categorical conventions

For all $d \geq 3$, let $\mathcal{R}^d$ be the moduli space of holomorphic disks with d marked points on the boundary, and let $\mathcal{S}^d \to \mathcal{R}^d$ be the corresponding universal family of disks. Let $\overline{\mathcal{R}}^d$ be Deligne-Mumford compactification of $\mathcal{R}^d$.

We will use Seidel's conventions for setting up the compact Fukaya category, with sign conventions described in Section 6. In this case:

- Standard coordinates on the complex numbers $\mathbb{C}$ are $z = s+it$. Let $\mathbb{R}^\pm = \{s \in \mathbb{R}; \pm s \geq 0\}$. Define the standard Riemann surfaces
  - $Z := \mathbb{R} \times [0, 1]$,

  - $Z^{\pm} := \mathbb{R}^{\pm} \times [0,1] \subset \mathbb{C}$, and
  - $C^{\pm} := \mathbb{R}^{\pm} \times S^1$, where the complex structure is, in the coordinates s, t on the first and second factors, given by

$$i\partial_s = \partial_t.$$

- A symplectic form gives a bijection between vector fields and one-forms

$$X_\theta \longleftrightarrow \theta = -i_{X_\theta}\omega;$$

if $\theta = dH$ for some function H, we write $X_H := X_{dH}$ and we call X_H the Hamiltonian vector field associated to H. On the cotangent bundle of $\mathbb{R}^n$, the flow $x' = X_H$ gives the standard form of Hamilton's equations.

- The symplectic manifold (M, ω) is a Liouville domain, namely, an *exact symplectic manifold with contact-type boundary*, see Seidel (7b). This comes equipped with a primitive θ_M for ω, and thus with a Liouville vector field X_M characterized by $i_{X_M}\omega = \theta$ (which points *outwards* along ∂M) and a smooth real-valued function h_M defined in a neighborhood of ∂M characterized by the property that $h_M^{-1}(1) = \partial M$ and $X_{\theta_M} h_M = h_M$. We choose some ω-compatible almost complex structure I on a sufficiently small neighborhood of ∂M which is invariant under the flow of X_θ and such that $dh_M \circ i = \theta_M$.We can complete (M, ω) to a *Liouville manifold* by adding on a collar to M of form $\mathbb{R}_r \times \partial M$ with $\theta_{\mathbb{R}\times\partial M} = e^r(\theta_M)|_{\partial M}$.

- Let Σ be a Riemann surface with boundary obtained by removing a finite number of points from a compact Riemann surface with boundary $\hat{\Sigma}$. We say points in $\hat{\Sigma} \setminus \Sigma$ are *boundary-marked points* if they lie in $\partial\hat{\Sigma}$, and otherwise we call them *interior marked points*. To define limiting conditions for Floer's equation, we need to equip marked points with *ends*. A *strip-like end* on Σ at a boundary marked point p of Σ is a proper holomorphic embedding u from $Z^{\pm}$ to Σ such that $u^{-1}(\partial\Sigma) = \mathbb{R}^{\pm} \times \{0,1\}$ and such that, viewing u as an embedding into $\hat{\Sigma}$, one has that $\lim_{s\to\pm\infty}(u(z)) = p$. Similarly, a *cylindrical end* on Σ at an interior marked point of Σ is a proper holomorphic embedding u from $C^{\pm}$ to Σ such that $\lim_{s\to\pm\infty} u(s,t) = p$. We say that u is an *outgoing*, or *positive* end, and that p is a *positive* puncture, if we used Z^+ or C^+ in the above definition, and that u is an *incoming*, or *negative* end, and that p is a *negative* puncture, if we instead used Z^- or C^-. Since Z^+ is biholomorphic to Z^-, and similarly C^+ is biholomorphic to C^-, there is a bijection between positive and negative ends at a marked point p. Nonetheless, we may think of the marked points as being divided into positive and negative marked points, and when we choose ends at the marked points, the type of end will respect this dichotomy. We will think of Z has having canonical strip like ends $Z^{\pm}$, with negative marked point p_0 and positive marked point p_1.

- To equip Σ with *Lagrangian labels* we assign a choice of Lagrangian submanifold L_C of M to each component $C \in \pi_0(\partial\Sigma)$.

- Write $\mathcal{J}$ for the space of almost complex structures on M which agree with I near the boundary of M, and write $\mathcal{H}$ for the space of smooth functions on M which vanish near the boundary of M. *Floer data* on Z are pairs $(H, J) \in C^\infty([0,1]; \mathcal{H}) \times C^\infty([0,1]; \mathcal{J})$. If Σ is a Riemann surface with boundary equipped with Lagrangian labels, then *perturbation* data for that Riemann surface are a pair $(K, J) \in \Omega^1(\Sigma, \mathcal{J}) \times C^\infty(\Sigma, \mathcal{J})$, such that $K(\xi)|_{L_C} = 0$ for all $\xi \in TC \subset T(\partial\Sigma)$, for all connected components C of the boundary of Σ.

- Floer's equation on a strip equipped with Floer data (H, J) with Lagrangian labels L_i on $\mathbb{R} \times \{i\}$, $i = 0, 1$, reads

$$\begin{aligned} u \in C^\infty(Z, M); u(s,i) \in L_i \text{ for } i = 0,1, \\ \partial_s u + J(t,u)(\partial_t u - X_H(t,u)) = 0. \end{aligned} \tag{14}$$

This is a special case of the *inhomogeneous pseudoholomorphic map* equation on a Riemann surface Σ equipped with Lagrangian labels $\{L_C\}_{C\in\pi_0(\partial S)}$ and perturbation data (K, J), the general form of which is

$$\begin{aligned} u \in C^\infty(\Sigma, M); u(C) \in L_C \text{ for all } C \in \pi_0(\Sigma); \\ (du - Y)^{0,1} = 0 \end{aligned} \tag{15}$$

where $Y \in \Omega^1(\Sigma, C^\infty(TM))$ is defined by $i_{Y(\xi)}\omega = K(\xi)$.

- We will assume that M is equipped with a $\mathbb{Z}$*-grading*, namely, a fiber bundle $\widetilde{LGr}(M) \to M$ equipped with a map of fiber bundles $\widetilde{LGr(M)} \to LGr(M)$ to the Lagrangian Grassmannian bundle of M, such that on every fiber the map of a universal covering. The obstruction to the existence of a $\mathbb{Z}$-grading is whether $2c_1(M) = 0$.
- The Fukaya category $Fuk(M)$ has objects *Lagrangian branes* $\tilde{L} = (L, \alpha)$ where $L \subset M$ is a closed exact Lagrangian submanifold of M and α is a lift of the Gauss map $L \to LGr(M)$ to $\widetilde{LGr(M)}$ (the *grading* of L). The obstruction to the existence of α the *Maslov class* of L, $\mu_L \in H^1(L, \mathbb{Z})$.
- The Fukaya category is only defined after a choice of *coherent peturbation data*, namely,
  1. A choice of *consistent universal choice of strip-like ends* in the sense of Lemma II.9.2 of [25]; this involves a choice for all $d \geq 2$, for each universal family
  $$\mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$$
  of disks with $d+1$ marked points $p_0, \dots, p_{d+1}$, ordered counterclockwise on the boundary of the disk, with p_0 the unique negative marked point and the rest positive marked points,
  2. For every ordered pair of Lagrangians L_0, L_1, a choice of Floer data (H, J) on the strip with $\mathbb{R} \times \{i\}$ labeled by L_i, for which the image of L_0 under the time-1 flow of H intersects L_1 transversally;
  3. A choice of *consistent universal choice of perturbation data* in the sense of Lemma II.9.5 of [25]; and moreover,
  4. the choices of Floer data and perturbation data are required to be *regular*, namely, that the linearization of Floer's equation 14 at any solution u is surjective, and likewise such that for any disk $\hat{\Sigma} \in \mathcal{R}^{k+1}$ with marked points $p_0, \dots p_k$, Lagrangian labels $\{L_i\}_{i=0}^k$ on the boundary components of $\Sigma = \hat{\Sigma} \setminus \{p_i\}_{i=0}^k$, with L_i on the component between p_i and p_{i+1} (with $p_{k+1} = p_0$) and any finite-energy solution u to the inhomogeneous map equation 15 on Σ with the above Lagrangian labels and the perturbation data induced by the choice of universal perturbation data above, the *extended linearized operator* is surjective (see equation (II.9.18) of [25] for a definition of the operator, and (II.9.26) of *(loc. cit.)* for an explicit formula; see (II.8.12) for a definition of the energy of a solution to (15).).

The universal choices of strip-like ends of perturbation data allows us to make sense of the the spaces of solutions to the pseudoholomorphic map equations *for the families* $\mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$. Namely, given any disk $\hat{\Sigma} \in \mathcal{R}^{d+1}$, $d \geq 2$, with marked points and Lagrangian labels as in the last bullet point above, write $\mathcal{M}(\hat{\Sigma}, L_0, \dots, L_k)$ for the set of finite energy solutions to the equation 15 on Σ with the Lagrangian labels and the perturbation data determined by the choices made, and define

$$\mathcal{M}_{\mathcal{R}^{k+1}}(L_0, \dots, L_k) := \{(\hat{\Sigma}, u) | \hat{\Sigma} \in \mathcal{R}^{k+1}, u \in \mathcal{M}(\hat{\Sigma}, L_0, \dots, L_k)\} \tag{16}$$

as a set; equipping this set with the Gromov topology on stable maps (e.g. [13]), this set is (in our setup) a manifold. Similarly, given closed exact Lagrangians (L_0, L_1), let

$$\widetilde{\mathcal{M}}(L_0, L_1), \mathcal{M}(L_0, L_1)$$

denote the set of solutions to Floer's equation with Lagrangian labels L_i and the Floer data chosen for these Lagrangians, and its corresponding quotient by the $\mathbb{R}$-action induced by the automorphisms of Z; these are both manifolds in the Gromov topology.

- Let $(\tilde{L}_0, \tilde{L}_1)$ be a pair of Lagrangian branes with (L_0, L_1) the underlying Lagrangian submanifolds. Let $\mathcal{C}(\tilde{L}_0, \tilde{L}_1)$ denote the set of time-1 Hamiltonian chords from L_0 to L_1, namely, maps $\gamma : [0,1] \to M$ such that $\gamma(i) \in L_i$ for $i = 0, 1$, and $d\gamma/dt = X_{H_t}(\gamma)$. For every $y \in \mathcal{C}(\tilde{L}_0, \tilde{L}_1)$, the gradings of the $\tilde{L}_i$ define the degrees $|y| \in \mathbb{Z}$ of the chords (see [4, Section 1.3], or [25, Section II.11]).
- The underlying graded vector space of the morphism cochain complex $CF^*(L_0, L_1)$ is the sum
$$CF^n(L_0, L_1) = \bigoplus_{\substack{y \in \mathcal{C}(\tilde{L}_0, \tilde{L}_1) \\ |y| = n}} \mathbb{Z}/2[y]$$
- Let $(\tilde{L}_i)_{i=0}^k$, $k \geq 1$, be a collection of objects of $Fuk(M)$, and
$$\gamma_i \in \mathcal{C}(\tilde{L}_{i-1}, \tilde{L}_i), i = 1, \ldots k.; \gamma_0 \in \mathcal{C}(\tilde{L}_0), \tilde{L}_k).$$
Denote by
$$\mathcal{M}(\gamma_0; \gamma_1, \ldots, \gamma_k)$$
the subspace of $\mathcal{M}(L_0, L_1, \ldots, L_k)$ for which u limits to $\gamma_0, \ldots, \gamma_k$ at the marked points $p_0, \ldots, p_k$ in the coordinates determined by the strip-like ends that have been chosen for the domain of the map. In our setup, whenever
$$\sum_{j=1}^k |\gamma_j| + (2-k) - |\gamma_0| = 0,$$
this is a compact manifold.The structure operation for the Fukaya category $Fuk(M)$,
$$\mu^k : CF^*(\tilde{L}_0, \tilde{L}_1) \otimes \ldots \otimes CF^*(\tilde{L}_{k-1}, \tilde{L}_k) \to CF^*(\tilde{L}_0, \tilde{L}_k)[2-k]$$
is given by
$$\mu^d(y_1, \ldots, y_k) = \sum_{\substack{\gamma_0 \in \mathcal{C}(L_0, L_k); \\ \sum_{j=1}^k |\gamma_j| + (2-k) - |\gamma_0| = 0}} \sum_{u \in \mathcal{M}(\gamma_0; \ldots, \gamma_k)} [\gamma_0]. \tag{17}$$
These operations satisfy the A_∞ equations (129).

2.5 Opposite Fukaya categories and negative symplectic forms

Now recall the following

Definition 4. Given an A_∞ category A over an arbitrary ring R, its *opposite category* is the A_∞ category A^{op} with the same objects as that of A, and

$$hom_{A^{op}}(X, Y) = hom_A(Y, X),$$

$$\mu^d_{A^{op}}(x_1, \ldots, x_d) = (-1)^{\sum_{1 \leq i < j \leq d} (|x_i| - 1)(|x_j| - 1)} \mu^d_A(x_d, \ldots, x_1).$$

Symplectic manifolds also admit a notion of 'opposites':

Definition 5. Given a symplectic manifold (possibly with boundary) (M, ω_M), its *opposite* is the symplecitcmanifold (M^-, ω_{M^-}) with $M^- = M, \omega_{M^-} = -\omega_M$. If M was a Liouville domain then M^- is also a Liouville domain - the primitive of ω_{M^-} is taken to be the negative of the primitive of ω_M, and the resulting Liouville flow is then the same as that for M.

The choices described in Section 2.4 needed to define $Fuk(M)$ for M a Liouville domain give a corresponding collection of choices needed to define $Fuk(M^-)$. By leaving the grading on $M = M^-$ and the consistent universal choice of strip-like ends unchanged, replacing the position-dependent almost complex structures in all perturbation data with their negatives, keeping all Hamiltonian-valued one-forms going into the perturbation data the same, one gets the data needed to define the Fukaya category of M^-. Composition with complex-conjugation then gives a bijection between maps from disks contributing to the operations of $Fuk(M)$ and of $Fuk(M^-)$ while reversing the order of the Lagrangian labels on the boundary of the disk, proving the

Lemma 3. *With the above choices of data needed to define $Fuk(M^-)$, if we are working over a field $\Bbbk$ of characteristic 2,*

$$Fuk(M^-) = Fuk(M)^{op}. \tag{18}$$

A much more refined statement was proven by Sheridan in [], which proved the following isomorphism for the variant of the Fukaya category that is defined over the integers, whenever a certain additional "grading datum" is specified. See Appendix 6 for a discussion of various notions of opposite A_∞ categories.

2.6 A moduli space of annuli

In this section we define a moduli space $\mathcal{M}_{n+1,m+1}$ of pseudoholomorphic maps from annuli with marked points which will play the central role in the proof of Theorem 2. First we fix our notation for the moduli of marked Riemann surfaces with boundary; then we specify how to choose coherent perturbation data; and finally we use the chosen perturbation data to define the moduli space.

2.6.1 Conventions for moduli of marked Riemann surfaces

For any $n, m \geq 1$, let $\mathcal{C}_{n,m}$ be the moduli space of holomorphic annuli with n marked points on one boundary component and m marked points on the other, up to marked-point-preserving biholomorphism. In our conventions for orienting the boundary ∂N of an oriented manifold N, an oriented bases of T_xN for $x \in \partial N$ is given by the outwards pointing vector followed by an oriented basis for $T_x\partial N$; in particular, the induced orientation on the boundary of the unit disk in $\mathbb{C}$ goes *counterclockwise.* Label the n marked points on one boundary component $\alpha_0, \ldots, \alpha_{n-1}$ such that the α_i are arranged with index i increasing *along with* the induced orientation of the boundary; likewise, label the m marked points on the other boundary component such that the β_i are arrange with the index i increasing *opposite to* the induced orientation of the boundary. A diagram of this labeling convention is given in Figure **??**. Let $\overline{\mathcal{C}}_{n,m}$ be the Deligne-Mumford compactification of this moduli space, which is a smooth manifold with corners (see Liu's thesis [21, Section 4]). There is a universal family of *stable marked bordered Riemann surfaces* $\hat{\Sigma}_{n,m} \to \overline{\mathcal{C}}_{n,m}$. The fibers $(\hat{\Sigma}_{n,m})_t$ of this family over strata of positive codimension are gluings of marked Riemann surfaces along certain pairs of marked points, i.e. *nodal* Riemann surfaces; the corresponding disjoint union of marked Riemann surfaces *before* the gluing is called the *normalization* of $(\hat{\Sigma}_{n,m})_t$, and is denoted by $\widetilde{(\hat{\Sigma}_{n,m})_t}$. Let $\Sigma_{n,m}$ be $\hat{\Sigma}_{n,m}$ with all marked points removed. The combinatorical types of the strata of $\overline{\mathcal{C}}_{n,m}$ and their adjacencies are diagrammed schematically in Figure 2.

2.6.2 Conventions for gluing

We briefly clarify our conventions for gluing Riemann surfaces. Namely, given a pair of Riemann surfaces with boundary $(\hat{\Sigma}_+,\ \hat{\Sigma}_-)$ equipped with a pair of interior marked points $p_\pm \in \hat{\Sigma}_\pm$ together with a choice of positive cylindrical end ϕ_+ for p_+ and negative cylindrical end ϕ_- for p_-, we may form the *glued Riemann surface*

$$\hat{\Sigma}_+ \#_{\ell,\theta} \hat{\Sigma}_-,$$

where $(\ell, \theta) \in (0,1] \times S^1$ is a *gluing parameter*, by removing $\phi_+((-\log(\ell)/\pi, \infty) \times S^1)$ from $\hat{\Sigma}_+$, removing $\phi_-((-\infty, log(\ell)/\pi) \times S^1)$ from $\hat{\Sigma}_-$, and identifying the resulting boundary circles $\phi_\pm(\{\mp \log(\ell)/\pi) \times S^1)$ via

$$\phi_+(-\log(\ell)/\pi), \theta + \tau) \sim \phi_-((+\log(\ell)/\pi), \tau), \quad \text{for } \tau \in S^1.$$

Similarly, given a pair of Riemann surfaces with boundary $(\hat{\Sigma}_+, \hat{\Sigma}_-)$ with equipped with a pair of boundary marked points $p_\pm \in \hat{\Sigma}_\pm$ and strip-like ends $\phi_\pm$ at $p_\pm$, respectively, we may form the glued Riemann surface

$$\hat{\Sigma}_+ \#_\ell \hat{\Sigma}_-$$

with gluing parameter $\ell \in (0,1]$ by removing $\phi_+((-\log(\ell)/\pi, \infty) \times [0,1])$ from Σ_+ and , $\phi_-((-\infty, \log(\ell)/\pi) \times [0,1])$ from $\hat{\Sigma}_-$ and identifying the resulting Riemann surfaces along

$$\phi_+(\{-\log(\ell)/\pi\} \times [0,1]) \sim \phi_-(\{\log(\ell)/\pi\} \times [0,1]).$$

The gluing construction gives families of nodal Riemann surfaces

$$\hat{\Sigma}_+ \#_{\ell,\theta} \hat{\Sigma}_- \to D := \{z = e^{2\pi(\ell/2 + i\theta)}\} \subset \mathbb{D}$$

$$\hat{\Sigma}_+ \#_\ell \hat{\Sigma}_- \to [0,1]$$

with smooth fibers away from zero; when $\hat{\Sigma}_\pm$ are stable, these families correspond to maps from D or $[0,1]$ to the Deligne-Mumford compactification of moduli of marked Riemann surfaces with boundary [21] which send zero into a boundary stratum of the compactification. Given a family of Riemann surfaces $\hat{\Sigma}^t_\pm$ with parameter t, together with a smoothly varying family of marked points $p^t_\pm \in \hat{\Sigma}^t_\pm$ and a smoothly varying family of ends at the $p^t_\pm$, the gluing construction applied to the whole family gives a *smooth map* [25, Lemma 9.2] to the neighborhood of a complex codimension 1 boundary of the appropriate Deligne-Mumford moduli space. More generally, given several distinct smoothly varying families of points $(p^t_i)_\pm \in \hat{\Sigma}^t_\pm$, for $i = 1, \ldots, k$ with corresponding smoothly varying families of ends, the gluing construction applied simultaneously to all the p^t_i at once gives a smooth map to a neighborhood of a higher codimension boundary stratum of the appropriate Deligne-Mumford moduli space; choosing the families to be the universal ones over the boundary strata, one gets a smooth parametrization, the *gluing coordiantes*, of the neighborhood of the boundary of the Deligne-Mumford space.

Given perturbation data in the sense of (15) on $\hat{\Sigma}_\pm$ for which the choices of Lagrangian labels near p and limiting Floer data at p agree, the gluing construction gives canonical choices of perturbation data on the gluings of $\hat{\Sigma}_\pm$, the *glued perturbation data*, which vary smoothly in the gluing coordinates.

2.6.3 Coherence data

To choose the coherent perturbation data needed to define the moduli space $\mathcal{M}_{n+1,m+1}$, we will need to choose a *universal choice of ends on* $\overline{\mathcal{C}}_{n,m}$. We will first state how to declare all marked points on all normalizations of curves in the universal family as either positive or negative, while satisfying the condition that any pair of identified points are of oppositite valence, so that we know whether to choose positive or negative ends in each case. We declare all the initial marked points in the top dimensional stratum of $\overline{\mathcal{C}}_{n,m}$ to be positive. As we degenerate to lower-dimensional strata (see Figure 2), there is either a component that is an annulus, in which case we impose the constraint that all the boundary marked points of the annulus are positive; or the curve has two disks glued at an interior point, in which case we require that all boundary marked points of each of these two disks are positive, and we choose that the interior marked point on the component with boundary marked points labeled by α_i is positive; or the stratum is a degeneration of the stratum described in Figure 2 (C), where a degeneration of the annulus occurs via a self-gluing of a disk via two boundary marked points. For that last stratum, we call the marked point involved in the self-gluing that we declare to be negative p^-, and the one we declare to be positive p^+; the valences are determined by the requirement that reading along induced orientation of the boundary of the disk, we have a marked point b_{j+1} (for some index $j+1$), followed by the negative marked point p^-, followed by a marked point α_i (for some index i); and for p^+, we have some α_{i-1}, then p^+, then some β_j (see Figure **??**). With these constraints, we have that

Lemma 4. *There exists a unique choice of valences for the marked points of the fibers of* $(\hat{\Sigma}_{n,m})$*, satisfying the constraints of the paragraph above, together with the condition that any pair of identified points are of opposite valence, and the condition that that the valences for marked points that are not to be glued are preserved by gluing. Moreover, with this choice, for any disk with no interior marked points occuring as a component of a fiber of one of the universal curves* $(\hat{\Sigma}_{n,m})$*, there is exactly one negative marked point, and so that disk may be viewed as an element of the associahedron* $\mathcal{R}^{k+1}$ *used to define* $Fuk(M)$*.*

A *universal choice of ends* for $\overline{\mathcal{C}}_{n,m}$ is a choice, for all $\widetilde{n, m, \text{for}}$ every stratum S of $\overline{\mathcal{C}}_{n,m}$, for every $t \in S$, for every marked point of the normalization $(\hat{\Sigma}_{n,m})_t$ of the fiber of the universal curve $\widetilde{\text{over } t}$,of an end at the marked point, such that the resulting choice of ends on any gluing Σ' of $(\hat{\Sigma}_{n,m})_t$ along a collection of points identified in $(\hat{\Sigma}_{n,m})_t$ agrees with the ends chosen for Σ' viewed as a fiber of the universal curve over another stratum of $\overline{\mathcal{C}}_{n,m}$, and such that these choices of ends vary smoothly over the strata.

A *universal and consistent choice of perturbation data on* $\overline{C}_{n,m}$, is a choice, for all n, m, for all possible choices of locally constant Lagrangian labelings of the fibers of $\hat{\Sigma}_{n,m}|_{C_{n,m})}$ (each of which induces Lagrangian labelings on all fibers of $\hat{\Sigma}_{n,m}$), for all strata S of $\overline{\mathcal{C}}_{n,m}$, choose perturbation data for the fibers of the universal curve $\mathcal{S}_S \to S$ varying smoothly over S, such on every component of $\mathcal{S}_s$ that is a disk with $n+1$ boundary marked points, the perturbation datum agrees with the perturbation datum on the corresponding disk in the Stasheff associahedron chosen to define the Fukaya category; and such that under any gluing of marked points, the perturbation data chosen for the glued surface agree with the glued perturbation data to infinite order zero in the gluing coordinates; and such that the Hamiltonian term of the perturbation data is zero near any interior marked points.

We call the combination of a universal choice of ends and a universal and consistent choice of perturbation data on $\overline{\mathcal{C}}_{n,m}$ to be a choice of *coherence data* for $\overline{\mathcal{C}}_{n,m}$.

For any $C \in C_{n+1,m+1}$ let ϵ_{α_i} and ϵ_{β_i} denote the end chosen on $(\Sigma_{n+1,m+1})_C$ at the corresponding marked point. Choose a pair of cyclically-composable tuples morphisms $a_0, \dots, a_n$ and $b_0, \dots, b_m$ in $Fuk(M)$. This induces a smooth family of Lagrangian labelings of the universal curve over $C_{n+1,m+1}$; we write L_x for the Lagrangian assigned to a point x of any boundary component of a fiber of the universal curve. We define the moduli space of stable maps

$$\mathcal{M}_{n+1,m+1}(a_0,\dots a_n; b_0,\dots,b_m) = \left\{ (u,C) \;\middle|\; \begin{array}{c} C \in C_{n+1,m+1};\ u \in C^\infty((\Sigma_{n+1,m+1})_C, M); \\ u(x) \in L_x \text{ for all } x \in \partial(\Sigma_{n+1,m+1})_C; \\ (du - Y_C)^{0,1} = 0; \\ \lim_{s\to+\infty} \epsilon_{\alpha_i}(s,t) = a_i(t); \\ \lim_{s\to+\infty} \epsilon_{\beta_i}(s,t) = b_i(t) \end{array} \right\} \tag{19}$$

This moduli space of stable maps has a Gromov-Floer bordification

$$\overline{\mathcal{M}}_{n+1,m+1}(a_0,\dots a_n; b_0,\dots,b_m)$$

which admits a stabilization map

$$\overline{\mathcal{M}}_{n+1,m+1}(a_0,\dots a_n;b_0,\dots,b_m)\to \overline{C}_{n+1,m+1}$$

which sends a stable map to the stabilization of its domain.

The standard inductive transversality and compactness arguments [25, (9i), (9k), (9l)] show that

Lemma 5. *Coherence data for $\overline{\mathcal{C}}_{n,m}$ exist, and moreover there are choices of coherence data such that the moduli spaces $\mathcal{M}_{n+1,m+1}(a_0,\dots a_n;b_0,\dots,b_m)$ are unions of smooth manifolds of dimensions equal to the indices ind(D_u) of the linearized Cauchy-Riemann operators associated to the elements $u\in\mathcal{M}_{n+1,m+1}(a_0,\dots a_n;b_0,\dots,b_m)$ of a given component. The number of elements $u\in\mathcal{M}_{n+1,m+1}(a_0,\dots a_n;b_0,\dots,b_m)$ for which ind$(D_u)=0$ is finite, and they form a discrete subspace. The Gromov-Floer bordifications $\overline{\mathcal{M}}_{n+1,m+1}(a_0,\dots a_n;b_0,\dots,b_m)$ are compact. For any 1-dimensional component A of $\mathcal{M}_{n+1,m+1}(a_0,\dots a_n;b_0,\dots,b_m)$, the maps added by the Gromov-Floer bordification never contain maps from domains with interior marked points, since those are of virtual codimension at least 2.*

Choose coherence data for $\overline{\mathcal{C}}_{n,m}$ as in the lemma above.

We define $\#\mathcal{M}_{n+1,m+1}(a_0,\dots a_n;b_0,\dots,b_m)$ to be the parity of the (finite) number of zero-dimensional components of $\mathcal{M}_{n+1,m+1}(a_0,\dots a_n;b_0,\dots,b_m)$.

2.7 Proof of Theorem 2

Having defined the moduli space $\mathcal{M}_{C_{n+1,m+1}}(a_0,\dots a_n;b_0,\dots,b_m)$, we can define an operation

$$R: C_\bullet(Fuk(M))\otimes C_\bullet(Fuk(M)^{op})\to \Bbbk$$

by setting

$$R((a_0\otimes\dots\otimes a_n)\otimes(b_0\otimes\dots\otimes b_m))=\#\mathcal{M}_{n+1,m+1}(a_0,\dots a_n;b_0,\dots,b_m).$$

Proposition 3. *Let ϕ be the expression in Equation 5 for M the diagonal bimodule of $Fuk(M)$ (see Remark 4), and let d be the differential on the domain of R. Then above operation satisfies*

$$R(d(x))=\phi(x). \tag{20}$$

Proof. Suppose $x=(a_0\otimes\dots\otimes a_n)\otimes(b_0\otimes\dots\otimes b_m)$. By linearity it suffices to prove the proposition for such x.

The expression $R(d(x))$ is a count of certain configurations of disks contributing to A_∞ operations in $Fuk(M)$ and in $Fuk(M)^{op}=Fuk(M^-)$ (Lemma 3) which lie in zero-dimensional moduli spaces, incident along their negative marked points to pseudoholomorphic annuli which lie in zero-dimensional components of some $\mathcal{M}_{C_{n+1,m+1}}(a'_0,\dots a'_n;b'_0,\dots,b'_m)$; in other words, it is a count of curve configurations as in (B) of Figure 2.

Because we have chosen coherence data as in Lemma 5, the gluing theorem for pseudoholomorphic maps says that every such configuration arises as a boundary point of a one-dimensional component A of the Gromov-Floer compactification of some moduli space of annuli $\mathcal{M}_{n+1,m+1}(a''_0,\dots a''_n;b''_0,\dots,b''_m)$ for some pair of tuples of cyclically composable morphisms $(a''_0,\dots,a''_n)$ and $(b''_0,\dots,b''_m)$. Combining the conjugation needed to define the A_∞ operations in $Fuk(M^-)$ (Section 2.5) with the reversed order of labels of the marked points $\{b_j\}$ relative to the boundary orientation (Section 2.6.1), we see that we must have $a''_i=a_i$ and $b''_i=b_i$ for all i.

The other curve configurations added by the Gromov-Floer bordification to compactify the component A must correspond to stable maps from a pre-stable domain D that upon stabilizing becomes one of the domains depicted in Figure 2. The Lagrangian boundary conditions we consider are exact, so any such stabilization cannot collapse any disks with only one boundary puncture, as such maps can never occur with an exact Lagrangian boundary. By Lemma 5, D cannot have any component containing an interior node. So D must either be a map from a prestable domain that is contributing to the count in Rdx, as in (B), Figure 2, or a map from a pre-stable domain with stabilization a self-gluing of a disk $\overline{\mathcal{C}}_{n,m}$. But, in the latter case, if the stabilization map did not preserve the domain (i.e. unstable strips were present), then gluing the collapsed components back on would mean that u was not a boundary point of a moduli space consisting of maps from annuli of index *exactly* 1, but instead of index *greater* than 1. So in the latter case u a map from an annulus with one self-gluing, exactly as in (C), Figure 2. But these latter map, due to our conventions for positive and negative marked points (Figure **??**) are *exactly* the maps which define ϕ, which is a supertrace, in (20). □

Theorem 2 follows immediately from Lemma 2 and the Proposition, since if x as in (10) represents a Hochschild class then (20) shows that $\phi(x) = 0$. ■

Remark 7. In the general case, where one studies the Fukaya category of compact Lagrangians on a non-exact symplectic manifold, the curves in $\mathcal{M}_{C_{n+1,m+1}}(a_0, \dots a_n; b_0, \dots, b_m)$ might degenerate via boundary disk bubbles, which would invalidate the above argument (ignoring for a moment the thorny issue of virtual cycles). I expect, however, that if the boundary disk bubbles are taken into account in the non-exact case, then the analysis of the degenerations of $\mathcal{M}_{C_{n+1,m+1}}(a_0, \dots a_n; b_0, \dots, b_m)$ would prove that *the un-curved Fukaya category Fuk(M) [**?**] satisfies Equation 2.*

3 Smooth Fukaya Categories

In this section we prove Theorem 3. The strategy is to replace the algebraically defined map

$$\begin{aligned} K_0(WF(M) \otimes WF(M)^{op}) &\xrightarrow{ch} (HH_\bullet(WF(M)) \otimes HH_\bullet(WF(M)^{op}))_0 \\ &\xrightarrow{id \otimes \delta} (HH_\bullet(WF(M)) \otimes HC^-_\bullet(WF(M)^{op}))_1, \end{aligned} \tag{21}$$

where $WF(M)$ denotes the Wrapped Fukaya category of M, with the simpler Floer-theoretic map

$$\begin{aligned} K_0(WF(M \times M^-)) &\xrightarrow{ch} HH_*(WF(M \times M^-)) \xrightarrow{\mathcal{OC}} SH^*(M \times M^-) \\ &= SH^*(M) \otimes SH^*(M^-) \to SH^*(M) \otimes (SC^*_{S^1})^*(M^-). \end{aligned} \tag{22}$$

Here, $\mathcal{OC}$ denotes the *open-closed map*, SH^* denotes *symplectic cohomology*, and $(SC^*_{S^1})^*$ denotes *negative symplectic cohomology*, a symplectic analog of negative cyclic homology. Certain straightforward Floer-theoretic opearations (Appendix 4) together with existing results, allow us to show that the image of the diagonal bimodule under the map in (21) is zero if and only if the image of the *diagonal Lagrangian submanifold* under the map in (22) is zero, whenever M is a *Weinstein domain.* This latter image has a relatively simple description in terms of moduli spaces of holomorphic curves, which we then manipulate using analytic methods to prove the Proposition.

In Section 3.1 we describe our conventions for Wrapped Floer cohomology, symplectic cohomology, the open-closed map, the construction of homology units in symplectic homology, as well as Ganatra's construction of positive/negative/cyclic symplectic cohomology and the corresponding open-closed maps. In Section 3.2 we recall alegbraic constructions such as the category of perfect complexes, and relate the maps (21) and in (22). Finally, in Section 3.3 we perform a sequence of bordisms on moduli spaces of curves to show that the image of the diagonal Lagrangian submanifold under the map in (94) is zero.

3.1 Conventions

Throughout this section, M will be a Liouville manifold equipped with a grading. We will write

$$M = \bar{M} \cup_{\partial \bar{M}} [0, \infty)_r \times \partial \bar{M} \tag{23}$$

for the decomposition of M into a Liouville domain $\bar{M}$ and its *conical end*. We will *assume that the Reeb flow on* $\partial \overline{M}$ *is nondegenerate*; this can be achieved by a generic perturbation of the Liouville vector field on M.

The Wrapped Fukaya category $WF(M)$ is an A_∞ category depending on M and its choice of grading; $WF(M)$ is also dependent on certain auxiliary choices, but it is independent of those choices up to quasi-equivalence. Our conventions for the definition of the Wrapped Fukaya category are those in [17] and [16], which largely agree with those in [1]; see Section 3.3, Definition 6 of [17] for a precise statement of these conventions. We will allow graded non-Pin Lagrangians as objects of $WF(M)$, in which case $WF(M)$ is only defined over a field $\Bbbk$ of characteristic 2. By relying on previous work, we will not need to work explicitly with the Wrapped Fukaya category beyond its formal categorical properties, so we will not describe the construction of the Wrapped Fukaya Category further. We will, however, have to manipulate certain moduli spaces arising in the ∞-mixed complex structure on symplectic cohomology defined in [17]; we describe our conventions, which also agree with those of [1], [17], in section 3.1.2.

3.1.1 Hamiltonians and almost complex structures

This section fixes notation for a set of Hamiltonian and complex-structure terms for perturbations of Floer's equation which are sufficient to achieve transversality for the moduli spaces we will consider and for which an integrated maximum principle [2] can prove compactness of the moduli spaces. We first describe the convenient class of Hamiltonians:

Definition 6. Let

$$\mathcal{H}(M) \tag{24}$$

denote the set of functions $H \in C^\infty(M)$ which, when restricted to the conical end $\mathbb{R} \times \partial \bar{M}$ of M, have the property that for every $r_0 >> 0$ there is an $R > r_0$ and an $\epsilon_R > 0$ such that

$$H(r, y)|_{(R-\epsilon_R, R+\epsilon_R) \times \partial \bar{M}} = r^2. \tag{25}$$

We now describe the convenient class of almost complex structures:

Definition 7. Let

$$\mathcal{J}(M) \tag{26}$$

be the set of almost complex structures of *rescaled contact type*, namely, those almost complex structures J on M such that

$$\lambda \circ J = -r dr. \tag{27}$$

Remark 8. This is the same as the class of almost complex structures used in [16], and a special case of those used in [17].

3.1.2 Conventions about wrapping

The *symplectic cohomology* of M is a chain complex

$$SC^*(M; H^{S^1}, J^{S^1}), \tag{28}$$

depending on the *closed-string Floer data*

$$H^{S^1}(t) \in C^\infty([0,1]_t, \mathcal{H}(M)), \text{a time-dependent family of Hamiltonians, and} \tag{29}$$

$$J^{S^1} \in C^\infty([0,1]_t, \mathcal{J}(M)),\ \text{a time-dependent almost complex structure ,} \tag{30}$$

which satisfy the following

Assumption 1.
The set of closed time-1 orbits of the Hamiltonian vector field $X_{H^{S^1}}$ *associated to the Hamiltonian* H^{S^1}*, denoted by*

$$\mathcal{C}(H^{S^1}), \tag{31}$$

consists entirely of nondegenerate *orbits.*

Throughout this section we will use a convenient notation for a natural set of coordinates on the punctured plane, thought of as a Riemann surface:

$$\begin{aligned} C = (-\infty, \infty)_s \times S^1_t \simeq \mathbb{C}^*_z, \\ z = e^{s+2\pi i t}. \end{aligned} \tag{32}$$

Whenever Assumption 1 holds, one can define, for every pair $y_\pm \in \mathcal{C}(H^{S^1})$, the moduli space of Floer trajectories

$$\mathcal{M}(y_-, y_+) = \left\{ u \in C^\infty(C, M) | (du - X_{H^{S^1}} \otimes dt)^{0,1}_{J^{S^1}} = 0, \lim_{s \to \pm\infty} u = y_\pm \right\} /\mathbb{R}, \tag{33}$$

where the $\mathbb{R}$ quotient is by translation in the s direction. This moduli space has a Gromov-Floer bordification

$$\overline{\mathcal{M}}(y-, y_+). \tag{34}$$

The graded abelian group underlying the symplectic cohomology chain complex is

$$SC^k(M; H^{S^1}, J^{S^1}) = \bigoplus_{\substack{y \in \mathcal{M} \\ CZ(y) = n-k}} \mathbb{Z}/2[y] \tag{35}$$

and to make sense of the differential on this chain complex, one must impose an additional condition, namely

Assumption 2. *For all* $y_\pm \in \mathcal{C}(H^{S^1})$ *the spaces* $\mathcal{M}(y_-, y_+)$ *are smooth manifolds of dimension one less than the Fredholm index of the curves representing points on these spaces, and their Gromov bordifications* $\overline{\mathcal{M}}(y-, y_+)$ *are compact.*

There is a fundamental and standard

Lemma 6. *There exist choices of closed-string Floer data satifying Assumptions 1 and 2. Given such choices, for every* $y_+ \in \mathcal{C}(H^{S^1})$ *the space* $\mathcal{M}(y_-, y_+)$ *is empty for all but a finite number of orbits* y_-*.*

Given such a choice of closed-string Floer data one defines the differential on the cochain complex $SC^k(M; H^{S^1}, J^{S^1})$ to be

$$d[y_+] = \sum_{y_- \in \mathcal{C}(H^{S^1})} \sum_{\substack{u \in \mathcal{M}(y_-, y_+) \\ ind u = 1}} [y_+]. \tag{36}$$

3.1.3 Some tools for action arguments

There is a particularly convenient class of Hamiltonians for symplectic cohomology which we wish to use, namely, the functions

$$H^0 + F(t) = H^{S^1}(t) \in C^\infty([0,1]_t, \mathcal{H}(M))$$

with H^0 a t-independent function on M such that

$$H = r^2 \text{ on } M \setminus \overline{M}, H \text{ a Morse function on } \overline{M} \text{ with no time-1 non-constant Hamiltonian orbits} \tag{37}$$

and $F(t)$ a function zero on an open neighborhood of $\overline{M}$.

In our conventions, action of an orbit of a Hamiltonian H on M is

$$\mathcal{A}_H(x) = -\int_x \lambda + \int_0^1 H(x(t))dt, \quad \text{for } x \in \mathcal{C}_M(H) \tag{38}$$

A standard calculation [23] shows that the action of any orbit of H contained in $M \setminus \bar{M}$ is negative, so for such $H^{S^1}(t)$, the only orbits of positive action are (constant) orbits of $H|_{\bar{M}}$. By choosing F sufficiently C^1-small and only supported in a neighborhood of the orbits of $H_{\bar{M}}$, and appealing to the standard transversality and compactness arguments for Floer trajectories, we can conclude that

Lemma 7. *There exist closed-string Floer data (H^{S^1}, J^{S^1}) with H^{S^1} as in (37, so that the Floer data satisfy Assumptions 1, 2, and moreover:*

1. *the only orbits of positive action are the constant orbits,*
2. *there are finitely many orbits of action greater than any fixed number;*
3. *If we look instead at orbits of x of $-H$, then there are only finitely such orbits of $\mathcal{A}_{-H}(x)$ less than any fixed number.*

3.1.4 The ∞-mixed complex structure on symplectic cohomology

In contrast to the very standard preceding section, this section describes a sequence of moduli spaces recently introduced by Ganatra [17] which equip symplectic cohomology with the structure of an ∞-mixed complex. The idea that symplectic cohomology should have such a structure is very old, with motivation being traced back to Floer's implementation [11] of Witten's use of S^1 localization on the free loop space [30]. While there are Morse-Bott models for symplectic cohomology for time-independent hamiltonians [5], for which the BV operator is strict, the general strategy of using the a convenient cell structure on BS^1 to build the ∞-mixed complex structure is very old [29] [23] [6]; the moduli spaces in [17] arise from a different cell structure for BS^1 than the most traditional one. A variant of the ∞-mixed complex structure is carefully constructed in Zhao's paper [31] (see also [3]). There, Zhao discusses the subtle interaction between the action filtration and the completions involved in cyclic homology; see Remark 9 for an example.

We begin by recalling the definition of the domains of Ganatra's moduli spaces:

Definition 8. An *r-point angle-decorated cylinder* is a sequence of points $p_1, \ldots, p_r \in C$ satisfying

$$(p_1)_s \leq \ldots \leq (p_r)_s. \tag{39}$$

The *heights* and *angles* associated to this data are

$$h_i = (p_i)_s, s = 1, \ldots, r, \quad \text{and } \theta_i = (p_i)_t, i = 1, \ldots, r, \text{ respectively.} \tag{40}$$

The *moduli space of r-point angle-decorated cylinders*

$$\mathcal{M}_r \tag{41}$$

is the space of r-point angle-decorated cylinders modulo simultaneous s-translation of all the points $\{p_i\}$. We think of this as a moduli space of cylinders equipped with certain auxiliary marked points.

Every r-point angle-decorated cylinder $\{p_i\}$ has an associated positive cylindrical end at $+\infty$ on C given by

$$\begin{aligned} \epsilon_+ : [0, \infty) \times S^1 &\to C \\ (s,t) &\mapsto (s + h_r + 1, t), \end{aligned} \tag{42}$$

and negative cylindrical end at $-\infty$ given by

$$\begin{aligned} \epsilon_+ : (-\infty, 0] \times S^1 &\to C \\ (s,t) &\mapsto (s - (h_1 + 1), t). \end{aligned} \tag{43}$$

The space of *broken r-point angle-decorated cylinders*

$$\overline{\mathcal{M}}_r = \bigsqcup_s \bigsqcup_{\substack{j_1,\ldots,j_s \\ \sum j_i = r}} \overline{\mathcal{M}}_{j_1} \times \ldots \times \overline{\mathcal{M}}_{j_s} \tag{44}$$

is a compactification of $\overline{\mathcal{M}}_r$ defined as a smooth manifold with corners, with the s-fold broken configurations comprising the codimension s boundary, and the manifolds-with-corners structure defined using the gluing maps determined by the ends (42) and (43). (This makes sense because the ends induced on a glued curve exactly agree with the ends in (42) and (43).)

The points defining an angle-decorated curve are allowed to have coinciding heights, giving rise to certain partially-defined forgetful maps between these moduli spaces which will be later important to state the consistency conditions for Floer data defining the ∞-mixed complex structure.

Definition 9. Define the space

$$\mathcal{M}_r^i, 1 \le i \le r-1 \tag{45}$$

as the locus in $\mathcal{M}_r$ for which $h_i = h_{i+1}$. Let

$$\overline{\mathcal{M}}_r^i \tag{46}$$

be the closure of $\mathcal{M}_r^i$ in $\overline{\mathcal{M}}_r$. There is a map

$$\pi_i : \mathcal{M}_r^i \to \mathcal{M}_{r-1};\ \pi_i(p_1, \ldots, p_r) = (p_1, \ldots, p_i, p_{i+2}, \ldots, p_r), \tag{47}$$

which extends uniquely to a continuous map

$$\pi_i : \overline{\mathcal{M}}_r^i \to \overline{\mathcal{M}}_{r-1}. \tag{48}$$

The manifold with corners $\overline{\mathcal{M}}_r$ thus has a boundary $\partial\overline{\mathcal{M}}_r$ covered the images of maps

$$\overline{\mathcal{M}}_k \times \overline{\mathcal{M}}_{r-k} \to \partial\overline{\mathcal{M}}_r, \text{ and} \tag{49}$$

$$\overline{\mathcal{M}}_r^i \to \partial\overline{\mathcal{M}}_r. \tag{50}$$

Definition 10. Let (H^{S^1}, J^{S^1}) be a choice of closed string Floer data satifying Assumptions 1 and 2. A *universal and consistent choice of Floer data for the S^1-action* is a choice, for every $r \ge 1$ and every representative $\hat{C} = (\hat{C}_1, \ldots, \hat{C}_s) \in \mathcal{M}_r$ with $\hat{C}_i = \{p_k\}_{k=1}^{j_i}$, of surface dependent Hamiltonians

$$H^{\hat{C}_i} : C \to \mathcal{H}(M) \tag{51}$$

and a surface-dependent almost complex structures

$$J^{\hat{C}_i} : C \to \mathcal{J}(M), \tag{52}$$

called the Floer data associated to $\hat{C}$, which are compatible with the closed-string Floer data in the sense that

$$\begin{aligned} (\epsilon_\pm^i)^* H^{\hat{C}}(s,t) &= H^{S^1}(t), \text{ and} \\ (\epsilon_\pm^i)^* J^{\hat{C}}(s,t) &= J^{S^1}(t) \end{aligned} \tag{53}$$

where $\epsilon^i_\pm$ is are the canonical cylindrical ends of (42) (43) associated to the angle-decorated cylinder $\hat{C}_i$.

These Floer data are required to vary smoothly over $\overline{\mathcal{M}}_r$, in the sense that the Floer data vary smoothly over the interiors of the $\mathcal{M}_r$, and they agree to infinite order at the boundary strata with: the Floer data near the boundary strata of the form (49) induced by gluing the product Floer data chosen lower-dimensional moduli spaces; and, with the Floer data induced near the boundary strata of the form (50) by pulling back via the map π_i (48) the Floer data choen on $\overline{\mathcal{M}}_{k-1}$.

Given a universal and consistent choice of Floer data for the S^1 action, Ganatra defines the moduli space

$$\mathcal{M}_r(y_-, y_+) = \left\{ u : S^1 \times \mathbb{R} \to M, p = (p_1, \dots, p_r) \in \mathcal{M}_k | (du - X_p \otimes dt)^{0,1}_{J_p} = 0; \lim_{s \to \pm\infty} u(s,t) = y_\pm(t); \right\} \tag{54}$$

with Gromov-Floer compactification

$$\overline{\mathcal{M}}_r(y_-, y_+) \tag{55}$$

and defines, for $k \geq 1$, the operation

$$\delta_k : SC^*(M) \to SC^{*-2k+1}(M)$$
$$\delta_k([y_+]) = \sum_{y_- \in \mathcal{C}(H^{S^1})} \sum_{\substack{u \in \mathcal{M}_r(y_-, y_+) \\ ind(D_u) = -\dim \mathcal{M}_k}} [y_-]. \tag{56}$$

He then proves

Proposition 4. *[17] Universal and consistent choices of Floer data for the S^1 action exist. Writing δ_0 for the differential on symplectic cohomology, the operations $\{\delta_i\}_{i=0}^\infty$ define the structure of an ∞-mixed complex on symplectic cohomology as in Equation 7. Moreover, given any pair of choices of closed-string Floer data $(H_i^{S^1}, J_i^{S^1})_{i=1,2}$ and corresponding Floer data for the S^1 action, there is continuation map*

$$SC^*(M; H_1^{S^1}, J_1^{S^1}) \to SC^*(M; H_2^{S^1}, J_2^{S^1}) \tag{57}$$

which is a quasi-isomorphism and which extends to a map of ∞-mixed complexes.

Moreover, if the closed-string Floer data are chosen as in Lemma 7 then the Hamiltonian terms of the Floer data for the S^1-action can be chosen to be independent of $p \in \overline{\mathcal{M}}^r$ when restricted to $\overline{M}$ (see [17, Section 4.4]).

The negative cyclic complex, periodic cyclic complex, and cyclic complex for the ∞-mixed complex structure described above define the underlying complexes for *negative symplectic cohomology* $(SC^-_{S^1})^*$, *perodic symplectic cohomology* $(SC^\infty_{S^1})^*$, and *cyclic symplectic cohomology* $(SC^+_{S^1})^*$, respectively. The cohomologies of the corresponding complexes will be denoted by $(SH^{-/\infty/+}_{S^1})^*$.

3.1.5 Recollection of Wrapped Floer cohomology

While we will avoid most of the technical details underlying the Wrapped Fukaya category, we must now recall the definition of its morphism groups.

We say that a Lagrangian submanifold $L \subset M$ is *conical at infinity* f

$$L \cap [r, \infty) \times \partial\bar{M} = [r, \infty) \times \partial L \tag{58}$$

where $r > 0$ and $\partial L \subset \partial\bar{M}$ is a Legendrian submanifold. The objects of the Wrapped Fukaya category are Lagrangians conical at infinity that are equipped with auxiliary data, e.g. gradings.

Given two Lagrangians L_0 and L_1 that are each conical at infinity, one chooses a function $H_t : [0,1]_t \to \mathcal{H}(M)$ such that the image of L_0 under the time 1 flow of H_t is transverse to L_1, and defines the abelian group underlying the Hom complex $Hom_{WF(M)}(L_0, L_1)$ to be free $\mathbb{Z}/2$ vector space on

$$\mathcal{C}(L_0, L_1; H_t) = \{\gamma : [0,1] \to M \mid \gamma'(t) = X_{H_t}(\gamma(t)), \gamma(i) \in L_i \text{ for } i = 0,1\}. \tag{59}$$

The differential then counts solutions to Floer's equation 14 for $J : [0,1] \to \mathcal{J}(M)$. It is standard that for generic J these moduli spaces of solutions consist of regular trajectories, and that for such (H, J), the Gromov-Floer bordifications of the moduli spaces are compact.

In fact, in [16], which we use heavily, one requires a number of additional constraints; namely, Hamiltonian terms H_Δ are all strictly quadratic on the conical end of M; and only a finite set of conical Lagrangians, specified a-priori, is chosen as objects of the Wrapped Fukaya category. However, given any finite set of conical Lagrangians, the transversality condition on the Hamiltonian chords between the Lagrangians can be achieved by a Hamiltonian perturbation of these conical Lagrangians, so this is not a constrant; and moreover, the condition of strong nondegeneracy implies that there exist a finite number of conical Lagrangians which generate the category in the sense that adding any other conical Lagrangian to the category does not change its category of twisted complexes. Since the truth of Theorem 3 only depends on isomorphism class of the category of perfect complexes of the Wrapped Fukaya category, these choices for the definition of the Wrapped Fukaya category do not affect our results.

3.1.6 Open-closed map

In this section we describe in detail one component of the open closed map [1], [17]

$$\mathcal{OC} : C^{nu}_{*-n}(WF(M)) \to SC^*(M). \tag{60}$$

Specifically, in Section 3.2 M will be a product of a Liouville domain with its opposite, and the object $\Delta \in WF(M)$ will have as underlying Lagrangian the graph of the identity map. We will describe how to define $\mathcal{OC}$ on

$$Hom^*(\Delta, \Delta) \subset C^{nu}_*(WF(M)). \tag{61}$$

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk, let $p^+ = 1$ be a positive boundary marked point and let $p^- = 0$ be a negative boundary marked point. Define a negative cylindrical end at p^- by

$$\begin{aligned} \epsilon_- &: (-\infty, 0]_s \times S^1_t \\ \epsilon_-(s,t) &= e^{(s-1)+i(2\pi t+\pi)} \end{aligned} \tag{62}$$

and choose a positive strip-like end ϵ_+ at p_+ with image disjoint from that of ϵ_-. Write

$$\mathbb{D}^{1|1} = \mathbb{D} \setminus \{p^\pm\} \tag{63}$$

and choose data

$$\begin{aligned} \alpha &\in \Omega^1(\mathbb{D}^{1|1}) \\ J &\in C^\infty(\mathbb{D}^{1|1}, \mathcal{J}(M)) \\ H &\in C^\infty(\mathbb{D}^{1,1}, \mathcal{H}(M)) \end{aligned} \tag{64}$$

and satisfying

Assumption 3.
- *The restriction of α to $\partial\mathbb{D}^{1|1}$ vanishes;*
- *The 1-form α satisfies $d\alpha \leq 0$ relative to the trivialization of $\Omega^2(\mathbb{D}^{1|1})$ coming from the standard symplectic form on the disk;*

- *Thinking of $H \otimes \alpha$ as a Hamiltonian-valued 1-form, and writing X_α for the corresponding Hamiltonian-vector-field-valued 1-form, one has that the data (H, α, J) are* compatible *with the Floer data (H^Δ, J^Δ) chosen for Δ in the sense that*

$$(\epsilon_+^* X_\alpha)(s,t) = H^\Delta(t), (\epsilon_+^* J)(s,t) = J^(\Delta)(t); \tag{65}$$

- *Writing X_α as before, one has that the data (H, α, J) are* compatible *with the closed-string Floer data (H^{S^1}, J^{S^1}) used to define symplectic cohomology, in the sense that*

$$(\epsilon_-^* X_\alpha)(s,t) = H^{S^1}(t), (\epsilon_-^* J)(s,t) = J^(S^1)(t); \tag{66}$$

Given (α, J) as in 64 satisfying Assumption 3 and elements $y_+ \in \mathcal{C}(L_\Delta, L_\Delta, H^\Delta)$, $y_- \in \mathcal{C}(H^{S^1})$, one write X_α for the vector field valued 1-form associated to $H \otimes \alpha$ and uses it to define the moduli space

$$\mathcal{M}^{1|1}(y_-, y_+) = \left\{ u \in C^\infty(\mathbb{D}^{1|1}, M) \;\middle|\; \begin{array}{c} (du - X_\alpha)_J^{0,1} = 0 \\ u(\partial \mathbb{D}^{1|1}) \subset L(\Delta) \\ \lim_{s \to \pm\infty} \epsilon_\pm u = y_\pm \end{array} \right\} \tag{67}$$

with Gromov-Floer bordification $\overline{\mathcal{M}}^{1,1}(y_-, y_+)$. Standard transversality and compactness arguments show:

Lemma 8. *For generic choices of (α, J) satisfying Assumption 3, the spaces $\mathcal{M}^{1|1}(y_-, y_+)$ of Eq. 67 are disjoint unions of smooth manifolds of expected dimension, and their Gromov-Floer bordifications $\overline{\mathcal{M}}^{1,1}(y_-, y_+)$ are compact. Moreover, given any $y_+ \in \mathcal{C}(L_\Delta, L_\Delta, H^\Delta)$, the space $\mathcal{M}^{1|1}(y_-, y_+)$ is empty for all but finitely many closed orbits y_-.*

Choosing (α, J) so that Lemma 8 applies, one defines $\mathcal{OC}$ on $Hom^*(\Delta, \Delta)$ by the equation

$$OC([y_+]) = \sum_{y_- \in \mathcal{C}(H^{S^1})} \sum_{\substack{u \in \overline{\mathcal{M}}^{1|1}(y_-, y_+) \\ \text{ind } u = 0}} [y_-]. \tag{68}$$

The following proposition is the main result of [17]:

Proposition 5. *The map defined in Eq. 68 admits an extension to a map of ∞-mixed complexes*

$$\mathcal{OC} : C^{nu}_{*-n}(WF(M)) \to SC^*(M).$$

The degree zero term of the map $\mathcal{OC}$ of the map of Proposition 5, i.e. the underlying map of chain complexes, may or may not be an equivalence.

Definition 11. We say that M is *nondegenerate* when the map $\mathcal{OC}$ of Proposition 5 is a quasi-isomorphism.

One has the following fundamental result

Proposition 6. *([7, Theorem 1.4], see also [18]) If M is a Weinstein domain then it is nondegenerate.*

Remark 9. The combination of Proposition 5 and Proposition 6 shows that the negative/periodic/cyclic symplectic cohomology described above are invariants of the Wrapped Fukaya category of a Liouville domain. In [31], Zhao introduces a different variant of periodic symplectic cohomology $(SH^*\infty, 2_{S^1})^*$ which satisfies a *localization theorem*: in fairly general circumstances, one has an isomorphism

$$(SH^{\infty,2}_{S^1})^* \simeq H^*(M)((u)). \tag{69}$$

Such a localization theorem cannot hold for the version of periodic symplectic cohomology used in this paper. Indeed, subcritical handle surgery on M can easily change the cohomology of M but leaves the Wrapped Fukaya category unchanged [18], thus changing the right hand side of (69) while leaving the left hand side the same.

3.1.7 Homology units

The final moduli space we describe in this preliminary section is the moduli space giving rise to *homology units* (Defintion 2) in the Wrapped Fukaya category.

Let Δ be an object of the category $WF(M)$ with underlying Lagrangian L_Δ.

Equip the unit disk $\mathbb{D} \subset \mathbb{C}$ with one negative boundary marked point $p^- = -1$ and choose a negative strip-like end ϵ_- at p^-. Define the Riemann surface

$$\mathbb{D}^{-1} = \mathbb{D} \setminus \{p^-\} \tag{70}$$

and choose data

$$\begin{aligned} \alpha &\in \Omega^1(\mathbb{D}^{-1}, C^\infty(M)) \\ J &\in C^\infty(\mathbb{D}^{-1}, \mathcal{J}(M)) \\ H &\in C^\infty(\mathbb{D}^{1,1}, \mathcal{H}(M)) \end{aligned} \tag{71}$$

satisfying

Assumption 4.
- *The restriction of α to $\partial\mathbb{D}^{1|1}$ vanishes;*
- *The 1-form α satisfies $d\alpha \leq 0$ (see 3)*
- *Thinking of $H \otimes \alpha$ as a Hamiltonian-valued 1-form, and writing X_α for the corresponding Hamiltonian-vector-field-valued 1-form, one has that the data (H, α, J) are* compatible *with the floer data (H^Δ, J^Δ) chosen for Δ in the sense that*

$$(\epsilon_-^* X_\alpha)(s,t) = H^\Delta(t), (\epsilon_-^* J)(s,t) = J^(\Delta)(t); \tag{72}$$

Given a pair (α, J) as in (71) satisfying Assumption 4, define for every $y_- \in \mathcal{C}(L_\Delta, L_\Delta; y_-)$, the moduli space

$$\mathcal{M}^{-1}(y_-) = \{u \in C^\infty((\mathbb{D}^{-1}, \partial\mathbb{D}^{-1}), (M, L_\Delta)) | (du - X_\alpha)_J^{0,1} = 0; \lim_{s \to -\infty} \epsilon_-^* u = y_-\} \tag{73}$$

where X_α is the vector-field valued 1-form associated to $H \otimes \alpha$, and write $\overline{\mathcal{M}}^{-1}(y_-)$ for the Gromov-Floer bordification of the moduli space above. Standard compactness and transversality theory proves the lemma below:

Lemma 9. *For generic choices of (α, J) as in (71) satisfying Assumpotion 4, the spaces $\overline{\mathcal{M}}^{-1}(y_-)$ are smooth manifolds of expected dimension and their bordifications $\overline{\mathcal{M}}^{-1}(y_-)$ are compact.*

Choosing (α, J) so that Lemma 9 applies, one defines

$$\begin{gathered} e_\Delta \in WF(M)(\Delta, \Delta) \\ e_\Delta = \sum_{y_- \in \mathcal{C}(L_\Delta, L_\Delta; H^\Delta)} \sum_{\substack{u \in \overline{\mathcal{M}}^{-1}(y_-) \\ \operatorname{ind} u = 0}} [y_-]. \end{gathered} \tag{74}$$

A standard gluing argument proves the

Lemma 10. *The element e_Δ is a homology unit for Δ.*

3.1.8 Floer homology on product manifolds

Shortly, we will wish to study Floer theory on $M \times M^-$, where M is a Liouville manifold. Unfortunately, there is a well-known technical difficulty that $\bar{M} \times \bar{M}^-$ is a manifold-with-corners, and thus *not* a Liouville domain. However, $M \times M-$ does have a Liouville structure with the radial coordinate

$$r_{M \times M^-} = r_M + r_{M^-}$$

and corresponding conical end

$$r^{-1}_{M \times M^-}((\epsilon, \infty)).$$

With this choice of Liouville structure, $\overline{M \times M^-}$ is a manifold with boundary that "smooths the corners" of $\overline{M} \times \overline{M^-}$. One can use this Liouville structure to define $WF(M \times M^-), SH^*(M, \times M^-)$; however, with this definition these invariants will not obviously satisfy any "Kunneth Formulae" because, for example, given a pair of Lagrangians $L \subset M, L^- \subset M^-$ that are conical at infinity, the product $L \times L^-$ is not conical at infinity with respect to the Liouville structure of $M \times M^-$. However, for product Lagrangians, one has a natural choice of "split Floer data": namely, one requires that Hamiltonian terms in Floer's equation are sums of Hamiltonian terms on M and on M^-, one requires that the almost complex structures are products, etc. The advantage of the definitions using "split Floer data" is that Kunneth-type formulae are manifest. There are then several approaches of compairing Fukaya categories defined using split Floer data with Fukaya categories compatible with the Lioville structure on $M \times M^-$ which we will review in Section 3.2.3. For technical simplicity, we will use the split variant of the Wrapped Fukaya category to define the Fukaya category of a product. In this section, we review what we need about Fukaya-categorical constructions using split Floer data.

Write π_M, π_{M^-} for the projections to the corresponding factor of $M \times M^-$. Given a choice of closed-string Floer data (H, J) on M satisfying Assumptions 1 and 2, the symplectic cohomology with the corresponding split Floer data is defined to be

$$SC^*_{split}(M \times M^-; \pi^*_M H + \pi^*_{M^-} H, (J, -J))$$

in the sense that one uses the time-dependent Hamiltonian and almost complex structure

$$\begin{aligned} H_{split} &= \pi^*_M H + \pi^*_{M^-} H \\ J_{split} &= J \oplus -J \end{aligned} \tag{75}$$

in the definition of symplectic cohomology as in Section 3.1.2; even thought H_{split} and J_{split} are not compatible with the Liouville structure on $M \times M^-$, the definitions still make sense.

Symplectic cohomology then almost tautologically satisfies the Kunneth isomorphism

$$SC^*_{split}(M \times M^-; \pi^*_M H + \pi^*_{M^-} H, (J, -J)) \simeq SC^*(M; H, J) \otimes SC^*(M^-; H, -J); \tag{76}$$

every orbit of H_{split} is the product of an orbit of H on M and an orbit of H on M^-, and the map simply sends orbits of H_{split} to the tensor products of the corresponding orbits on M and M^-.

Using the same closed-string Floer data on M, one can define the Wrapped Floer homology of the diagonal $\Delta \subset M \times M^-$

$$WC^*_{split}(\Delta, \Delta; \pi^*_M(H_{t/2}) + \pi^*_{M^-} H_{1-t/2}; (J, -J)) \tag{77}$$

which satisfies a "Kunneth formula"

$$WC^*_{split}(\Delta, \Delta; \pi^*_M(H_{t/2}) + \pi^*_{M^-} H_{1-t/2}; (J, -J)) \simeq SH^*(M; H, J). \tag{78}$$

There is a corresponding constructions for the wrapped Floer homology of split Lagrangians

$$WC^*_{M \times M^-}(L_1 \times L_2, L'_1 \times L'_2) \simeq WC^*_M(L_1, L'_1) \times WC^*_{M^-}(L_2, L'_2). \tag{79}$$

In [16, Section 8], Ganatra packages the split wrapped Floer homologies (77) and (79) into an A_∞ category $\mathcal{W}^2$ containing as objects the product Lagrangians $L_1 \times L_2 \subset M \times M^-$, possibly equipped with Pin structures and gradings, and also the (suitably decorated) diagonal Lagrangian $\Delta_L \subset M \times M^-$. Moreover, this category admits a cohomologically fully faithful A_∞ functor

$$M : \mathcal{W}^2 \to WF(M) - mod - WF(M). \tag{80}$$

The functor M plays the role of the categorical Kunneth map in that paper.

Given a choice of the data (α, J, H), as described in Section 3.1.6, that is needed to define the maps

$$\mathcal{OC} : WF(M)^*(L_i, L_i) \to SH^*(M) \tag{81}$$

for all objects $L_i \in WF(M)$, one has the data $(\alpha, -J, H)$ needed to define the corresponding maps

$$\mathcal{OC} : WF(M^-)^*(L_i, L_i) \to SH^*(M^-) \tag{82}$$

$$\mathcal{OC}_{split} : WC^*_{M\times M^-}(L_1 \times L_2, L_1 \times L_2) \to SC^*_{split}(M \times M^-). \tag{83}$$

defined in the two equivalent ways: either by composing the isomorphisms 79 and (76) with the tensor product of the maps (81) and (82), or alternatively by using the data $(\alpha, (J, -J), \pi^*_M H + \pi^*_{M^-} H)$ to define such a map using the moduli space (67) with target $M \times M^-$ exactly as in Section 3.1.6 via the formula (68).

In Appendix 4, we show the

Proposition 7. *The maps $\mathcal{OC}_{split}$ defined in (83) extend to a chain map*

$$\mathcal{OC}_{split} : C_{*-2n}(\mathcal{W}^2) \to SC^*(M \times M^-).$$

3.2 Algebraic constructions

In this section address some technicalities about about K theory and the noncommutative chern character, and then explain how to reduce the proof of 3 to a computation involving equivariant symplectic cohomology.

3.2.1 Perfect complexes

In this section we briefly review the notion of perfect module over an A_∞-category A.

There is an A_∞ category $Mod(A)$ of right A_∞ modules over A. Its cohomology category $H^0(Mod(A))$ is a Karoubi-complete triangulated category. We say that an element $X \in H^0(Mod(A))$ is *perfect* if it is isomorphic in $H^0(Mod(A))$ to an image of a projector $p \in End(X')$, where $X' \in H^0(Mod(A))$ is isomorphic to an iterated cone on $A \in H^0(Mod(A))$. We say that X is *generated* by some other elements of $H^0(Mod(A))$ if it is isomorphic to an iterated cone on sums of those elements. The A_∞ category $Perf(A)$ is the full A_∞ subcategory of $Mod(A)$ with objects those which lift perfect objects of $H^0(Mod(A))$.

Given a quasi-equivalence $\phi : A' \to A$ between A_∞ categories A', A, we have a pullback A_∞-functor $\phi^* : ModA \to ModA'$ and corresponding triangulated functor $\phi^* : H^0(ModA) \to H^0(ModA')$. It is a theorem that the pullback of a perfect module by a quasi-equivalence ϕ is perfect, and that the restriction

$$\phi^* : H^0(PerfA) \to H^0(PerfA')$$

is an equivalence of triangulated categories, and thus

$$\phi^* : PerfA \to PerfA'$$

is a quasi-equivalence.

The group $K_0(A)$ is the K-group of the triangulated category $H^0(Perf(A))$.

Remark 10. Efimov states conjecture 2 when A is a *dg*-algebra, rather than an A_∞ category, and nominally uses the definition of $K_0(A)$ as the Waldhausen K group of the Waldhausen category of cofibrant perfect A modules. When we work over a field there is no cofibrancy assumption; moreover, this definition gives the same group as the one described above.

3.2.2 The noncommutative chern character

In this section we recall the definition of the noncommutative chern character. Given a unital A_∞ algebra A and a perfect unital A-module M, recall that Shklyarov, in [27], defines the *noncommutative Chern character* $ch(M)$ to be the image of the unit morphism of M in the cyclic bar complex of $Perf(A)$. An similar definition is given in [26, Definition 5.13] for c-unital categories: namely, the definition of $ch(M)$ for perfect modules M over c-unital categories A is the image of the cohomological unit of M in the cyclic bar complex of $Perf(A)$. This agrees with Skhlyarov's definition, since unital A_∞ categories are c-unital, and moreover, by Proposition 2, c-unital categories A admit homotopy units

$$\phi : A \to A',$$

which induce maps

$$\phi_* : Perf(A) \to Perf(A'),$$

$$\phi_* : C_\bullet(A) \to C_\bullet(A')$$

and the image of a cohomological unit for $M \in Perf(A)$ in $HH_*(A)$ is sent, under the induced map $\phi_* : HH_*(A) \to HH_*(A')$ to the image of a strict unit for $\phi_* M$. Finally, this notion of of the noncommutative chern character generalizes Connes' definition of the Chern character map for projective modules M over associative algebras [22]. It is a theorem that the noncommutative chern character of M descends to its class in $K_0(A)$.

Remark 11. Efimov cites [8] for his definition of the chern character, which is rather inexplicit. It is (presumably) obvious to experts that the definition given by Shklyarov agrees with all other definitions. In the calculations in his paper he just uses Shklyarov's definition, (or rather, the image of Skhlyarov's definition in the reduced bar complex,). In any case, in our paper we simply interpret conjecture 2 using Shklyarov's definition, which is consistent with Efimov's computations.

There is a Kunneth map

$$K : HH_*(WF(M) - mod - WF(M)) \to HH_*(WF(M)) \otimes HH_*(WF(M)^{op}) \tag{84}$$

which can be defined as follows. Let $WF(M)_L$ and $WF(M)_R$ be the categories of Yoneda images of objects of $WF(M)$ in the *dg*-category of left or right A_∞ modules over $WF(M)$, respectively. Inside the bimodule category $WF(M) - mod - WF(M)$, which is a *dg-category*, there is the subcategory $WF(M)_L \otimes_k WF(M)_R$ of tensor products of elements of $WF(M)_L$ and $WF(M)_R$; this isomorphic to the tensor product of the *dg*-categories $WF(M)_L$ and $WF(M)_R$, and is Morita-equivalent to $WF(M) - mod - WF(M)$ via the inclusion. The map K is then the composition of the isomorphism

$$HH_*(WF(M) - mod - WF(M)) \to HH_*(WF(M)_L \otimes_k WF(M)_R)$$

induced by the Morita equivalence, with the standard Kunneth map for the tensor product of dg algebras, together with the Yoneda isomorphisms

$$Y_L : WF(M) \simeq WF(M)_L \tag{85}$$

$$Y_R : WF(M)^{op} \simeq WF(M)_R \tag{86}$$

3.2.3 Reducing the conjecture to symplectic cohomology

We now explain how to compare the maps of (21) and (22), at least on the diagonal. See Appendix **??** for a discussion regarding different definitions of opposite categories.

We wish to prove the following

Proposition 8. *If M is strongly nondegenerate, then image of the diagonal bimodule under the map (21) is zero if and only if the image of the diagonal Lagrangian under the map (22) is zero.* □

First, as stated in section 3.1.8, there is a cohomologically full and faithful functor [16]

$$\mathbf{M} : \mathcal{W}^2 \to WF(M) - mod - WF(M). \tag{87}$$

Moreover, [16] proves that this sends the diagonal Lagrangian to the diagonal bimodule, and sends product Lagrangians $L_1 \times L_2$ to the tensor products of the corresponding Yoneda modules

$$Y_L(L_1) \otimes Y_R(L_2) \in WF(M)_L \otimes_k WF(M)_R. \tag{88}$$

The first order term of the A_∞ functor M induces a map

$$SH^*(M) = Hom_{\mathcal{W}^2}(\Delta, \Delta) \to Hom_{WF(M)-mod-WF(M)}(\Delta, \Delta) = HH^*(WF(M)) \tag{89}$$

where the first equality is the Kunneth isomorphism (78) and the last equality is the definition of Hochschild cohomology. By [16], this map agrees on homology with the *open-closed map* [**?**], and this map is a ring isomorphism whenever M is nondegenerate. This, the image of the unit for diagonal Lagrangian, is sent to the unit of the diagonal bimodule, $1_\Delta \in HH^*(WF(M))$, by M.

The functor $\mathbf{M}$ and the Kunneth map on Hoschild homology induce the following diagram:

$$\begin{array}{ccc} HH_*(\mathcal{W}_2) & \xrightarrow{K \circ \mathbf{M}} & HH_*(WF(M)) \otimes HH_*(WF(M)^{op}) \\ \downarrow{\scriptstyle OC} & & \downarrow{\scriptstyle OC \otimes OC} \\ SH^*(M \times M^-) & \longrightarrow & SH(M) \otimes SH(M^-). \end{array} \tag{90}$$

The top left-corner contains the unit of the diagonal Lagrangian, which is sent by the top arrow to $K \circ ch_0(\Delta)$ by the previous discussion. Because vertical maps OC on the right extend to equivalences of ∞-mixed complexes, we have the immediate

Lemma 11. *If the diagram (90) commutes then Proposition 8 holds .*

Unfortunately, it is technically elaborate to check that the above diagram commutes. Some comments which indicate possible methods for checking the comutativity of the above diagram can be found in Remarks **??**, **??** of [16]. Furthermore, there are other diagrams of this sort that one might write down which would suffice to prove Propostion 8; for example, one could use the Kunneth map defined in [18]. We will content ourselves with giving a proof of Proposition 8 under the strong nondegeneraccy assumption, by proving that

$$OC(1_\Delta) = OC \otimes OC \circ K \circ ch_0(\Delta).$$

Indeed, $\Delta \in K_0(WF(M) - mod - WF(M))$ is a linear combination of the classes of the Yoneda bimodules, so $ch_0(\Delta)$ is a linear combination of the images of the units of the Yoneda bimodules on Hochschild homology. Therefore, to prove the above it suffices to check that

$$OC(ch_0(L_1 \times L_2)) = OC \otimes OC \circ K \circ ch_0(Y_L(L_1) \otimes_k Y_R(L_2)). \tag{91}$$

Now, The first order term of $\mathbf{M}$ applied to $L_1 \times L_2$ also induces maps

$$\begin{gathered} WF_M(L_1, L_1) \otimes WF_{M^-}(L_2, L_2) = End_{W^2}(L_1 \times L_2) \xrightarrow{\mathbf{M}} \\ End_{WF(M)-mod-WF(M)}(Y_L(L_1) \otimes_k Y_R(L_2)) \simeq WF(M)(L_1, L_1) \otimes WF(M^-)(L_2, L_2) \end{gathered} \tag{92}$$

where the first equality is coming from the fact that W^2 is defined using split floer data, and the last equality follows from the (algebraic!) Kunneth formula for endomorphisms of Yoneda bimodules. All of these maps are unital maps. The fact that $\mathcal{OC}$ respects the first Kunneth isomorphism in (92) by (83) then proves (91). This concludes the proof or Proposition 8.

3.2.4 Opposite ends, Hamiltonians, and forms

In this completely elementary section, we carefully explain how to think of $SH^*(M^-)$ as defined using Floer trajectories in M, as well as a certain relation between positive and negative ends of holomorphic curves, as preparation for the core argument in Section 3.3.

Let H_t, J_t be the closed-string Floer data (29) chosen to define symplectic cohomology of M. Then the maps defining the differential satisfy Floer's equation on M with Hamiltonian term H_t, which in our conventions can be written in local coordinates as

$$u : \mathbb{R}_s \times S^1_t \to M, \;\; \partial_s u + J_t(\partial_t u - X^M_H) = 0$$

where we have introduced the notation X^M_H to denote the Hamiltonian vector field associated to H *using the symplectic form of* M. In section 3.1.8, we use the Floer data $(H_t, -J_t)$ to define the symplectic cohomology $SH^*(M^-; H_t)$ of M^-. Then the maps defining the differential satisfy the equation

$$u : \mathbb{R}_s \times S^1_t \to M^- = M, \;\; \partial_s u - J_t(\partial_t u - X^{M^-}_H) = \partial_s u - J_t(\partial_t u - X^M_{-H}) = 0. \tag{93}$$

Given a solution u of (93), the map $\tilde{u}(s,t) = u(-s,t)$ solves the Floer equation on M with Hamiltonian $-H$, and the differential for $SH^*(M^-; H_t)$ counts such maps to M in the sense that the differential of an orbit y^- of $-H_t$ on M is given by a sum of solutions u to

$$\partial_s u + J_t(\partial_t u - X^M_{-H})$$

with the coefficient of y^+ given by the count of solutions u with

$$\lim_{s \to \pm\infty} u(s,t) = y^{\pm}$$

which are the *opposite* asymptotics from what our conventions would dictate for the definition of the hypothetical group $SH^*(M; -H)$.

Now, consider a solution to Floer's equation on M with Hamiltonian term $-H_t$ on a positive end:

$$u : [0, \infty)_s \times S^1 \to M, \;\; \partial_s u + J_t(\partial_t u - X^M_{-H}).$$

We can equivalently think of this as a solution to Floer's equation on M with Hamiltonian term H_t on a *negative* end,

$$u : (-\infty, 0]_s \times S^1 \to M, \;\; \partial_s u + J_t(\partial_t u - X^M_H)$$

via the (holomorphic!) identification $\bar{u}(s,t) = u(-s,-t)$.

3.3 A sequence of bordisms

The arguments of Section 3.2.3 show that the conjecture follows by showing that image of the diagonal Lagrangian $\Delta \subset M \times M^-$ under the map

$$\begin{aligned} K_0(WF(M \times M^-)) &\xrightarrow{ch} HH_*(WF(M \times M^-)) \xrightarrow{OC} SH^*(M \times M^-) \\ &= SH^*(M) \otimes SH^*(M^-) \to SH^*(M) \otimes (SC^*_{S^1})^*(M^-). \end{aligned} \tag{94}$$

is zero. Let

$$K : SH^*(M \times M^-) \to SH^*(M) \otimes SH^*(M^-) \tag{95}$$

denote the Kunneth map. There is a canonical "acceleration" map

$$a_M : H^*(M) \to SH^*(M) \tag{96}$$

for any Liouville manifold M. [23, Section (3e)]. We choose the closed-string Floer data needed to define $SH^*(M)$ as in ((37), see also Lemma 7) then the acceleration map is just the inclusion of the constant orbits of $H|_{\overline{M}}$.

We begin by proving

Lemma 12. *The element*

$$K(\mathcal{OC}(ch(\Delta))) \in SH^*(M) \times SH^*(M^-) \tag{97}$$

lies in the image of

$$a_M \otimes a_{M^-} : H^*(M) \otimes H^*(M^-) \to SH^*(M) \otimes SH^*(M^-), \tag{98}$$

where a_M, a_{M^-} are the acceleration maps of (96).

Proof. Recall that we are using split Hamiltonians for the definition of $SH^*(M \times M^-)$ and of $WF(\Delta, \Delta)$ as in Section 3.1.8. Let (H, J) be the choice of closed-string Floer data used to define $SH^*_{split}(M \times M^-)$ and $WF(\Delta, \Delta)$. Write

$$(y_1, y_2) \in \mathcal{C}(H_{split}) \tag{99}$$

for a general pair of Hamiltonian orbits of H on M and on M^-, respectively, corresponding to a single Hamiltonian orbit of $H_{split} = \pi_1^* H + \pi_2^* H$ on $M \times M^-$.

The gluing theorem for solutions to the inhomogeneous pseudoholomorphic map equations, together with definition of open-closed map (see Sections 3.1.6 and 3.1.8) and the cohomological unit $ch(\Delta)$ (see Sections 3.2.2 and 3.1.7) show that there exists

$$\begin{gathered}\alpha \in \Omega^1((-\infty, 0] \times S^1, C^\infty(M \times M^-)), \\ J_\alpha \in C^\infty((-\infty, 0] \times S^1, \mathcal{J}(M) \times \mathcal{J}(M^-))\end{gathered} \tag{100}$$

satisfying

$$\begin{gathered}\alpha(s,t) = (\pi_1^* H + \pi_2^* H) \otimes dt \text{ for } s << 0 \\ J_\alpha(s,t) = (J, -J) \text{ for } s << 0 d\alpha \leq 0\end{gathered} \tag{101}$$

such that the moduli space

$$\mathcal{M}^{glued}(y_1, y_2) = \left\{ \quad u = (u_1, u_2) : (-\infty, 0] \times S^1 \to M \times M^- \;\middle|\; \begin{array}{c} (du - X_\alpha)^{0,1}_{J_\alpha} = 0 \\ u(0, \cdot) \in \Delta \\ \lim_{s \to -\infty} u_j(s,t)) = y_j, j = 1, 2 \end{array} \right\} \tag{102}$$

is a smooth manifold (with underlying topology induced by the inclusion into the space of smooth maps u), such that $K(\mathcal{OC}(ch(\Delta)))$ is equal to the *finite* sum

$$\sum_{(y_1,y_2) \in \mathcal{C}(H_{split})} \sum_{\substack{u \in \mathcal{M}^{glued}(y_1,y_2) \\ \mathrm{ind}(D_u)=0}} [y_1] \otimes [y_2]. \tag{103}$$

(where $ind(D_u)$ denotes the index of the linearized Cauchy-Riemann operator associated to an map u).

Now, we will choose certain auxiliary data to construct a bordism from $\mathcal{M}^{glued}$ to a moduli space that is easy to understand. We will choose a

$$\begin{gathered}\phi \in C^\infty(\mathbb{R}, [1, -1]) \\ J_\phi \in C^\infty(\mathbb{R} \times S^1, J(M))\end{gathered} \tag{104}$$

such that

$$\begin{aligned} \phi(s) &= 1 \text{ for } s << 1 \\ \phi(s) &= -1 \text{ for } s >> 1 \\ J_\phi(s,t) &= J \text{ for } |s| >> 1 \\ \phi([-1,1]) &= 0 \\ \partial_s \phi &\le 0 \end{aligned} \tag{105}$$

together with a smooth family, ranging over $\tau \in [0,1]$ of data

$$\begin{aligned} \beta^\tau = (\beta^\tau_M, \beta^\tau_{M^-}) &\in \Omega^1((-\infty,0] \times S^1, C^\infty(M \times M^-)) \\ J^\tau_\beta &\in C^\infty((-\infty,0], \mathcal{J}(M) \times \mathcal{J}(M^-)) \end{aligned} \tag{106}$$

such that

$$\begin{aligned} \beta^1 &= \alpha, \\ \beta^1_M(s,t) &= \phi(s) H dt, \\ \beta^1_{M^-}(s,t) &= -\phi(-s) H dt, \\ J^\tau_\beta(s,t) &= (J, -J) \text{ for } s << 0 \\ J^1_\beta(s,t) &= (J_\phi(s,t), -J_\phi(-s,t)) \\ d\beta_\tau &\le 0 \text{ for all } \tau. \end{aligned} \tag{107}$$

Given such data we may consider the moduli spaces

$$\mathcal{M}^{bordism}(y_1,y_2) = \left\{ \begin{array}{c} \tau \in [0,1] \\ u = (u_1,u_2) : (-\infty,0] \times S^1 \to M \times M^- \end{array} \middle| \begin{array}{c} (du - X_{\beta^\tau})^{0,1}_{J^\tau_\beta} = 0 \\ u(0,\cdot) \in \Delta \\ \lim_{s \to -\infty} u_j(s,t) = y_j \text{ for } i = 1,2 \end{array} \right\} \tag{108}$$

and

$$\mathcal{M}^{final}(y_1,y_2) = \left\{ u : \mathbb{R} \times S^1 \to M \middle| \begin{array}{c} (du - \phi X_H)^{0,1}_{J_\phi} = 0 \\ \lim_{s\to-\infty} u(s,t) = y_1(t) \\ \lim_{s\to\infty} u(s,t) = y_2(t) \end{array} \right\} \tag{109}$$

where we think of ϕ as a function on $\mathbb{R} \times S^1$ via pullback from the first factor. We have the subspace

$$\mathcal{M}^{bordism}_1(y_1,y_2) := \{((\tau,u) \in \mathcal{M}^{bordism}(y_1,y_2) | \tau = 1\}. \tag{110}$$

There is a map

$$\begin{aligned} F : \mathcal{M}^{final}(y_1,y_2) \ni u \mapsto F(u) = (\tau,(u_1,u_2)) &\in \mathcal{M}^{bordism}_1(y_1,y_2) \\ \tau &= 1 \\ u_1(s,t) &= u(s,t) \\ u_2(s,t) &= u(-s,t). \end{aligned} \tag{111}$$

For each of $\mathcal{M}^{glued}(y_1,y_2)$, $\mathcal{M}^{bordism}(y_1,y_2)$, and $\mathcal{M}^{final}(y_1,y_2)$, write $ind(D_u)$ for the index of the linearized Cauchy-Riemann operator of an inhomogeneous map u in the moduli space.

We have the elementary

Lemma 13. *The map F is a bijection.*

Proof. The inverse to F is given by

$$u(s,t) = \begin{cases} u_1(s,t), & s \le 0 \\ u_2(-s,t) & s \ge 0 \end{cases}$$

To show that this is an inverse, it suffices to check that the map u defined above is smooth. For $s \in (-1,1)$, both cases of the definition of u above satisfy

$$\partial_s u + J_\phi \partial_t u = 0$$

Moreover, the t-derivatives of both cases of the definition of u agree, so their s derivatives must as well; an inductive argument shows that u is indeed smooth, and thus F has an inverse. □

We will think of the moduli spaces $\mathcal{M}^{final}(y_1, y_2)$ and $\mathcal{M}^{bordism}(y_1, y_2)$ as equipped with the topology induced by their inclusion into spaces of smooth maps (together with a factor of $[0,1]_\tau$ for $\overline{\mathcal{M}}^{bordism}$). Standard transversality methods [12], the maximum principle (for $\mathcal{M}^{final}$) [23], and the integrated maximum principle (for $\mathcal{M}^{bordism}$) [2], prove the following

Proposition 9. *There exist choices as in (104) and (106), satisfying the constraints (**??**), (**??**), such*

- *The elements of $\mathcal{M}^{final}$ for which $ind(D_u) = 0$ are a finite discrete set;*
- *The elements $\mathcal{M}^{bordism}$ for which $ind(D_u) = -1$ are a finite discrete set;*
- *The connected components of $\mathcal{M}^{bordism}$ for which $ind(D_u) = 0$ form 1-manifolds, with Gromov-Floer compactification given by adding the elements of*

$$\mathcal{M}((y_1, y_2), (y_1', y_2')) \times \mathcal{M}^{bordism}(y_1', y_2')$$

where the first factor refers to the Floer moduli space (33) defining $SC(M \times M^-)$ with the split Floer data, ranging over all $(y_1', y_2') \in \mathcal{C}(H_{split})$, for which the index of the linearized Cauchy riemann operator of the first term is 1 and for the second is -1; as well as the elements of

$$\mathcal{M}^{final}(y_1, y_2) \text{ and } \mathcal{M}^{glued}(y_1, y_2)$$

for which $ind(D_u) = 0$.

The above proposition then shows that $K(\mathcal{OC}(ch(\Delta)))$ is cohomologous to

$$\sum_{(y_1,y_2) \in \mathcal{C}(H_{split})} \sum_{\substack{u \in \mathcal{M}^{final}(y_1,y_2) \\ ind(D_u)=0}} [y_1] \otimes [y_2] \tag{112}$$

via a cocycle given by the count of elements of $\mathcal{M}^{final}(y_1, y_2)$ for which $ind(D_u) = -1$.

We now argue that for action reasons, the above element is in the image of the acceleration map, i.e. consists of constant orbits. Indeed, we have the usual inequality of geometric and topological energies, which in our conventions reads

$$0 \leq \int_{\mathbb{R} \times S^1} |du|^2_{J_\phi} \leq \mathcal{A}_H(y_1) - \mathcal{A}_{-H}(y_2) + \int \phi'(s) H ds dt \leq \mathcal{A}_H(y_1) - \mathcal{A}_{-H}(y_2);$$

see (38) for the definition of $\mathcal{A}_H(x)$. By Lemma 7, the right hand side is strictly negative unless y_1 and y_2 are constant orbits.

So the topological energy, and thus the geometric energy, must both be zero, and the elements $u \in \mathcal{M}^{final}(y_1, y_2)$ contributing to the sum in (112) must all be constant maps, which can only asymptote to constant orbits. So the sum of (112) is in the image of the acceleraiton map. □

Remark 12. It is straightforward to adapt this argument to non-split floer-theoretic invariants. In that case, the Kunneth map on symplectic cohomology becomes a certain continuation map (see **??**); after gluing, one again gets a bijection between terms of $K(\mathcal{OC}(ch(\Delta)))$ and certain maps into M with a seam on $0 \times S^1$ labeled by a Lagrangian boundary condition; using a parameterized moduli space, one then performs a bordism to make all the Floer data near this seam be split as in the above proof, after which bootstrapping an analytic continuation allow one to remove the seam.

But now we state a lemma, essentially proven in [17, Section 4.4], and for convenience recall the proof:

Lemma 14. *The composition of the acceleration map and the Connes map*

$$\delta \circ a : H^*(M^-) \to (SC^-_{S^1})^*(M^-) \tag{113}$$

is zero.

Proof. We must first review one more fact about the Connes map from the homology to the negative-cyclic homology of a mixed complex M; namely, it is the boundary map of the exact sequence of complexes [22, Equation 5.1.4.2]

$$M((u)) \xrightarrow{u} M((u)) \to M((u))/uM((u)) = M$$

which a short calculation shows is the zero map whenever the map

$$\delta : M \to M((u)), y \mapsto \sum_{k \geq 1} u^{k-1} \delta^k y$$

is zero.

Recall that we are using closed-string Floer data as in Lemma 7, and so the constant orbits form a subcomplex of $SC^*(M) \otimes SC^*(M^-)$. By Proposition 4, we may have chosen the Floer data for the S^1 action to be independent of of $p \in \overline{\mathcal{M}}_k$ inside $\overline{M}^-$. The maps

$$\delta_k : SC^*(M^-; H, -J) \to SC^*(M^-; H, -J) \tag{114}$$

are action-nondecreasing, and so must preserve the subcomplex of constant orbits, since those are the only ones with positive action (Lemma 7). In fact, each of these maps must be zero for $k \geq 1$, since the corresponding moduli spaces (54) cannot have zero-dimensional components – given an element $(u, p) \in \mathcal{M}_k(y_-, y_+)$ with y_-, y_+ constant orbits, any other choice of $p' \in \mathcal{M}_k$ will give an element $(u, p') \in \mathcal{M}_k(y_-, y_+)$ (Figure 6). Since the image of the acceleration map is exactly the constant orbits, and the map

$$\delta : SC^*(M) \to (SC^-_{S^1})^*(M) \tag{115}$$

is

$$y \mapsto \sum_{k \geq 1} u^{k-1} \delta^k y, \tag{116}$$

this proves the proposition. □

Combining the lemmata of this section, we see that we have proven Theorem 3.

4 Split open-closed map

In this section we outline the construction of the split open-closed map, the existence of which is mentioned in Remark 11.1 of [16].

In [16], Ganatra defines moduli spaces of *genus-0 open-closed strings*

$$\mathcal{N}^{\mathbf{I}, \tilde{\mathbf{K}}}_{h,n,\vec{m}}, \vec{m} = (m^1, \ldots, m^h), \mathbf{I} \subset \{1, \ldots, n\}, \tilde{\mathbf{K}} = (K^1, \ldots, K^h), K^j \subset \{1, \ldots, m^j\}$$

which parametrize complex spheres with h disjoint disks removed, n interior marked points, m^j boundary marked points on the j-th boundary component of the Riemann surface, the interior marked points labeled by $\mathbf{I}$ and the boundary marked points on the j-th component labeled by K^j declared to be negative. He imposes the constraint that there is at most one negative interior marked point or at most two negative boundary marked points. He then defines notions of *Floer data* for genus-0 open-closed strings, as well as compactifications

$$\overline{\mathcal{N}}^{\mathbf{I},\tilde{\mathbf{K}}}_{h,n,\vec{m}}$$

of these moduli spaces.

Let

$$\mathcal{R}^{n,1}, \overline{\mathcal{R}}^{n,1}$$

denote the moduli space of disks with n positive boundary marked points and one negative interior marked point, and its Gromov compactification, respectively. We will label the marked points $p_1, \dots, p_n$ running counterclockwise. Suppose that we have chosen a *Lagrangian labeling* $\mathbb{L} = (\mathbb{L}_1, \dots, \mathbb{L}_n)$ with each $\mathbb{L}_i$ equal either to a product Lagrangian $L_i^1 \times L_i^2 \subset M \times M^-$ or the diagonal Lagrangian $\Delta \subset M \times M^-$. Given a disk $C \in \mathcal{R}^{n,1}$, we think of each $\mathbb{L}_i$ as labeling the component $\partial_i C$ of the boundary of $C \setminus \{p_1, \dots, p_n\}$ lying in between the marked points p_i and p_{i+1}, with $p_{n+1} = p_1$ by convention. Then there is a map

$$\phi_{\mathbb{L}} : \mathcal{R}^{n,1} \to \mathcal{N}^{\mathbf{I}^{\mathbb{L}},\tilde{\mathbf{K}}^{\mathbb{L}}}_{h^{\mathbb{L}},n^{\mathbb{L}},\vec{m}^{\mathbb{L}}}$$

for certain data $\mathbf{I}^{\mathbb{L}}, \tilde{\mathbf{K}}^{\mathbb{L}}, h^{\mathbb{L}}, n^{\mathbb{L}}, \vec{m}^{\mathbb{L}}$ determined by $\mathbb{L}$, defined by *gluing pairs of disks* as follows. (The analogous map for the case where the domain is the associahedron is defined in Section 7 of [16] and is used to define the A_∞ operations on the category $\mathcal{W}^2$ defined therein.) Given a representative C of a point $[C] \in \mathcal{R}^{n,1}$, let $\bar{C}$ denote the same disk but with the opposite complex structure and the interior negative marked point now thought of as a positive marked point. The map $\phi_{\mathbb{L}}$ assigns to $[C]$ the isomorphism class of a genus-zero open-closed string, denoted by

$$C \cup_\Delta \bar{C}$$

constructed by partially gluing C to $\bar{C}$ along their boundaries: if $x \in \partial_i C$ and $\mathbb{L}_i = \Delta$ then x is identified with x in $\partial\bar{C}$, and boundary marked points of C are identified with the corresponding boundary marked points of $\bar{C}$. The boundary marked points of C that lay in between two boundary components labeled by Δ now appear as interior marked points of $C \cup_\Delta \bar{C}$, and are to be considered negative; the remaining boundary marked points continue to appear as boundary marked points of $C \cup_\Delta \bar{C}$ and are also considered negative. Finally, let p_0^- and p_0^+ denote the two interior marked points of $C \cup_\Delta \bar{C}$ coming from the interior marked points of C and $\bar{C}$ respectively; they are respectively considered to be negative and positive interior marked points of $C \cup_\Delta \bar{C}$.

An inductive argument involving the compactifications of the domains and codomains of the $\phi_{\mathbb{L}}$ shows that

Lemma 15. *$\phi_{\mathbb{L}}$ extends to a continuous map*

$$\phi_{\mathbb{L}} : \overline{\mathcal{R}}^{n,1} \to \overline{\mathcal{N}}^{\mathbf{I}^{\mathbb{L}},\tilde{\mathbf{K}}^{\mathbb{L}}}_{h^{\mathbb{L}},n^{\mathbb{L}},\vec{m}^{\mathbb{L}}}.$$

Now, Ganatra defines a notion of *Floer data* (Definition 4.11) $\mathbf{F}_S$ for an open-closed string S, given by *weighted strip and cylinder data* $\mathfrak{S}$, a sub-closed one form α_S, a primary Hamiltonian $H_S : S \to (H)(M)$, an $\mathfrak{S}$-adapted rescaling function a_S, an almost complex structure J_S, and an S^1-perturbation F_S, all satisfying mutual compatibility conditions as well as compatibility conditions with the background choice of closed-string Floer data $(H_t = H + F_t, J_t)$ made as in Lemma 7. Given an identification of positive marked points of one open-closed string and corresponding negative marked points of another open-closed string, one can glue Floer data chosen for each of the open-closed string as in Section **??**.

A *universal and consistent choice of split Floer data* for the moduli spaces $\overline{\mathcal{R}}^{n,1}$ is a choice of Floer data for every element of every $\overline{\mathcal{R}}^{n,1}$ equipped with every possible choice of Lagrangian labels $\mathbb{L}$ such that the Floer data vary smoothly over each $\overline{\mathcal{R}}^{n,1}$ and such that the Floer data associated to the boundary strata of $\overline{\mathcal{R}}^{n,1}$ agree to infinite order in the gluing coordinates near the boundary strata with the the Floer data obtained by gluing the Floer data associated to the disks representing the points in the boundary strata.

Given a Floer datum for an open-closed string S, one can consider the set of maps

$$u : S \to M$$

satisfying the Floer-type equation

$$(du - X_S \otimes \alpha_S)^{0,1}$$

with asymptotic and boundary conditions defined by the rescaling function and the weighted strip and cylinder data, and where X_S is the S-dependent Hamiltonian vector field corresponding to $H_S + F_S$. We wish to modify, slightly, the equation defining the moduli spaces that we consider.

If $S = \phi_{\mathbb{L}}([C])$ for some $C \in \mathcal{R}^{n,1}$, we will let p_o^+ and p_o^- the positive and negative interior marked points of S arising from doubling the unique interior marked point of C, respectively. The Floer datum for S provides us with a positive cylindrical end $\delta_{p_o^-} : [0,\infty) \times S^1 \to S$, along with a weight $\eta > 0$, associated to p_o^+. We choose, once and for all, a smooth function

$$\tau : [0,\infty) \to [1,-1]$$

such that

$$\tau|_{[0,1]} = 1; \tau|_{[2,\infty]} = 0, \tau' \leq 0;$$

with all derivatives bounded. From the definition of a Floer datum we get that

$$\delta^*_{p_o^-} X_S \otimes \alpha_S = \frac{(\psi^\eta)^* X_{H_t}}{\eta} \otimes dt$$

where ψ^η is the flow for time $\log(\eta)$ of the Liouville vector field of M. We let $\widetilde{X_S \otimes \alpha_S}$ denote the Hamiltonian-vector-field-valued one-form on S (associated to the implicitly defined function $\tilde{H}_S : S \to C^\infty(M)$) which agrees with $X_S \otimes \alpha_S$ away from the image of $\delta_{p_o^-}$, and satisfies

$$\delta^*_{p_o^-} \widetilde{X_S \otimes \alpha_S} = \tau(s) \frac{(\psi^\eta)^* X_{H_t}}{\eta} \otimes dt.$$

Suppose now that we have chosen a universal and consistent choice of split Floer data for the moduli spaces $\mathcal{R}^{n,1}$.

Given an element $C \in \mathcal{R}^{n,1}$ with boundary marked points $p_1, \ldots, p_n$, Lagrangian labels $\mathbb{L}$, and writing $S = \phi_{\mathbb{L}}(C)$, we will say that a boundary marked point p_i *succeeds* $\mathbb{L}_{i-1}$ and *precedes* $\mathbb{L}_i$, with the indices written cyclically so $\mathbb{L}_0 = \mathbb{L}_n$. We will write w_k for the weight associated to the end corresponding to p_k by the Floer data chosen for C, and $w_\pm$ for the weights associated to $p_0^\pm$. Given a Hamiltonian G on a symplectic manifold M, we will write $\mathcal{C}_M(G)$ for the time-1 orbits of the Hamiltonian, and given Lagrangian submanifolds L and L', we will write $\mathcal{C}_M(L, L'; G)$ for the time-1 chords of G starting at L and ending at L'. There are isomorphisms

$$\mathcal{C}_M(G) \simeq \mathcal{C}_M(G/\rho \circ \psi^\rho),$$

$$\mathcal{C}_M(L, L'; G) \simeq \mathcal{C}_M(\psi^\rho L, \psi^\rho L'; G/\rho \circ \psi^\rho).$$

We let $d_1, \ldots, d_{r_d} \in \{1, \ldots n\}$ denote the indices of boundary marked points lying in between two boundary components labeled by Δ; for each index, say we have choices

$$x_{d_j} \in \mathcal{C}(\Delta, \Delta) := \mathcal{C}_M(H_t).$$

Let $a_1, \ldots, a_{r_a}$ be the indices of boundary marked points succeeding Δ and preceding a product Lagrangian; for each index, say we have choices

$$y_{a_j} \in \mathcal{C}(\Delta, L_1 \times L_2) := \mathcal{C}_M(L_1, L_1; H_t).$$

Let $b_1, \ldots, b_{r_b}$ be the indices of points preceding Δ and succeding a product Lagrangian; for each index, say we have choices

$$y_{b_j} \in \mathcal{C}(L_1 \times L_2, \Delta) := \mathcal{C}_M(L_1, L_2; H_t).$$

Finally, let $c_1, \ldots c_d$ be the indices of the remaining boundary marked points, which must lie in between two product Lagrangians. For each index, say we have choices

$$y_{c_j} = (y^1_{c_j}, y^2_{c_j}) \in \mathcal{C}(L_1 \times L_2, L'_1 \times L'_2) := \mathcal{C}_M(L_1, L'_1; H_t) \times \mathcal{C}_M(L_2, L'_2; H_t).$$

These choices correspond to a set of choices

$$z_i \in \mathcal{C}(\mathbb{L}_{i-1}, \mathbb{L}_i), i = 1, \ldots, n$$

Say we also have choices

$$x_- \in \mathcal{C}_M(H_t), x_+ \in \mathcal{C}_{M^-}(H_t).$$

Write ϵ^k or δ^k for the strip-like or cylindrical positive end associated to the marked point of S coming from $p_k \in \partial C$; and write $\delta^{\pm}$ for the positive/negative cylindrical end associated to the marked points $p_o^-, p_o^+ \in S$ arising from doubling the interior marked point of C.

Define the moduli space

$$\mathcal{R}^{n,1}(\{x_j\}, \{y_j\}; x_-, x_+) = \mathcal{R}^{n,1}_*(z_1, \ldots, z_n; x_-, x_+)$$

to be the collection of maps

$$u : S \to M; S = \phi_{\mathbb{L}}(C); C \in \mathcal{R}^{n,1}$$

such that

$$(du - \widetilde{X_S \otimes \alpha_S})^{0,1} = 0$$

and satisfying the boundary and asympotic conditions

$$\lim_{s \to +\infty} u \circ \epsilon^k(s,t) = \psi^{w_k} y_k(t), k = a_1, \ldots, a_{r_a}, b_1, \ldots, b_{r_b}, c_1, \ldots, c_{r_c};$$

$$\lim_{s \to +\infty} u \circ \delta^k(s,t) = \psi^{w_k} x_k(t), k = d_1, \ldots, d_{r_d};$$

$$\lim_{s \to +\infty} u \circ \delta^+(s,t) = \psi^{w_+} x_+(t);$$

$$\lim_{s \to -\infty} u \circ \delta^-(s,t) = \psi^{w_-} x_-(t);$$

$$u(z) \in \phi^{a_S(z)} \mathbb{L}_\alpha, \alpha \in \pi_0(\partial S).$$

The topological energy of any solution u is given by

$$\int_S \omega - d(u^*(\tilde{H}_S))\alpha_S$$

which is equal by Stokes theorem to

$$\sum_j \mathcal{A}(y_{a_j}) + \sum_j \mathcal{A}(b_j) + \sum_j \mathcal{A}(c_j) - \mathcal{A}(x_-) + \mathcal{A}(x_+)$$

which upper bounds the (nonnegative) geometric energy of a solution. Here the above terms are defined exactly as in [16, (A.8), (A.11)], except that $\mathcal{A}(x_+)$, due to the modification we have made to Floer's equation, is taken to be (A.11) with the terms H and F_t replaced by their negatives.

In particular, because of Lemma 7, the topological energy is actually upper bounded by

$$\sum_j \mathcal{A} y_{a_j} + \sum_j \mathcal{A}(b_j) + \sum_j \mathcal{A}(c_j) + K \tag{117}$$

for some constant $K > 0$. Now, we have the following π_r denote an extension of the collar coordinate of the collar of M to a function

$$\pi_r : M \to [0, \infty)$$

Proposition 10. *Given an element of* $\mathcal{R}^{n,1}(\{x_j\}, \{y_j\}; x_-, x_+)$ *there is a constant* C *depending only on* $\{x_j\}, \{y_j\}, x_-, x_+$ *such that*

$$\pi_r \circ u \leq C.$$

Proof. One may follow the proof as in [16, Theorem A.1], except that one redefines the unperturbed region S^u to not include the image of the strip-like end associated to p_0^+. On the image of the strip-like end associated to p_0^+, we have chosen our Floer data explicitly so that Floer's equation takes the form of the continuation map equation between the Floer homology groups $SH^*(-H')$ and $SH^*(H')$ for a Hamiltonian H'; and the maximum principle argument as in [23] suffices to prove compactness in that region. □

By Gromov compactness, the above proposition, and the bound in (117)

Proposition 11. *Fixing* $\{x_j\}, \{y_j\}$ *as in the above proposition, the number of* $x_- \in \mathcal{C}_M(H_t)), x_+ \in \mathcal{C}_{M^-}(H_t)$ *for which* $\mathcal{R}^{n,1}(\{x_j\}, \{y_j\}; x_-, x_+)$ *is nonempty is finite.*

The salient point of the above proposition is that although $\bar{p}_0$ is a positive marked point, we wish to think of it as an *output* of the open-closed map, because it arises artificially from the negative marked point $\tilde{p}$ via the doubling construction.

The standard inductive construction of Floer data works in our setting and we have the

Proposition 12. *Universal and consistent choice of split Floer data for the moduli spaces* $\overline{\mathcal{R}}^{n,1}$ *exist. Moreover, given the choices needed to define the split open closed maps (83) (which*

Chose split Floer data as in the above proposition. Recall the Kunneth map for symplectic cohomology with split Floer data, together with the isomorphism

$$SC^*(M^-) \simeq SC^*(M, -H)$$

described in section 3.2.4. We then define

$$\mathcal{OC} : C^{nu}_{*-2n}(\mathcal{W}^2) \to SC^*(M \times M^-)$$

via the formula

$$\mathcal{OC}(z_1 \otimes \ldots \otimes z_n) = \sum_{(x_-,x_+) \in \mathcal{C}(H_{split})} \sum_{\substack{u \in \mathcal{R}^{n,1}_*(z_1,\ldots,z_n; x_-,x_+) \\ ind(D_u)=0}} [x_-] \otimes [x_+]$$

(where $ind(D_u)$ is of course the index of the linearized Cauchy-Riemann opeator of u).

The continuity of ϕ and the standard argument based on the combinatorics of $\overline{\mathcal{R}}^{n,1}$ showing that the open-closed maps defined in [], [] are chain maps show that $\mathcal{OC}$ is in fact a chain map. This proves Proposition 7

5 Appendix: Proof of proposition about homotopy units

This proof is written in the convention for A_∞ categories of [25] rather than the one we use in this paper; as we are currently working in characteristic 2, the proof goes through as written, but see Appendix 6 for a description of how to convert the proof to the conventions used in the rest of this paper.

Proof of Proposition 2. Recall that a *formal diffeomorphism* of an A_∞category A with A_∞ operations μ_A^d is an arbitrary sequence of maps

$$\Phi^d : hom_A(X_{d-1}, X_d) \otimes \ldots \otimes hom_A(X_0, X_1) \to hom_A(X_0, X_d)[1-d] \tag{118}$$

such that Φ^1 is a linear automorphism. The difeomorphism Φ defines a new A_∞ structure $\Phi_* A = \mu^1_{\Phi_* A}, \mu^2_{\Phi_* A}, \ldots$ on the linear category A; the new operations are computed as the solution to the system of equations

$$\begin{aligned} &\sum_r \sum_{s_1,\ldots,s_r} \mu^r_{\Phi_* A}(\Phi^{s_r}(a_d,\ldots,a_{d-s_r+1}),\ldots,\Phi^{s_1}(a_{s_1},\ldots,a_1)) = \\ &\sum_{m,n} (-1)^* \Phi^{d-m+1}(a_d,\ldots,a_{n+m+1},\mu^m_A(a_{n+m},\ldots,a_{n+1}),a_n,\ldots a_1) \end{aligned} \tag{119}$$

where $(-1)^*$ is a certain Koszul sign [25, Section I.1c]. This makes Φ into an A_∞ functor between A equipped with the original A_∞ structure and A equipped with the A_∞ structure $\Phi_* A$; formal diffeomorphisms then form a *group* under composition of functors.

We now make use of [25, Lemma I.2.1]:

Lemma 16. *Let A be a c-unital A_∞ category. Then there exists a formal diffeomorphism Φ with $\Phi^1 = Id$, such that the modified A_∞ structure $\Phi_* A$ is strictly unital. Moreover, Φ sends the chosen c-units of A to the strict units of $\Phi_* A$.*

Equivalently, given a c-unital A_∞ category A, there is a strictly unital A_∞ category B (with the same underlying graded linear category as A) and a formal diffeomorphism $\Phi : B \to A$. Since B is strictly unital, it admits a canonical homotopy unit $B \to B'$. The underlying graded linear category of a homotopy unit for A is the same as that for B; thus we define the graded linear category $A' := B'$. The point is that Φ extends to a formal diffeomorphism $\Phi : B' \to A'$ such that the induced A_∞ structure on A' makes the canonical inclusion $A \to A'$ into a homotopy unit for A'.

Specifically, for any $X \in Ob(A') = Ob(B') = Ob(A) = Ob(B)$, one writes

$$Hom_{A'}(X, X) = Hom_{B'}(X, X) = Hom_B(X, X) \oplus \Bbbk f_X[1] \oplus \Bbbk e^+_X \tag{120}$$

and one writes $e^0_{B,X} \in Hom_B(X, X)$ for the strict unit and $e^0_{A,X} \in Hom_A(X, X)$ for the chosen c-units of A. One then defines $\Phi : B' \to A'$ by the conditions that

$$\begin{gathered} \Phi(e^+_X) = e^+_X, \\ \Phi(f_X) = f_X, \\ \Phi(\ldots, f_X, \ldots) = 0 \text{ whenever there is more than one term, and} \\ \Phi(\ldots, e^+_X, \ldots) = 0, \text{meaning that } \Phi \text{ is strictly unital.} \end{gathered} \tag{121}$$

This is a formal diffeomorphism, so there is a unique A_∞ structure $\mu_{A'}$ satisfying the equations

$$\begin{aligned} &\sum_r \sum_{s_1,\ldots,s_r} \mu^r_{A'}(\Phi^{s_r}(b_d,\ldots,b_{d-s_r+1}),\ldots,\Phi^{s_1}(b_{s_1},\ldots,b_1)) = \\ &\sum_{m,n} (-1)^* \Phi^{d-m+1}(b_d,\ldots,b_{n+m+1},\mu^m_{B'}(b_{n+m},\ldots,b_{n+1}),b_n,\ldots b_1). \end{aligned} \tag{122}$$

Since Φ extends a diffeomorphism from B to A, the uniqueness of the solutions to the above equations implies that $\mu_{A'}|_A = \mu_A$. Specializing the above equations we see that

$$\mu^1_{A'}(f_X) = \mu^1_{A'}(\Phi^1(f_X)) = \Phi^1(\mu^1_{B'}(f_X)) = \Phi^1(e^0_{B,X} - e^+_X) = e^0_{A,X} - e^+_X. \tag{123}$$

So it remains to show that e_+ is a strict unit for $\mu_{A'}$. The condition $\mu^1_{A'}(e^+_X)$ trivially follows from (122). Likewise, (122) implies that

$$\begin{gathered}\mu^2_{A'}(e^+_X, \Phi^1(a)) = \mu^2_{A'}(\Phi^1(e^+_X), \Phi^1(a)) = \\ -\mu^1_{A'}(\Phi^2(e^+_X, a)) + \Phi^1(\mu^2_{B'}(e^+_X, a)) + \Phi^2(\mu^1_{B'}(e^+_X), a) + \Phi^2(e^+_X, \mu^1_{B'}(a)),\end{gathered} \tag{124}$$

where all the second line all terms are zero except for the second term. Writing $|a|$ for the degree of a the above computation together with a similar computation of $\mu^2_{A'}(\Phi^1(a), e^+_X)$ show that

$$\mu^2_{A'}(e^+_X, a) = a = (-1)^{|a|}\mu^2_{A'}(a, e^+_X) \tag{125}$$

where the sign comes from the suppressed Koszul signs in (122).

Thus it remains to show that $\mu^k_{A'}(\ldots, e^+_X, \ldots) = 0$ for $k \geq 3$. This is proven by induction on k. Consider applying 122 to a tuple $(b_k, \ldots, b', e^+_X, b'', \ldots, b_1)$. In the left hand side of 122 by the inductive hypothesis and the properties of Φ (121), all terms vanish except for

$$\mu^k_{A'}(\Phi^1(b_k), \ldots, \Phi^1(e^+), \ldots, \Phi^1(b_1)). \tag{126}$$

On the right hand side, by the inductive hypothesis and the properties of Φ, one sees that the only non-zero terms are

$$(-1)^*\Phi^{k-1}(\ldots, \mu^2_{B'}(b', e^+), b'', \ldots) + (-1)^*\Phi^{k-1}(\ldots, b', \mu^2_{B'}(e^+, b'')). \tag{127}$$

Computing the suppressed Koszul signs one sees that this sum cancels, and thus the quantity in (126) is zero. One must give slightly different arguments if e^+_X lies all the way to the left or all the way to the right in the tuple plugged into (122). For example, if one applies (122) to $(e^+_X, b'', \ldots)$ (where there k entries in the tuple), the inductive hypothesis and the properties of Φ show that all terms vanish except for

$$\mu^k_{A'}(\Phi^1(e^+_X), \Phi^1(b'), \ldots) + \mu^2_{A'}(\Phi^1(e^+_X), \Phi^{k-1}(b'', \ldots)) = (-1)^*\Phi^{k-1}(\mu^2_{B'}(e^+_X, b''), \ldots) \tag{128}$$

on the left and right hand side, respectively. A computation of the suppressed Koszul sign shows that two terms containing μ^2 cancel, showing that $\mu^k_{A'} e^+_X, \ldots) = 0$ for $k > 3$ (assuming the base case $k = 3$). An analogous computation of (122) applied to $(\ldots, b', e^+_X)$ completes the inductive step. Finally, the arguments for the inductive step also adapt to prove the base case $k = 3$ by explicitly expanding (122) when applied to tuples (e^+_X, b'', b), (b', e^+_X, b'') and (b, b', e^+_X). This completes the proof. □

6 Appendix: A_∞ conventions

There are many different sign conventions for A_∞-algebraic equations in the literature.

The convention in this paper is that the A_∞ operations

$$\mu^k : Hom(L_0, L_1) \otimes Hom(L_1, L_2) \otimes \ldots \otimes Hom(L_{k-1}, L_k) \to Hom(L_0, L_k)$$

must satisfy the A_∞ equations

$$\sum_{i+j+k=n+1} (-1)^{|a_1|+\ldots+|a_i|+i}\mu_{i+k+1}(a_1, \ldots, a_i, \mu_j(a_{i+1}, \ldots a_{i+j}), a_{i+j+1}, \ldots, a_n) = 0 \tag{129}$$

These agree with the conventions of Efimov [10], Seidel [24], and [26], but disagree with those of [25] and [16]. To convert from one convention to the other, one simply reverses the order of the inputs into the μ^k operations.

Relatedly, there are two different conventions for opposite A_∞ categories: the one given in Definition 4, and one used in [25] and in [16]. They only differ by signs in the A_∞ operations, so over a field of characteristic 2 they agree. A discussion of these differences can be found in [26].

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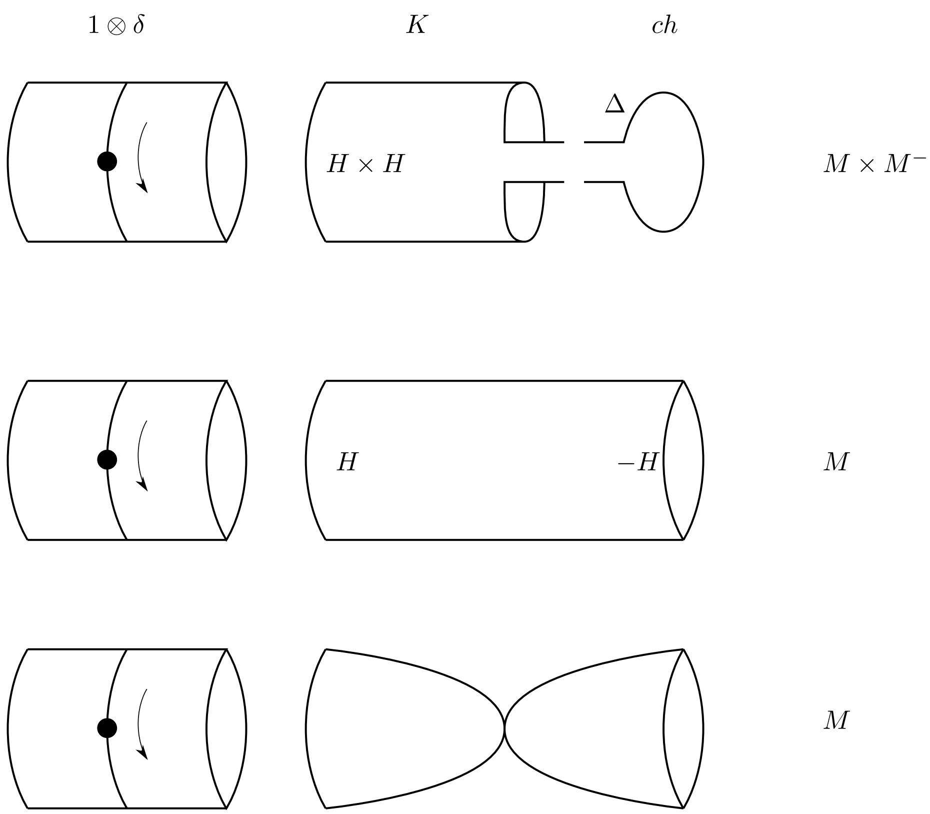

Figure 1: A summary of the proof of Theorem 3. Each piece of the initial moduli space corresponds to a morphism in the statement of the conjecture, as labeled. By passing from curves in $M \times M^-$ to curves in M, one can then pull apart the piece of the moduli space consisting with curves without marked points, showing that the image of $K \circ ch$ factors through the "constant loops" in symplectic cohomology, and is thus annihilated by the Connes map. (Technically, we do not 'pull apart" the last moduli space, because an action argument suffices.) To get a formally dual proof of Theorem 2 (not detailed in this paper) one reads the second diagram from left-to-right rather than right-to-left; the moduli space corresponding to the right cylinder, with H chosen to be a small Morse function, gives a pairing on $HF(H)$ that pulls back to Shklyarov pairing on Hochschild homology of compact Lagrangians under the open closed map. The bordism in the third diagram shows that the Shklyarov pairing annihilates the image of $1 \otimes \delta$.

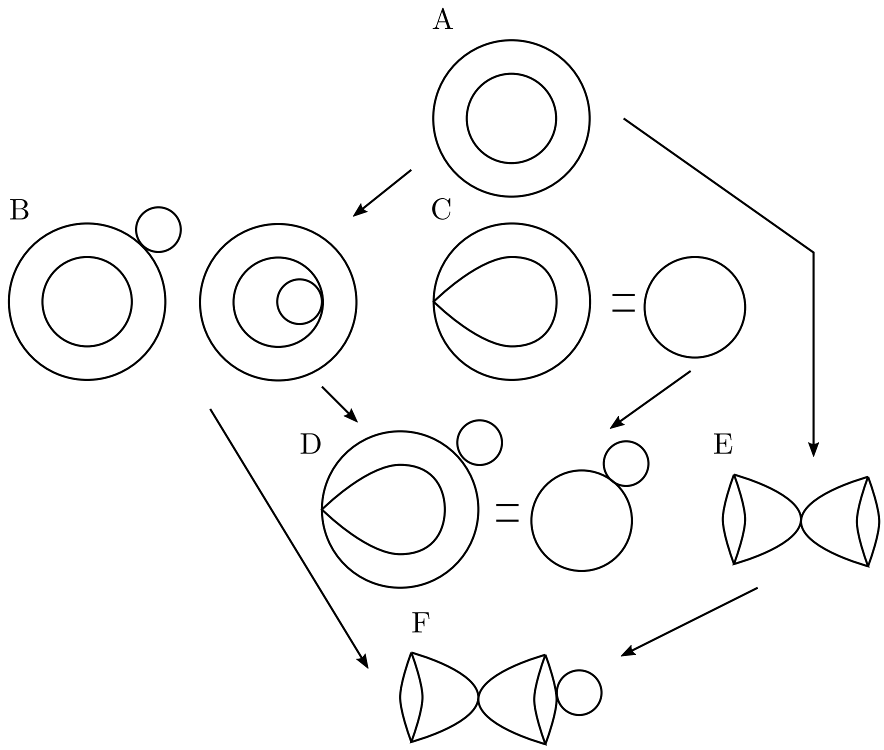

Figure 2: The moduli space giving rise to $R(x)$ and the relation between $R(x)$ and $\phi(x)$. The moduli space is depicted in the center; its degenerations follow the arrows, with an increase in virtual codimension for every arrow. The boundary components of 1-dimensional moduli spaces of this type can only be either contributions to $\phi(x)$ (C) or contirbutions to $dR(x)$ (B); the other possible degenerations do not contribute becasuse their real codimension is greater than 1. This moduli space is different from the moduli space giving rise to the Cardy relation [1] because in this moduli space, there is no constraint on the relative angle between a_0 and b_0, so the degeneration from A to E is codimension 2. In these degenerations, boundary disk bubbles interior sphere bubbles are excluded by exactness.

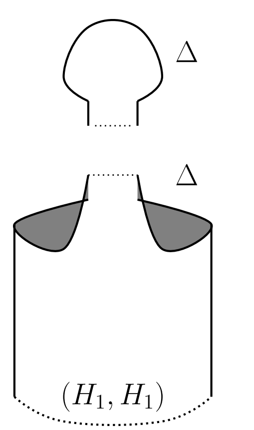

Figure 3: The moduli space defining $K(\mathcal{OC}(ch(\Delta))$.

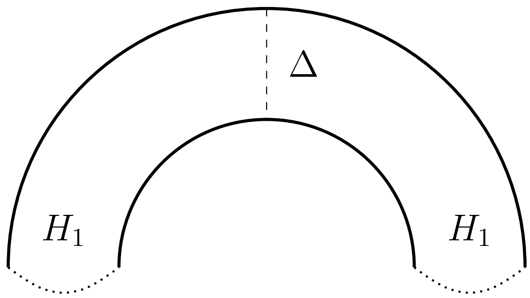

Figure 4: An alternative description of $K(\mathcal{OC}(ch(\Delta))$, see Lemma 12.

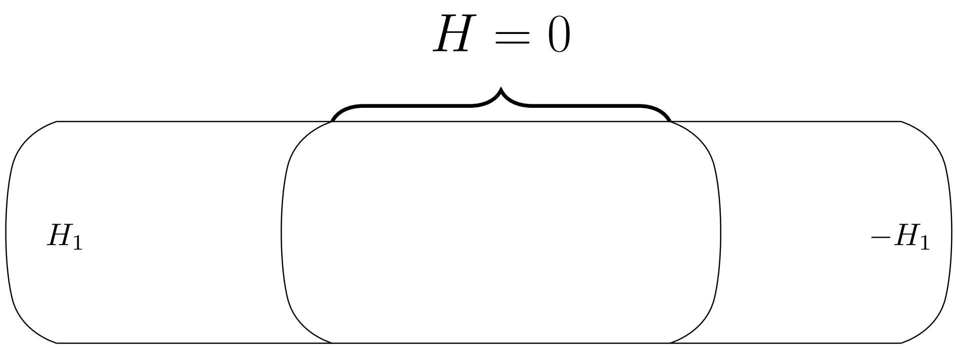

Figure 5: The moduli space we get after performing a bordism to the gluing of 3. From a TQFT perspective, one should imagine "stretching" the region where $H = 0$ until the domains degenerate into a union of two disks joined at an interior point; we replace this additional bordism of moduli spaces with an action argument, for convenience.

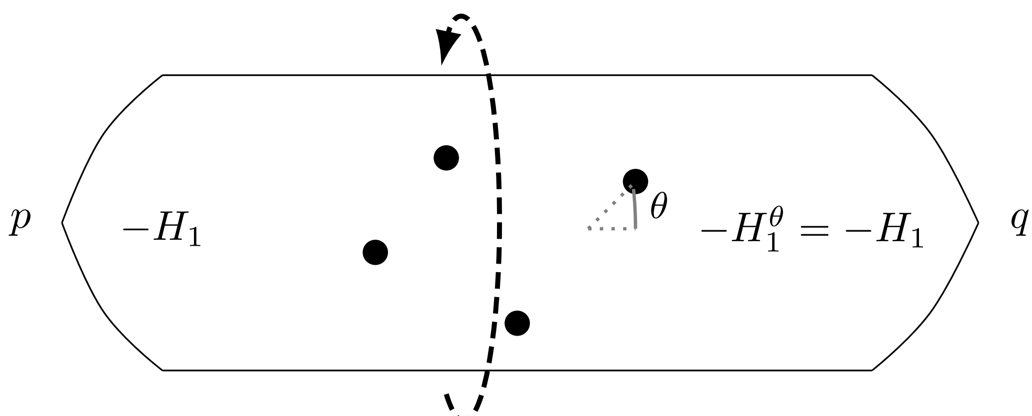

Figure 6: A symmetry of the moduli space arising after applying $1 \otimes \delta_k$ to $K(\mathcal{OC}(ch(\Delta)))$. We have drawn the moduli space as if it is a curve mapping to M, rather than to M^-. See Lemma 14.