Bijective proofs for Fibonacci identities related to Zeckendorf's Theorem

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Abstract

In *Proofs that Really Count*, Benjamin and Quinn wrote that there were no known bijective proofs for certain identities that give instances of Zeckendorf's Theorem, for example, $5f_n = f_{n+3} + f_{n-1} + f_{n-4}$, where $n \geq 4$ and where f_k is the k-th Fibonacci number (there are analogous identities for ℓf_n for every positive integer ℓ). In this paper, we provide bijective proofs for $5f_n = f_{n+3} + f_{n-1} + f_{n-4}$ and the seven other examples of such identities listed in *Proofs that Really Count*. We interpret f_k as the cardinality of the set F_k consisting of all ordered lists of 1's and 2's whose sum is k. We then demonstrate bijections that prove the eight identities listed in *Proofs that Really Count*; for example, to prove $5f_n = f_{n+3} + f_{n-1} + f_{n-4}$, we give a bijection between $\{1,2,3,4,5\} \times F_n$ and $F_{n+3} \cup F_{n-1} \cup F_{n-4}$. A few possible directions for future research are also given.

1 Introduction

We will interpret the n-th Fibonacci number f_n as the cardinality of the set F_n of all ordered lists of 1's and 2's that have sum n. Thus, $(f_0, f_1, f_2, f_3, f_4, f_5, \ldots) = (1, 1, 2, 3, 5, 8, \ldots)$. For an integer ℓ, the number ℓf_n will be interpreted as the cardinality of the Cartesian product $[\ell] \times F_n$, where $[\ell] := \{1, 2, 3, \ldots, \ell\}$. We will use the notation $[\![a_1, a_2, \ldots, a_k]\!]$ to denote an element of F_n.

On page 15 of *Proofs that Really Count* [1], the following eight identities are given under the heading of identities in need of combinatorial proofs:

$$5f_n = f_{n+3} + f_{n-1} + f_{n-4} \quad \text{for } n \geq 4 \tag{5}$$
$$6f_n = f_{n+3} + f_{n+1} + f_{n-4} \quad \text{for } n \geq 4 \tag{6}$$
$$7f_n = f_{n+4} + f_{n-4} \quad \text{for } n \geq 4 \tag{7}$$
$$8f_n = f_{n+4} + f_n + f_{n-4} \quad \text{for } n \geq 4 \tag{8}$$
$$9f_n = f_{n+4} + f_{n+1} + f_{n-2} + f_{n-4} \quad \text{for } n \geq 4 \tag{9}$$
$$10f_n = f_{n+4} + f_{n+2} + f_{n-2} + f_{n-4} \quad \text{for } n \geq 4 \tag{10}$$

$$11f_n = f_{n+4} + f_{n+2} + f_n + f_{n-2} + f_{n-4} \qquad \text{for } n \geq 4 \qquad (11)$$
$$12f_n = f_{n+5} + f_{n-1} + f_{n-3} + f_{n-6} \qquad \text{for } n \geq 6 \qquad (12)$$

(Note that each of the above identities is easily seen to be true by induction; and also each identity is true for all integers n by extending the definition of Fibonacci numbers recursively to negative indices.)

In Section 2.1, we construct a map

$$\phi_5 : [5] \times F_n \longrightarrow F_{n+3} \cup F_{n-1} \cup F_{n-4},$$

and we explain why ϕ_5 is bijective, thus providing a bijective proof of Equation (5). In Appendix A, we define bijections ϕ_ℓ for $\ell = 6, 7, 8, 9, 10, 11, 12$ which are similar to ϕ_5 and which provide bijective proofs of the identities given in, respectively, Equations (6), (7), (8), (9), (10), (11), and (12) above. For $(i, X) \in [\ell] \times F_n$, the general approach for each bijection (including ϕ_5) is to prepend a short list of 1's and 2's to X depending on i, and in a few more complicated cases the short list being prepended to X also depends on the initial elements in X (which are deleted before prepending). A Maple implementation [8] of all the bijections in this paper along with programs to check bijectivity may be found online at:

`http://www.math.rutgers.edu/~matchett/Publications/ZeckFibBijections`

In Section 3 we discuss how the identities relate to Zeckendorf's Theorem and give some possible directions for further research.

2 A bijection for $5f_n = f_{n+3} + f_{n-1} + f_{n-4}$

2.1 The bijection ϕ_5

For $n \geq 4$, we define the bijection $\phi_5 : [5] \times F_n \longrightarrow F_{n+3} \cup F_{n-1} \cup F_{n-4}$ below. For $(i, X) \in [5] \times F_n$, we define ϕ_5 by cases depending on the value of i. For each case, the output is X with whatever action is described, for example "prepend $[\![1, 1, 1]\!]$" means that $\phi_5(i, X) = Y$, where Y is the list starting with three 1's followed by elements of X (see below for examples). Similarly, saying "change the $[\![1]\!]$ to $[\![2, 2]\!]$" means that $\phi(i, X) = Y$ where Y consists of two 2's followed all the elements in X except the first element (which must have been 1).

If $i = 1$, then prepend $[\![1, 1, 1]\!]$ $(\hookrightarrow F_{n+3})$
If $i = 2$, then prepend $[\![1, 2]\!]$ $(\hookrightarrow F_{n+3})$
If $i = 3$, then prepend $[\![2, 1]\!]$ $(\hookrightarrow F_{n+3})$
If $i = 4$, then
- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![2, 2]\!]$. $(\hookrightarrow F_{n+3})$
- if X starts with $[\![2, 1]\!]$, then change the $[\![2, 1]\!]$ to $[\![1, 1, 2, 2]\!]$. $(\hookrightarrow F_{n+3})$
- if X starts with $[\![2, 2]\!]$, then change the $[\![2, 2]\!]$ to $[\![\,]\!]$ $(\hookrightarrow F_{n-4})$

If $i = 5$, then
- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![\,]\!]$. $(\hookrightarrow F_{n-1})$
- if X starts with $[\![2]\!]$, then change the $[\![2]\!]$ to $[\![1, 1, 2, 1]\!]$. $(\hookrightarrow F_{n+3})$

We defer the description of the bijections ϕ_ℓ for $\ell = 6, 7, 8, 9, 10, 11, 12$ to Section A, since they are very similar to the bijection ϕ_5 described above.

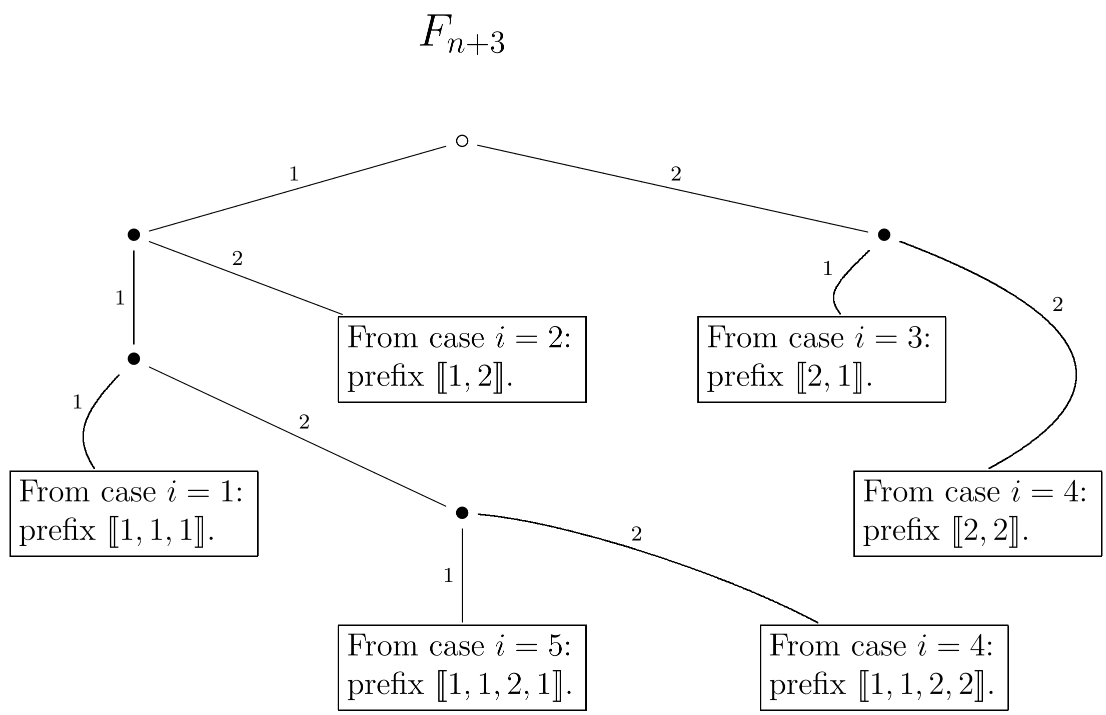

Figure 1: All of the elements of F_{n+3} fit into the tree above. To find the placement of a particular ordered list $X \in F_{n+3}$, move down the tree from the root, $\circ$, at the top, moving left in the k-th level if the k-th entry in X is 1 and moving right if the k-th entry is 2.

Examples for $n = 4$:

$$\begin{array}{rrcrl}
\phi_5: & (1, [\![1,2,1]\!]) & \mapsto & [\![1,1,1,1,2,1]\!] & \in F_7 \\
\phi_5: & (2, [\![1,2,1]\!]) & \mapsto & [\![1,2,1,2,1]\!] & \in F_7 \\
\phi_5: & (3, [\![1,2,1]\!]) & \mapsto & [\![2,1,1,2,1]\!] & \in F_7 \\
\phi_5: & (4, [\![1,2,1]\!]) & \mapsto & [\![2,2,2,1]\!] & \in F_7 \\
\phi_5: & (4, [\![2,1,1]\!]) & \mapsto & [\![1,1,2,2,1]\!] & \in F_7 \\
\phi_5: & (4, [\![2,2]\!]) & \mapsto & [\![]\!] & \in F_0 \\
\phi_5: & (5, [\![1,2,1]\!]) & \mapsto & [\![2,1]\!] & \in F_3 \\
\phi_5: & (5, [\![2,1,1]\!]) & \mapsto & [\![1,1,2,1,1,1]\!] & \in F_7
\end{array}$$

Note that for $n = 4$, the domain for ϕ_5 is $\{1, 2, 3, 4, 5\} \times F_4$ and the range is $F_7 \cup F_3 \cup F_0$.

2.1.1 Showing ϕ_5 is bijective

Each of the eight cases for ϕ_5 is injective. Thus, it is sufficient to check that none of the eight images overlap and that together they are surjective, which we now do.

There is exactly one case mapping to F_{n-4} (namely, when $i = 4$ and X starts with $[\![2, 2]\!]$), and it is clearly surjective. Also, there is exactly one case mapping to F_{n-1} (namely, when $i = 5$ and X starts with $[\![1]\!]$), and it is also clearly surjective. There are six cases mapping to F_{n+3}, and one can compare the prefixes attached in each case to see that the six images are disjoint and together comprise all of F_{n+3} (see Figure 1).

Thus, ϕ_5 is a bijection that provides a combinatorial proof of Equation (5). It is clear that a combinatorial inverse for ϕ_5 may be constructed using the tree in Figure 1.

3 Further directions

As mentioned in [1], the identities in Equations (5), (6), ..., (12) give instances of Zeckendorf's Theorem that every positive integer can be uniquely represented as a sum of non-consecutive Fibonacci numbers (for more on Zeckendorf's Theorem, see [4]; a clever combinatorial proof has also been given by Gessel [5]).

We may define the ℓ-th *Zeckendorf family identity* to be an equation with the form

$$\ell f_n = \sum_{t \in S_\ell} f_{n+t} \tag{13}$$

that holds for all integers n, where S_ℓ is a finite subset of integers that depends only on ℓ and contains no two adjacent integers. Note that the set S_ℓ is unique (by Zeckendorf's Theorem). Also, in [1], the authors mention that the ℓ-th Zeckendorf family identity exists for every ℓ (which can be proven by induction). Plugging in values for ℓ and n in the ℓ-th Zeckendorf family identity gives a formula that is an instance of Zeckendorf's Theorem so long as n is sufficiently large—Zeckendorf's Theorem only uses Fibonacci numbers with positive indices, so we need $n + t > 0$ for all $t \in S_\ell$. However, that fact that Zeckendorf family identities exist for all ℓ is not simply a consequence of Zeckendorf's Theorem, since the ℓ-th Zeckendorf family identity holds for *all* sufficiently large n while the set S_ℓ is fixed.

The bijections described in this paper may appear to involve some arbitrary choices, and indeed they do. For the bijection ϕ_ℓ with domain $[\ell] \times F_n$, the elements in $[\ell]$ can be regarded as formal symbols; and thus there are $\ell!$ bijections possible by renaming the symbols in $[\ell]$, which simply permutes the main cases of the bijection ϕ_ℓ. Also, by reversing lists in the range or domain or both it is trivially possible to construct new bijections; however, these new bijections do not contribute to understanding the structure of the Zeckendorf family identities. Thus, we will define two bijections for the ℓ-th Zeckendorf family identity to be *formally equivalent* if one may be derived from the other by permuting the symbols in $[\ell]$ and possibly reversing lists in the range or domain.

Interestingly, one may also construct numerous bijections for the ℓ-th Zeckendorf family identity that are not formally equivalent. For example, we can construct a bijection ϕ_5' for Equation (5) that is not formally equivalent to ϕ_5. For $(i, X) \in [5] \times F_n$, we define ϕ_5' as follows, using the notation of Subsection 2.1:

If $i = 1$, then prepend $[\![1,1,1]\!]$ $(\hookrightarrow F_{n+3})$
If $i = 2$, then prepend $[\![1,2]\!]$ $(\hookrightarrow F_{n+3})$
If $i = 3$, then prepend $[\![2,1]\!]$ $(\hookrightarrow F_{n+3})$
If $i = 4$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![1,1,2]\!]$. $(\hookrightarrow F_{n+3})$
- if X starts with $[\![2,1]\!]$, then change the $[\![2,1]\!]$ to $[\![2]\!]$. $(\hookrightarrow F_{n-1})$
- if X starts with $[\![2,2]\!]$, then change the $[\![2,2]\!]$ to $[\![\,]\!]$ $(\hookrightarrow F_{n-4})$

If $i = 5$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![2,2]\!]$. $(\hookrightarrow F_{n+3})$
- if X starts with $[\![2]\!]$, then change the $[\![2]\!]$ to $[\![1]\!]$. $(\hookrightarrow F_{n-1})$

Note that ϕ_5' maps into F_{n-1} from two cases, whereas ϕ_5 maps into F_{n-1} from only one case; thus, ϕ_5 and ϕ_5' are not formally equivalent. In fact, it is possible to construct (at least) $3! = 6$ bijections for Equation (5) that are distinct with respect to formal equivalence (two of which are ϕ_5 and ϕ_5'). These bijections provide similar, but distinct, bijective proofs of

Equation (5). Analogously, counting only bijections that are not formally equivalent, one can construct (at least) 6 possible bijections for each of Equations (6), (7), and (8); (at least) $4! = 24$ possible bijections for each of Equations (9), (10), and (11); and (at least) $8! = 40320$ possible bijections for Equation (12).

Paul Raff [7] proposed an approach that can be used to construct a bijection for Equation (5), and it turns out that the resulting bijection is formally equivalent to ϕ'_5.

The following questions are of interest for further work.

1. Is there an approach for constructing bijections for Equations (5), (6), ..., (12), or other Zeckendorf family identities that explicitly uses Zeckendorf's condition that the Fibonacci numbers not be adjacent?

   Note that the bijection ϕ_5 (see Section 2.1) does *not* make explicit use of the fact that the sets F_{n+3}, F_{n-1}, and F_{n-4} have indices that differ by at least 2.

2. Is there an approach for constructing bijections for Equations (5), (6), ..., (12), or other Zeckendorf family identities where $[\ell]$ is interpreted combinatorially, instead of as just a collection of formal symbols?

3. Is there a systematic way to construct a bijection for the ℓ-th Zeckendorf family identity (see Equation (13))?

4. Is there a canonical bijection for the ℓ-th Zeckendorf family identity?

5. Is there a simple combinatorial proof of the existence and uniqueness of the ℓ-th Zeckendorf family identity?

   A possible starting point might be Gessel's elegant combinatorial proof for Zeckendorf's Theorem and a few of its generalizations ([5]). The basic idea of Gessel's proof is to form two simple bijections—one from Zeckendorf representations to the ordered list L of positive binary numbers with no adjacent 1's, and another from L to the positive integers—and then to prove that the composition of the two bijections is the canonical map taking a Zeckendorf representation to the integer it represents.

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Appendix

A Bijections for Equations (6), (7), (8), (9), (10), (11), and (12)

In this appendix we describe bijections ϕ_ℓ for $\ell = 6, 7, 8, 9, 10, 11, 12$, which provide bijective proofs of, respectively, Equations (6), (7), (8), (9), (10), (11), and (12). It is clear that the ϕ_ℓ defined below are bijections by drawing trees analogous to the one in Figure 1.

A.1 A bijection for $6f_n = f_{n+3} + f_{n+1} + f_{n-4}$

We define the bijection $\phi_6 : [6] \times F_n \longrightarrow F_{n+3} \cup F_{n+1} \cup F_{n-4}$ below. This bijection is a combinatorial interpretation of Equation (6). For $(i, X) \in [6] \times F_n$, we define ϕ_6 by cases depending on the value of i.

If $i = 1$, then prepend $[\![1]\!]$ $(\hookrightarrow F_{n+1})$
If $i = 2$, then prepend $[\![1, 1, 1]\!]$ $(\hookrightarrow F_{n+3})$
If $i = 3$, then prepend $[\![1, 2]\!]$ $(\hookrightarrow F_{n+3})$
If $i = 4$, then prepend $[\![2, 1]\!]$ $(\hookrightarrow F_{n+3})$
If $i = 5$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![2, 2]\!]$ $(\hookrightarrow F_{n+3})$
- if X starts with $[\![2, 1]\!]$, then change the $[\![2, 1]\!]$ to $[\![1, 1, 2, 2]\!]$. $(\hookrightarrow F_{n+3})$
- if X starts with $[\![2, 2]\!]$, then change the $[\![2, 2]\!]$ to $[\![]\!]$. $(\hookrightarrow F_{n-4})$

If $i = 6$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![2]\!]$. $(\hookrightarrow F_{n+1})$
- if X starts with $[\![2]\!]$, then change the $[\![2]\!]$ to $[\![1, 1, 2, 1]\!]$. $(\hookrightarrow F_{n+3})$

A.2 A bijection for $7f_n = f_{n+4} + f_{n-4}$

We define the bijection $\phi_7 : [7] \times F_n \longrightarrow F_{n+4} \cup F_{n-4}$ below. This bijection is a combinatorial interpretation of Equation (7). For $(i, X) \in [7] \times F_n$, we define ϕ_7 by cases depending on the value of i.

If $i = 1$, then prepend $[\![1, 1, 1, 1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 2$, then prepend $[\![1, 1, 2]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 3$, then prepend $[\![1, 2, 1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 4$, then prepend $[\![2, 1, 1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 5$, then prepend $[\![2, 2]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 6$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![1, 2, 2]\!]$ $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2, 1]\!]$, then change the $[\![2, 1]\!]$ to $[\![1, 1, 1, 2, 2]\!]$. $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2, 2]\!]$, then change the $[\![2, 2]\!]$ to $[\![]\!]$. $(\hookrightarrow F_{n-4})$

If $i = 7$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![2, 1, 2]\!]$. $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2]\!]$, then change the $[\![2]\!]$ to $[\![1, 1, 1, 2, 1]\!]$. $(\hookrightarrow F_{n+4})$

A.3 A bijection for $8f_n = f_{n+4} + f_n + f_{n-4}$

We define the bijection $\phi_8 : [8] \times F_n \longrightarrow F_{n+4} \cup F_n \cup F_{n-4}$ below. This bijection is a combinatorial interpretation of Equation (8). For $(i, X) \in [8] \times F_n$, we define ϕ_8 by cases depending on the value of i.

If $i = 1$, then prepend $[\![1, 1, 1, 1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 2$, then prepend $[\![1, 1, 2]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 3$, then prepend $[\![1, 2, 1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 4$, then prepend $[\![2, 1, 1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 5$, then prepend $[\![2, 2]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 6$, then do nothing $(\hookrightarrow F_{n})$
If $i = 7$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![1, 2, 2]\!]$ $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2, 1]\!]$, then change the $[\![2, 1]\!]$ to $[\![1, 1, 1, 2, 2]\!]$. $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2, 2]\!]$, then change the $[\![2, 2]\!]$ to $[\![]\!]$. $(\hookrightarrow F_{n-4})$

If $i = 8$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![2, 1, 2]\!]$. $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2]\!]$, then change the $[\![2]\!]$ to $[\![1, 1, 1, 2, 1]\!]$. $(\hookrightarrow F_{n+4})$

A.4 A bijection for $9f_n = f_{n+4} + f_{n+1} + f_{n-2} + f_{n-4}$

We define the bijection $\phi_9 : [9] \times F_n \longrightarrow F_{n+4} \cup F_{n+1} \cup F_{n-2} \cup F_{n-4}$ below. This bijection is a combinatorial interpretation of Equation (9). For $(i, X) \in [9] \times F_n$, we define ϕ_9 by cases depending on the value of i.

If $i = 1$, then prepend $[\![1, 1, 1, 1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 2$, then prepend $[\![1, 1, 2]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 3$, then prepend $[\![1, 2, 1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 4$, then prepend $[\![2, 1, 1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 5$, then prepend $[\![2, 2]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 6$, then prepend $[\![1]\!]$ $(\hookrightarrow F_{n+1})$
If $i = 7$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![1, 1, 1, 2]\!]$. $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2]\!]$, then change the $[\![2]\!]$ to $[\![]\!]$. $(\hookrightarrow F_{n-2})$

If $i = 8$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![1, 2, 2]\!]$ $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2, 1]\!]$, then change the $[\![2, 1]\!]$ to $[\![2, 2]\!]$. $(\hookrightarrow F_{n+1})$
- if X starts with $[\![2, 2]\!]$, then change the $[\![2, 2]\!]$ to $[\![]\!]$. $(\hookrightarrow F_{n-4})$

If $i = 9$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![2, 1, 2]\!]$. $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2, 1]\!]$, then change the $[\![2, 1]\!]$ to $[\![2, 1, 1]\!]$. $(\hookrightarrow F_{n+1})$
- if X starts with $[\![2, 2]\!]$, then change the $[\![2, 2]\!]$ to $[\![2, 1, 2]\!]$. $(\hookrightarrow F_{n+1})$

A.5 A bijection for $10f_n = f_{n+4} + f_{n+2} + f_{n-2} + f_{n-4}$

We define the bijection $\phi_{10} : [10] \times F_n \longrightarrow F_{n+4} \cup F_{n+2} \cup F_{n-2} \cup F_{n-4}$ below. This bijection is a combinatorial interpretation of Equation (10). For $(i, X) \in [10] \times F_n$, we define ϕ_{10} by cases depending on the value of i.

If $i = 1$, then prepend $[\![1,1,1,1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 2$, then prepend $[\![1,1,2]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 3$, then prepend $[\![1,2,1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 4$, then prepend $[\![2,1,1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 5$, then prepend $[\![2,2]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 6$, then prepend $[\![1,1]\!]$ $(\hookrightarrow F_{n+2})$
If $i = 7$, then prepend $[\![2]\!]$ $(\hookrightarrow F_{n+2})$
If $i = 8$, then
- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![1,1,1,2]\!]$. $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2]\!]$, then change the $[\![2]\!]$ to $[\![]\!]$. $(\hookrightarrow F_{n-2})$

If $i = 9$, then
- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![1,2,2]\!]$ $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2,1]\!]$, then change the $[\![2,1]\!]$ to $[\![1,2,2]\!]$. $(\hookrightarrow F_{n+2})$
- if X starts with $[\![2,2]\!]$, then change the $[\![2,2]\!]$ to $[\![]\!]$. $(\hookrightarrow F_{n-4})$

If $i = 10$, then
- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![2,1,2]\!]$. $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2]\!]$, then change the $[\![2]\!]$ to $[\![1,2,1]\!]$. $(\hookrightarrow F_{n+2})$

A.6 A bijection for $11f_n = f_{n+4} + f_{n+2} + f_n + f_{n-2} + f_{n-4}$

We define the bijection $\phi_{11} : [11] \times F_n \longrightarrow F_{n+4} \cup F_{n+2} \cup F_n \cup F_{n-2} \cup F_{n-4}$ below. This bijection is a combinatorial interpretation of Equation (11). For $(i, X) \in [11] \times F_n$, we define ϕ_{11} by cases depending on the value of i.

If $i = 1$, then prepend $[\![1,1,1,1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 2$, then prepend $[\![1,1,2]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 3$, then prepend $[\![1,2,1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 4$, then prepend $[\![2,1,1]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 5$, then prepend $[\![2,2]\!]$ $(\hookrightarrow F_{n+4})$
If $i = 6$, then prepend $[\![1,1]\!]$ $(\hookrightarrow F_{n+2})$
If $i = 7$, then prepend $[\![2]\!]$ $(\hookrightarrow F_{n+2})$
If $i = 8$, then do nothing $(\hookrightarrow F_n)$
If $i = 9$, then
- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![1,1,1,2]\!]$. $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2]\!]$, then change the $[\![2]\!]$ to $[\![]\!]$. $(\hookrightarrow F_{n-2})$

If $i = 10$, then
- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![1,2,2]\!]$ $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2,1]\!]$, then change the $[\![2,1]\!]$ to $[\![1,2,2]\!]$. $(\hookrightarrow F_{n+2})$
- if X starts with $[\![2,2]\!]$, then change the $[\![2,2]\!]$ to $[\![]\!]$. $(\hookrightarrow F_{n-4})$

If $i = 11$, then
- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![2,1,2]\!]$. $(\hookrightarrow F_{n+4})$
- if X starts with $[\![2]\!]$, then change the $[\![2]\!]$ to $[\![1,2,1]\!]$. $(\hookrightarrow F_{n+2})$

A.7 A bijection for $12f_n = f_{n+5} + f_{n-1} + f_{n-3} + f_{n-6}$

We define the bijection $\phi_{12} : [12] \times F_n \longrightarrow F_{n+5} \cup F_{n-1} \cup F_{n-3} \cup F_{n-6}$ below. This bijection is a combinatorial interpretation of Equation (12). For $(i, X) \in [12] \times F_n$, we define ϕ_{12} by cases depending on the value of i.

If $i = 1$, then prepend $[\![1,1,1,1,1]\!]$ $(\hookrightarrow F_{n+5})$
If $i = 2$, then prepend $[\![1,1,1,2]\!]$ $(\hookrightarrow F_{n+5})$
If $i = 3$, then prepend $[\![1,1,2,1]\!]$ $(\hookrightarrow F_{n+5})$
If $i = 4$, then prepend $[\![1,2,1,1]\!]$ $(\hookrightarrow F_{n+5})$
If $i = 5$, then prepend $[\![2,1,1,1]\!]$ $(\hookrightarrow F_{n+5})$
If $i = 6$, then prepend $[\![1,2,2]\!]$ $(\hookrightarrow F_{n+5})$
If $i = 7$, then prepend $[\![2,1,2]\!]$ $(\hookrightarrow F_{n+5})$
If $i = 8$, then prepend $[\![2,2,1]\!]$ $(\hookrightarrow F_{n+5})$
If $i = 9$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![1,1,1,1,2]\!]$ $(\hookrightarrow F_{n+5})$
- if X starts with $[\![2,1]\!]$, then change the $[\![2,1]\!]$ to $[\![]\!]$. $(\hookrightarrow F_{n-3})$
- if X starts with $[\![2,2,1]\!]$, then change the $[\![2,2,1]\!]$ to $[\![2,2,2,2,2]\!]$. $(\hookrightarrow F_{n+5})$
- if X starts with $[\![2,2,2]\!]$, then change the $[\![2,2,2]\!]$ to $[\![]\!]$. $(\hookrightarrow F_{n-6})$

If $i = 10$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![]\!]$. $(\hookrightarrow F_{n-1})$
- if X starts with $[\![2]\!]$, then change the $[\![2]\!]$ to $[\![2,2,2,1]\!]$. $(\hookrightarrow F_{n+5})$

If $i = 11$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![2,1,1,2]\!]$ $(\hookrightarrow F_{n+5})$
- if X starts with $[\![2]\!]$, then change the $[\![2]\!]$ to $[\![1,1,2,2,1]\!]$. $(\hookrightarrow F_{n+5})$

If $i = 12$, then

- if X starts with $[\![1]\!]$, then change the $[\![1]\!]$ to $[\![1,2,1,2]\!]$. $(\hookrightarrow F_{n+5})$
- if X starts with $[\![2,1]\!]$, then change the $[\![2,1]\!]$ to $[\![1,1,2,2,2]\!]$. $(\hookrightarrow F_{n+5})$
- if X starts with $[\![2,2]\!]$, then change the $[\![2,2]\!]$ to $[\![2,2,2,2,1]\!]$. $(\hookrightarrow F_{n+5})$

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