

ADAMS-TYPE MAPS ARE NOT STABLE UNDER COMPOSITION

ROBERT BURKLUND, ISHAN LEVY, AND PIOTR PSTRAGOWSKI

(Communicated by Julie Bergner)

ABSTRACT. We give a simple counterexample to the plausible conjecture that Adams-type maps of ring spectra are stable under composition. We then show that over a field, this failure is quite extreme, as any map to an \mathbb{E}_∞ - k -algebra is a transfinite composition of Adams-type maps.

The Adams spectral sequence is a fundamental tool in stable homotopy theory which, given a map $A \rightarrow B$ of ring spectra, lets us compute the homotopy groups of A in terms of information living over B . Unfortunately, the E_2 -page of the Adams spectral sequence can be difficult to identify in general. The standard additional assumption which makes the E_2 -page computable in terms of homological algebra is that the map $A \rightarrow B$ is flat in one of the senses below.

Definition 1. We say a map $A \rightarrow B$ of \mathbb{E}_1 -rings is

- (1) *Adams-type* if we can write $B \simeq \varinjlim B_\alpha$ as a filtered colimit of perfect A -modules with the property that $\pi_*(B \otimes_A B_\alpha)$ is projective as a π_*B -module,
- (2) *descent-flat*¹ if $\pi_*(B \otimes_A B)$ is flat as a left π_*B -module.

If $A \rightarrow B$ is descent-flat, then for any pair M, N of A -modules such that $\pi_*(B \otimes_A N)$ is projective π_*B -module, the associated Adams spectral sequence computing homotopy classes of A -module maps from M to N has signature

$$\mathrm{Ext}_{(\pi_*B, \pi_*(B \otimes_A B))}^{s,t}(\pi_*(B \otimes_A N), \pi_*(B \otimes_A M)) \Longrightarrow [N, M]_A.$$

If $A \rightarrow B$ is furthermore Adams-type, then one can construct an Adams spectral sequence with the above signature with no projectivity assumption on $\pi_*(B \otimes_A N)$, as in the work of Devinatz [Dev97, §1].

As both of the above notions are a form of flatness, it is natural to expect that they are stable under composition, and the authors of this note have spent considerable amount of time trying to show that this is indeed the case. To our surprise, this is very far from being true; our first result is the following simple counterexample.

Received by the editors March 7, 2022, and, in revised form, July 2, 2022.

2020 *Mathematics Subject Classification.* Primary 55T15.

The second author was supported by the NSF Graduate Research Fellowship under Grant No. 1745302.

¹In older literature, what we call descent-flat is often simply referred to as *flat*. We avoid the latter term as it is also often used to refer to maps such that $\pi_*A \rightarrow \pi_*B$ is flat, a much stronger condition than the one we work with.

Theorem 2. *Both of the maps $\mathbb{S} \rightarrow \text{MU}$ and $\text{MU} \rightarrow \mathbb{Z}$ are Adams-type, but the composite $\mathbb{S} \rightarrow \mathbb{Z}$ is not even descent-flat. In particular, neither Adams-type nor descent-flat maps of \mathbb{E}_∞ -ring spectra are stable under composition.*

Proof. It is well-known that $\mathbb{S} \rightarrow \text{MU}$ is Adams-type [Ada74, 13.4(iv)]. Similarly, $\mathbb{Z}_*\mathbb{Z}$ is known to contain torsion elements in positive degrees and so is not flat over \mathbb{Z}_* [Car56], hence $\mathbb{S} \rightarrow \mathbb{Z}$ is not descent-flat.

To see that $\text{MU} \rightarrow \mathbb{Z}$ is Adams-type, note that if we write $\text{MU}_* \simeq \mathbb{Z}[b_1, b_2, \dots]$ for some choice of generators b_i , then

$$\mathbb{Z} = \text{colim}_n \bigotimes_{1 \leq i \leq n}^{\text{MU}} \text{MU}/b_i$$

in MU -modules. We claim this is the needed expression of \mathbb{Z} as a filtered colimit. To see this, note that we have an equivalence

$$\mathbb{Z} \otimes_{\text{MU}} \text{MU}/b_i \simeq \mathbb{Z}/b_i \simeq \mathbb{Z} \oplus \Sigma^{|b_i|+1}\mathbb{Z}$$

as b_i vanishes in \mathbb{Z} and hence for each n we have

$$\mathbb{Z} \otimes_{\text{MU}} \bigotimes_{1 \leq i \leq n}^{\text{MU}} \text{MU}/b_i \simeq \bigotimes_{1 \leq i \leq n}^{\mathbb{Z}} \mathbb{Z} \oplus \Sigma^{|b_i|+1}\mathbb{Z}$$

which is a perfect \mathbb{Z} -module whose homotopy groups form a finitely generated free abelian group, as needed. □

The counterexample of Theorem 2 was contrary to our expectations, and led us to ask which maps of ring spectra can be written as compositions (perhaps infinite) of Adams-type maps. In Theorem 6 we show that, surprisingly, *any* map of \mathbb{E}_∞ -rings whose target admits a structure of an \mathbb{E}_∞ -algebra over a field has this property.

Lemma 3. *Let $R \rightarrow R'$ be a map of \mathbb{E}_2 -algebras. If A is an \mathbb{E}_1 - R -algebra such that the unit $R \rightarrow A$ is of Adams-type, then so is $R' \rightarrow R' \otimes_R A$.*

Proof. We start with an observation about the Adams-type condition: If $B \rightarrow C$ is a map of \mathbb{E}_1 -algebras, and M is a B -module, then $\pi_*(C \otimes_B M)$ is projective as a π_*C -module iff the C -module $C \otimes_B M$ is a retract of a sum of suspensions of copies of C . We call a B -module which satisfies this condition C -projective.

Now we prove the lemma. Write A as a filtered colimit $A \simeq \text{colim} A_\alpha$ of A -projective R -modules. Then $R' \otimes_A R \simeq \text{colim} R' \otimes_R A_\alpha$ is a filtered colimit with the required property, as

$$(R' \otimes_R A) \otimes_{R'} (R' \otimes_R A_\alpha) \simeq R' \otimes_R (A \otimes_R A_\alpha)$$

which implies that $R' \otimes_R A_\alpha$ is $(R' \otimes_R A)$ -projective. □

Corollary 4. *Adams-type maps of \mathbb{E}_∞ -ring spectra are stable under pushouts.*

Lemma 5. *Let $A \rightarrow B$ be a map of \mathbb{E}_1 -rings such that either π_*A or π_*B is a graded skew-field (that is, every non-zero homogeneous element is invertible). Then $A \rightarrow B$ is Adams-type.*

Proof. Suppose that the first condition holds. Then, every A -module is a direct sum of shifts of A , and hence for any way to write $B \simeq \varinjlim B_\alpha$ as a filtered colimit

of perfect A -modules, $B \otimes_A B_\alpha$ is a finite direct sum of shifts of B and hence $\pi_*(B \otimes_A B_\alpha)$ is a free B_* -module.

Similarly, if π_*B is a graded skew-field, then the perfect B -modules $B \otimes_A B_\alpha$ are also finite sums of shifts of B , as needed. \square

Theorem 6. *Let $f : A \rightarrow B$ be a map of \mathbb{E}_∞ -rings such that B admits a structure of an \mathbb{E}_∞ -algebra over a field k . Then f can be factored as transfinite composite of Adams-type maps.*

Proof. We have a canonical factorization

$$A \rightarrow k \otimes A \rightarrow B,$$

where the second arrow is a map of \mathbb{E}_∞ - k -algebras. The first arrow is a base-change of $\mathbb{S} \rightarrow k$ and hence is a Adams-type by a combination of Corollary 4 and Lemma 5. It follows that to prove the result we can assume that $A \rightarrow B$ is a morphism of \mathbb{E}_∞ - k -algebras.

Let $k\{x_n\}$ denote the free \mathbb{E}_∞ - k -algebra on a variable of degree $|x_n| = n$. Let \mathcal{G} denote the set of maps $i_n : k \rightarrow k\{x_n\}$ together with the maps $p_n : k\{x_n\} \rightarrow k$ determined by $x_n \mapsto 0$. As a consequence of Lemma 5, all of the i_n and p_n are Adams-type, as either the source or target is a field. By the small object argument, the given map $A \rightarrow B$ can be factored as

$$A \rightarrow B' \rightarrow B$$

where $B' \rightarrow B$ has the right lifting property with respect to the maps in \mathcal{G} , and $A \rightarrow B'$ is a transfinite composition of maps $A_\alpha \rightarrow A_{\alpha+1}$ which fit into pushout squares

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \downarrow & \lrcorner & \downarrow \\ A_\alpha & \longrightarrow & A_{\alpha+1} \end{array}$$

where $f \in \mathcal{G}$. Each such map is Adams-type by Corollary 4, and it follows that $A \rightarrow B'$ is a transfinite composite of such maps. On the other hand, one may observe that the right lifting property with respect to i_n implies surjectivity on $\pi_n(-)$ and the right lifting property with respect to p_n implies injectivity on $\pi_n(-)$, therefore $B' \rightarrow B$ is an equivalence. \square

Remark 7. By using a more careful argument, where we also allow pushouts along maps $k\{S\} \rightarrow k$ and $k \rightarrow k\{S\}$ for S a (graded) set of generators, one can show that in the context of Theorem 6 the given map factors as an ω -indexed composite of Adams-type maps.

We believe the above two results show that the somewhat unexplored theory of Adams-type ring spectra still has a few surprises up its sleeve. To further emphasize this point, we share with the reader a few natural questions which we believe are open.

- (1) Does there exist a descent-flat map $A \rightarrow B$ which is not Adams-type?²

²If we had to guess, we expect that a descent-flat map of ring spectra which is not Adams-type does indeed exist, although we couldn't find one. A curious variant of this question (which we also do not know the answer to) would be to ask if exists a descent-flat map $A \rightarrow B$ such that the associated homology theory $\pi_*(B \otimes_A -) : \text{Mod}(A) \rightarrow \text{Comod}_{\pi_*(B \otimes_A B)}$ is adapted in the sense

- (2) Is every map of \mathbb{E}_1 -algebras (over the sphere) a transfinite composite of Adams-type maps? Is it true for every map of \mathbb{E}_∞ -algebras?
- (3) Let $A \rightarrow B$ be an Adams-type map and M be an A -module such that $\pi_*(B \otimes_A M)$ is a flat B_* -module. Can we write $M \simeq \varinjlim M_\alpha$ as a filtered colimit of perfect A -modules such that $\pi_*(B \otimes_A M_\alpha)$ is a projective B_* -module?

ACKNOWLEDGMENT

We would like to thank the anonymous referee for suggesting an improvement to Theorem 6, observing that it is enough to have a k -algebra structure on the target rather than the whole map.

REFERENCES

- [Ada74] J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, Ill.-London, 1974. MR0402720
- [Car56] H. Cartan, *Algèbres d'Eilenberg-MacLane et Homotopie*, Exposés 2 à 16, Séminaire Henri Cartan, Ecole Normale Supérieure, Paris (1956).
- [Dev97] Ethan S. Devinatz, *Morava modules and Brown-Comenetz duality*, Amer. J. Math. **119** (1997), no. 4, 741–770. MR1465068
- [PP21] Irakli Patchkoria and Piotr Pstragowski, *Adams spectral sequences and Franke's algebraicity conjecture*, Preprint [arXiv:2110.03669](https://arxiv.org/abs/2110.03669), 2021.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, DENMARK
Email address: rb@math.ku.dk

DEPARTMENT OF MATHEMATICS, MIT, CAMBRIDGE, MASSACHUSETTS
Email address: ishanl@mit.edu

DEPARTMENT OF MATHEMATICS, HARVARD, CAMBRIDGE, MASSACHUSETTS
Email address: pstragowski.piotr@gmail.com

of [PP21, Definition 2.19], which is not Adams-type? In other words, does possessing a modified Adams spectral sequence based on $\pi_*(B \otimes_A B)$ -comodules characterize Adams-type maps?