Worksheet 8

July 22, 2019

1. **(Problem 6 from Worksheet 7.)** A monoidal category is said to be **strict** if the natural transformations giving associativity of $\otimes$ and unitality of I are all the identity natural transformation. Show that the braid category B is strict monoidal. Show that B is braiding is natural but not symmetric.

2. Following Reshetikin–Turaev, we will say that a braided monoidal category is **compact** if, for all objects V in the category, there is an object V^* together with maps

$$\epsilon_V \colon V \otimes V^* \to I,$$

and

$$\eta_V \colon V^* \otimes V \to I$$

such that

$$l \circ \epsilon \otimes 1 \circ a^{-1} \circ 1 \otimes \eta \circ r^{-1} = 1_V,$$

and

$$r \circ 1 \otimes e \circ a \circ \eta \otimes 1 \circ l^{-1} = 1_{V^*},$$

natural in V.

   (a) Write down the requirements on the composites diagrammatically, making sure you have the domain/codomain for the right/left units (r and l) and associator (a), correct. (Note that these maps r, l, and a are part of the data of a monoidal category as defined in class. See Reshetikin–Turaev for the definition, using same notation.)

   (b) Show (by example) that it's not necessarily that $V^{**} = V$.

NOTE: Reshetikin–Turaev use the word "compact" to describe the property explored in this problem. In more modern terminology, we might say that every object of the category is **fully dualizeable**.

3. In category theory, the correct notion of "isomorphism" is not the naive one (e.g. a functor that is a bijection of objects and morphisms). It is that of an **equivalence of categories**: a functor $G\colon \mathcal{C} \to \mathcal{D}$ is said to be an equivalence if there is a functor $F\colon \mathcal{D} \to \mathcal{C}$ together with natural transformations

$$F \circ G \implies 1_{\mathcal{C}}$$

and

$$G \circ F \implies 1_{\mathcal{D}}$$

with each component of the respective transformations an isomorphism in the category $\mathcal{C}$ (resp. $\mathcal{D}$.) Such a natural transformation with each component an isomorphism is called a **natural isomorphism**.

   (a) We can consider a category $\mathcal{V}_0$ with objects $\{0, 1, 2, 3, \ldots$ and $\hom_{\mathcal{V}_0}(m, n) = \mathrm{Mat}\, m \times n(k)$. Show that the category $\mathrm{Vect}_k^{\mathrm{fin}}$ of finite-dimensional k-vector spaces and k-linear transformations is equivalent to the category $\mathcal{V}_0$, but is not naively isomorphic.

   (b) Show that a functor G is an equivalence of categories if and only if it is fully faithful and essentially surjective on objects.
   (See Worksheet 5, problem 5 for the definition of fully faithful. A functor is **essentially surjective on objects** if for all $x \in \mathcal{D}$, there is a $y \in \mathcal{C}$ such that Fy is isomorphic to x as an object in $\mathcal{D}$.)

4. **The role of the braid category** B**.** Given monoidal category $C = (C_0, \times, I, a, r, l)$ and $D = (D_0, +, J, a', l', r')$, we say that a functor $F\colon C_0 \to D_0$ is bf (strong) moniodal if:

$$F(I) = J,$$

and for any $U, V, W \in C_0$,:

$$F(U \times V) = F(U) + F(V),$$
$$a'_{FU,FV,FW} = F(a_{U,V,W})$$
$$r'_{FU} = Fr_U$$

$$l'_{FU} = Fl_U.$$

That is, if F "takes all the monoidal data of C to that of D."

If the categories are braided, with braidings

$$c = \{c_{U,V}\}_{U,V \in C_0}$$

and

$$d = \{d_{H,K}\}_{H,K \in D_0},$$

then we say that F is a **strong braided monoidal functor** if, for any U, V in C_0,

$$d_{F(U),F(V)} = F(c_{U,V}).$$

Let BMF denote the category of small braided monoidal categories and strong braided monoidal functors between them. Let $M = (M_0, \otimes, I, a, r, l, c)$ be a small braided monoidal category. Show that, as sets, $\hom_{BMF}(B, M) \simeq M_0$, following the outline below:

(a) Argue that we may assume M is strict, using the fact from class that every monoidal category is equivalent to a strict one in a way compatible with any given braiding.

(b) Show that the association

$$(F\colon B \to M) \mapsto F(1)$$

is surjective by showing that, for any $a \in M_0$, there is a strong braided monoidal functor with $F(1) = a$.

(c) Conversely, show that two strong braided monoidal functors with the same value on the object 1 of B must agree on all of B.

5. Show how to extend a compact structure on a monoidal category C to one on its strictification $C_\square$ (recall the relevant definitions from class).

6. (a) Explain why every knot can be considered as a ribbon graph (consisting of only one copy of $S^1 \times [0, 1]$ embedded in $\mathbb{R}^3$ between the planes $z = 1$ and $z = 0$.)

   (b) Explain in what sense a graph is a ribbon graph (don't try to make this part too precise/ "functorial").