Worksheet 7

July 3, 2019

1. This problem deals with duality (aka the process of adding "co-" to various definitions/ constructions).

   (a) Show that a functor $F: J \to \mathcal{C}$ uniquely determines a functor $F^{\mathrm{op}}: J^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ which is "the same" on objects and morphisms. In particular, a diagram of shape J in $\mathcal{C}$ can be viewed as a diagram of shape J^{op} in $\mathcal{C}^{\mathrm{op}}$.

   (b) Using the previous item, the colimit of a diagram of shape J in C should be the limit of the corresponding diagram of shape J^{op} in $\mathcal{C}^{\mathrm{op}}$. Write down the precise definition of a colimit in terms of universal properties, without referring to opposites.

   (c) Define a cocone and category of cocones over a fixed diagram in J. What is the universal property of the colimit cocone in this auxiliary category?

2. Let A, B, and C be sets, and let $\times$ denote the cartesian product of sets (which we know also gives a categorical product). Show that there is a bijection $(A \times B) \times C \simeq A \times (B \times C)$, without explicitly defining maps between the sets.

3. Practice with limits and colimits.

   (a) Determine the (binary) product and coproduct in the category of rings.

   (b) Describe (arbitrary) small products and coproducts in the category of sets.

   (c) Describe (arbitrary) small limits and colimits in teh category of sets in terms of your description of products and coproducts.

4. Consider the category of commutative, unital rings and ring homomorphisms. Fixing a ring R, we get a functor $R \otimes (-)\colon \text{Rng} \to \text{Rng}$ given by $S \mapsto R \otimes_{\mathbb{Z}} S$. Show that $R \otimes (-)$ admits a right adjoint.

5. Verify that any category admitting finite products has a symmetric monoidal structure given by $\times\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ given by sending a pair (A, B) to the categorical product $A \times B$. (The point is that you'll need to use the universal property of the product multiple times to check the axioms.)

6. A monoidal category is said to be **strict** if the natural transformations giving associativity of $\otimes$ and unitality of I are all the identity natural transformation. Show that the braid category B is strict monoidal. Show that B is braiding is natural but not symmetric.

7. Given a monoidal category $\langle \mathcal{C}, \otimes, I\rangle$ and an object X of $\mathcal{C}$, we can define a functor $X \otimes (-)\colon \mathcal{C} \to \mathcal{C}$ given by restricting the monoidal product $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ to the full subcategory $X \times \mathcal{C}$. (I.e. given on objects by $Y \mapsto X \otimes Y$.)

   A category C is said to be **closed** if the functor $X \otimes (-)$ admits a right adjoint. That is, if there is a functor $[R, -]$ such that we get natural isomorphisms

   $$\hom(R \otimes S, T) \simeq \hom(S, [R, T])$$

   naturally in $S, T \in \text{ob}(\text{Rng})$.

   Show that the monoidal category $\langle \text{Rng}, \otimes_Z, \mathbb{Z}\rangle$ is not closed.

8. Let A be a be a not necessarily commutative algebra over a field k, and consider the category $\text{Rep}(A)$ of its finited-dimensional linear representations. That is, $\text{Rep}(A)$ has objects left A-modules with underlying k-vector spaces finite-dimensional. Morphisms in $\text{Rep}(A)$ are A-linear maps.

   (a) One can define a monoidal structure on $\text{Rep}(A)$ by "lifting" that on Vect_k: given M, N in the representation category, one must equip $M \otimes_k N$ with the structure of a left A-module. Show that this can be done.

   (b) Can you given $\text{Rep}(A)$ the structure of a braided monoidal category? What additional structure might be helfpul?