Worksheet 4

Problem 1. The Alexander polynomial $\Delta(L) \in \mathbf{Z}[t^{1/2}, t^{-1/2}]$, an invariant of an oriented link L, is characterized by the Skein relation

$$\Delta(K_+) - \Delta(K_-) = (t^{1/2} - t^{-1/2})\Delta(K_0)$$

and the identity $\Delta(U) = 1$ where U is the unknot.

(a) Calculate the Alexander polynomial of the two component unlink.

(b) Show that the value of the Alexander polynomial of any oriented knot at $t = 1$ is 1. What is its value on an oriented link with more than one component?

Problem 2. Given a space X and open subsets $A, B \subset X$ for which $A \cup B = X$, the Mayer-Vietoris sequence is a long exact sequence

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0.$$

What this means is that for any pair of consecutive maps, the kernel of the latter is equal to the image of the former (e.g. the kernel of the map $H_n(A) \oplus H_n(B) \to H_n(X)$ is the image of the map $H_n(A \cap B) \to H_n(A) \oplus H_n(B)$).

Establish the "decategorification" of this long exact sequence:

$$\chi(A) + \chi(B) = \chi(X) + \chi(A \cap B)$$

where χ is the Euler characteristic. *Hint:* Problem 6 of Worksheet 2.

Problem 3. Tensor product of graded vector spaces.

Given a finite-dimensional vector space W_n for each $n \in \mathbf{Z}$ where all but finitely many are zero, we may form their direct sum $W = \bigoplus_n W_n$. The finite-dimensional vector space W along with its direct-sum decomposition is called a graded vector space. An element of W_n, thought of as an element of W, is called *homogeneous* of degree n, and a general element of W is a finite linear combination of homogeneous elements (of various degree). An example of a graded vector space is the homology $H_*(X) = \bigoplus_n H_n(X)$ of a space.

Recall that if W is vector space with basis $e_1, \ldots, e_n$, then the tensor product $W \otimes W$ has basis $e_i \otimes e_j$ for $1 \le i, j \le n$. If W is graded, then $W \otimes W$ may be given the structure of a graded vector space in the following way: choose a basis $e_1, \ldots, e_n$ for W consisting of homogeneous elements, and declare that $e_i \otimes e_j$ is homogeneous of degree $\deg(e_i) + \deg(e_j)$. Another way of saying this is if $W = \bigoplus_n W_n$, then

$$W \otimes W = \bigoplus_n (W \otimes W)_n \qquad \text{where} \qquad (W \otimes W)_n = \bigoplus_j W_j \otimes W_{n-j}.$$

The *graded dimension* of a graded vector space W is by definition the Laurent polynomial

$$q\text{-dim}(W) = \sum_{n \in \mathbf{Z}} (\dim W_n)\, q^n$$

in $\mathbf{Z}[q, q^{-1}]$ (the notation q-dim is used only in the context of the Jones polynomial). Show that

$$q\text{-dim}(W \otimes W) = q\text{-dim}(W) \cdot q\text{-dim}(W)$$

where the right-hand side of the equality is multiplication of Laurent polynomials.

Problem 4. The number of complete smoothings of a diagram with n crossings is 2^n. Computing Khovanov homology by hand quickly becomes too tedius, but for very small n it's reasonable and helpful to do the computations.

(a) How many oriented links with 1 or 2 components admit a diagram with only two crossings?

(b) Compute their Khovanov homologies using $\mathbf{F} = \mathbf{Z}/2$ coefficients.