Worksheet 10

July 24, 2019

1. **Affine algebraic groups as Hopf algebras.** Let G be an affine algebraic group over a field k: that is, G is an affine scheme over k equipped with a group structure such that the multiplication, inversion, and identity maps are all morphisms of affine schemes over k. Since G is affine, $G = \operatorname{spec} S$ for some commutative k-algebra S, and the structure maps of the group correspond to k-algebra maps. Show that S is a Hopf algebra.

2. Let (A, R) be a quasitriangular Hopf algebra (so, in particular, $R \in A \otimes A$ is invertible). Verify the identity

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where, if P is the swap map $A \otimes A \to A \otimes A$, R_{ij} are defined by:

$$R_{12} = R \otimes 1 \in A \otimes A \otimes A,$$

$$R_{13} = (id \otimes P)(R_{12}),$$

$$R_{23} = 1 \otimes R.$$

This and many other exercises on this worksheet are done in Reshetikin–Turaev. Try to do them yourself, looking to the paper as needed.

3. Let $\epsilon \colon A \to k$ be the counit of the Hopf algebra structure on A over a field k. Let $\Delta \colon A \to A \otimes A$ be the comulitplication. Show that

$$(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta.$$

4. **The braided monoidal structure on** $\operatorname{Rep} A$**.** Let A be a Hopf algebra over a field k.

(a) Verify that the tensor product $V \otimes W := V \otimes_k W$ of two left A-modules $V, W \in \operatorname{Rep} A$ can be equipped with the structure of a left A-module, using the comultiplication on A.

(b) The braiding structure on $\operatorname{Rep}(A)$ is given

$$c_{V,W} = P_{V,W} \circ (\rho_V \otimes \rho_W)(R) \colon V \otimes W \to W \otimes V.$$

This makes sense as follows: R is an invertible element of $A \otimes A$, so $(\rho_V \otimes \rho_W)(R)$ acts as invertible map $V \otimes W \to V \otimes W$. We can the compose with the A-linear "swap" isomorphism $P_{V,W}$ induced by $(v, w) \mapsto w \otimes v$. We get isomorphisms $V \otimes W \to W \otimes V$ in $\operatorname{Rep} A$, which are in fact natural in V and W and define a braiding structure on $\operatorname{Rep} A$.
Explain why it is not necessarily true that this braiding is symmetric. That is, why it is not immediate that $c_{V,W} \circ c_{W,V} = id$, even though $P_{V,W} \circ P_{W,V} = id$.

5. Let (A, R) be a quasitriangular Hopf algebra over a field k, with antipode map s. If $R = \sum \alpha_i \otimes \beta_i \in A \otimes A$ for some $\alpha_i, \beta_i \in A$, then we define $u = \sum_i s(\beta_i)\alpha_i$. Show that u is invertible and satisfies:

   (a) $s^2(a) = uau^{-1}$ for any $a \in A$.

   (b) $u \cdot s(u)$ is in the center of A (commutes with all other elements).

   (c) $u^{-1} = \sum \beta_i s^2(\alpha_i)$. (You might show that u is invertible by showing this is an inverse!).

6. **Double Duals.** Let A be a Hopf algebra over a field k. Show that all objects in $\operatorname{Rep} A$ are reflexive: that is, they are isomorphic to their double dual. To do this, first take any $V \in \operatorname{Rep} A$ and understand the left A-module structure on the finite-dimensional k-vector space $\hom_{\mathrm{Vect}_k}(V, k)$.
You'll need to use the element u defined in the previous exercise.

7. **Ribbon Hopf algebras.** Let (A, R) be a quasitriangular Hopf algebra over a field k, with antipode map s. If $R = \sum \alpha_i \otimes \beta_i \in A \otimes A$ for some $\alpha_i, \beta_i \in A$, then we define $u = \sum_i s(\beta_i)\alpha_i$, as before. Let v be an element in the center of A. We say that (A, R, v) is a ribbon Hopf algebra if:

$$v^s = us(u), \;\; s(v) = v, \;\; \epsilon(v) = 1, \;\; \Delta(v) = (R_{21}R_{12})^{-1}(v \otimes v).$$

Show that every quasitriangular Hopf algebra is a Hopf subalgebra of a ribbon Hopf algebra.