Definition of Derivative

Definition (5.1)

Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$ for the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b, t \neq x)$$

and define

$$f'(x) = \lim_{t \to x} \phi(t)$$

provided this limit exists.

We associate with a function f a function f' called the *derivative* of f defined on the points where the above limit exists.

Definition of Derivative Continued

Definition

If f' is defined at a point x we say f is *differentiable* at x. If f' is defined everywhere on $E \subseteq [a, b]$ we say f is differentiable on E.

It is possible to use right and left hand limits to get the notion of right and left hand derivatives. But we won't do this.

Differentiation implies Continuous

Theorem (5.2)

Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$ then f is continuous at x.

The converse of this theorem is not true.

Differentiation and Operations

Theorem (5.3)

Suppose f and g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g$, fg, and f/g are differentiable at x, and

(a) $(f + g)'(x) = f'(x) + g'(x)$

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

(c) $(\frac{f}{g})'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$

Examples

- The derivative of any constant function is 0
- If $f(x) = x$ then $f'(x) = 1$
- If $f(x) = x^n$ then $f'(x) = nx^{n-1}$

So every polynomial is differentiable.

Chain Rule

Theorem (5.5)

Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If

$$h(t) = g(f(t)) \quad (a \leq t \leq b)$$

then f is differentiable at x, and

$$h'(x) = g'(f(x))f'(x)$$