Bounded Maps

Definition

A mapping $\mathbf{f}$ of a set E into $\mathbb{R}^k$ is said to *bounded* if there is a real number M such that $|\mathbf{f}(x)| \leq M$ for all $x \in E$

Continuous Maps of Compact Spaces

Theorem

Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

Closed and Bounded Sets and Compact Spaces

Theorem

If $\mathbf{f}$ *is a continuous mapping of a compact metric space* X *into* $\mathbb{R}^k$, *then* $\mathbf{f}(X)$ *is closed and bounded. Thus* $\mathbf{f}$ *is bounded.*

Real Functions on Compact Metric Spaces

Theorem

Suppose f is a continuous real valued function on a compact metric space X and

$$M = sup_{p \in X}\ f(p) \quad m = inf_{p \in X}\ f(p)$$

Then there exists points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$

Here $\sup_{p \in X} f(p)$ is the least upper bound of $\{f(p) : p \in X\}$ and $\inf_{p \in X} f(p)$ is the greatest lower bounded of $\{f(p) : p \in X\}$.

Continuous 1-1 mappings

Theorem

Suppose f is a continuous $1-1$ mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by

$$f^{-1}(f(x)) = x \quad (x \in X)$$

is a continuous mapping of Y onto X.

Uniform Continuity

Definition

Let f be a mapping of a metric space X into a metric space Y. We say that f is *uniformly continuous* on X if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \epsilon$$

for all p and q in X for which $d_X(p, q) < \delta$.

Uniform Continuity and Compact Metric Spaces

Theorem

Let f be a continuous mapping of a compact space X into a metric space Y. Then f is uniformly continuous on X.

Non-Compact Sets of Reals

Theorem

Let E be a noncompact set in $\mathbb{R}^1$. Then

(a) *There exists a continuous function on E which is not bounded*

(b) *There exists a continuous bounded function on E which has no maximum*

If in addition E is bounded then

(c) *There exists a continuous function on E which is not uniformly continuous.*

Continuity and Connectedness

Theorem

If f is a continuous mapping of a metric space X into a metric space Y and if E is a connected subset of X then $f(E)$ is a connected subset of Y.

Continuity and Connectedness

Theorem

Let f be a continuous real valued function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$ then there exists a point $x \in (a, b)$ such that $f(x) = c$

Note this theorem does not have a converse. I.e. it is not always the case that fact that if

> For all c with $f(a) < c < f(b)$ then there exists a point $x \in (a, b)$ such that $f(x) = c$

that f is then continuous.