

Limit Points of Functions

Definition

Suppose

- ▶ (X, d_X) and (Y, d_Y) are metric spaces
- ▶ $E \subseteq X$ and f maps E into Y .
- ▶ p is a limit point of E

We then write $f(x) \rightarrow q$ as $x \rightarrow p$ or $\lim_{x \rightarrow p} f(x) = q$ if $q \in Y$ satisfies the following property:

For every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ whenever $0 < d_X(x, p) < \delta$

Notice that we don't require $p \in E$ and it is possible that $f(p) \neq \lim_{x \rightarrow p} f(x)$.

Theorem on Limit Points of Functions

Theorem

Let X, Y, E, f and p be as in the previous definition. Then

$$\lim_{x \rightarrow p} f(x) = q$$

if and only if

$$\lim_{n \rightarrow \infty} f(p_n) = q$$

for every sequence $\{p_n\}$ in E such that

$$p_n \neq p \quad \lim_{n \rightarrow \infty} p_n = p$$

Corollary

If f has a limit point at p , then this limit is unique.

Sums and Products of Functions

Definition

Suppose we have two functions f and g which take values in the complex numbers, both defined on E . By $f + g$ we mean the function such that for every $x \in E$

$$(f + g)(x) = f(x) + g(x)$$

Similarly we define the difference $f - g$, the product fg , and the quotient f/g (when g does not have 0 in its range).

If f assigns to each element $x \in E$ the same value c we say f is a constant function and write $f = c$.

If f, g are real valued functions and for all $x \in E$ we have $f(x) \geq g(x)$ we write $f \geq g$

Sums and Products of Functions

Definition

If \mathbf{f}, \mathbf{g} map E into \mathbb{R}^k we define $\mathbf{f} + \mathbf{g}$ by

$$(\mathbf{f} + \mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x)$$

and $\mathbf{f} \cdot \mathbf{g}$ by

$$(\mathbf{f} \cdot \mathbf{g})(x) = \mathbf{f}(x) \cdot \mathbf{g}(x)$$

And if γ is a real number

$$(\gamma \mathbf{f})(x) = \gamma \mathbf{f}(x)$$

Sums and Products of Functions

Theorem

Suppose X is a metric space with $E \subseteq X$ and p a limit point of E . Further suppose f and g are complex valued functions on E with

$$\lim_{x \rightarrow p} f(x) = A \quad \lim_{x \rightarrow p} g(x) = B$$

Then

- (a) $\lim_{x \rightarrow p} (f + g)(x) = A + B$
- (b) $\lim_{x \rightarrow p} (fg)(x) = AB$
- (c) $\lim_{x \rightarrow p} \frac{f}{g}(x) = \frac{A}{B}$ if $B \neq 0$

Note that if \mathbf{f} and \mathbf{g} map E into \mathbb{R}^k then (a) remains true and (b) becomes

$$\lim_{x \rightarrow p} (\mathbf{f} \cdot \mathbf{g})(x) = \mathbf{A} \cdot \mathbf{B}$$

Continuous Functions

Definition

Suppose (X, d_X) and (Y, d_Y) are metric spaces, $E \subseteq X$, $p \in E$ and f maps E into Y . f is said to be *continuous* at p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(y)) < \epsilon$$

for all $x \in E$ for which $d(x, p) < \delta$.

If f is continuous at every point of E then f is said to be *continuous on E* .

Continuous Functions Continued

Notice that f has to be defined at p for it to be continuous at p .

Notice that if p is an isolated point of E then any function which has E as its domain is continuous at p because for any $\epsilon > 0$ we can choose a $\delta > 0$ such that the only point $x \in E$ for which $d_X(x, p) < \delta$ is $x = p$

Continuous Functions and Limits

Theorem

Suppose all the conditions of the previous definition hold and that p is a limit point of E . Then f is continuous at p if and only if $\lim_{x \rightarrow p} f(x) = f(p)$.

Composition of Continuous Functions

Theorem

Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y , g maps the range of f , $f(E)$, into Z , and h is the mapping of E into Z defined by

$$h(x) = g(f(x)) \quad (x \in E)$$

If f is continuous at a point p and if g is continuous at $f(p)$ then h is continuous at p

We call the function h the *composition* or the *composite* of f and g and we write it as

$$h = g \circ f$$

Continuous Functions and Open Sets

Theorem

A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(V)$ is open in X for every set V which is open in Y .

Corollary

A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed C in Y .

Complex Continuous Functions

Theorem

Let f and g be continuous functions from a metric space X into \mathbb{C} . Then $f + g$, fg , f/g are continuous on X .

Continuous Vector Valued Functions

Theorem

- (a) Let f_1, \dots, f_k be real functions on a metric space X and let \mathbf{f} be the mapping of X into \mathbb{R}^k defined by

$$\mathbf{f}(x) = (f_1(x), \dots, f_k(x)) \quad (x \in X)$$

Then \mathbf{f} is continuous if and only if each of f_i are continuous.

- (b) If \mathbf{f} and \mathbf{g} are continuous mappings of X into \mathbb{R}^k then $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ are continuous.

The functions f_1, \dots, f_k are called the components of \mathbf{f} .

Note that $\mathbf{f} + \mathbf{g}$ is a mapping into \mathbb{R}^k whereas $\mathbf{f} \cdot \mathbf{g}$ is a mapping into the real numbers.

Examples

If x_1, \dots, x_k are the coordinates of the point $\mathbf{x} \in \mathbb{R}^k$ then the functions φ_i defined by

$$\varphi_i(\mathbf{x}) = x_i \quad (\mathbf{x} \in \mathbb{R}^k)$$

are continuous since

$$|\varphi_i(\mathbf{x}) - \varphi_i(\mathbf{y})| \leq |\mathbf{x} - \mathbf{y}|$$

(and we can take $\delta = \epsilon$). The functions φ_i are called the *coordinate functions*

Examples

If

$$P(\mathbf{x}) = \sum c_{n_1, \dots, n_k} x_1^{n_1} \cdots x_k^{n_k} \quad (\mathbf{x} \in \mathbb{R}^k)$$

Where the c_{n_1, \dots, n_k} are complex numbers and there are only finitely many terms then $P(\mathbf{x})$ is continuous.

The mapping $\mathbf{x} \rightarrow |\mathbf{x}|$ with $\mathbf{x} \in \mathbb{R}^k$ is continuous.

5 Similarly if \mathbf{f} is a continuous function from a metric space X into \mathbb{R}^k then $|\mathbf{f}(p)|$ is continuous.

Remarks

While we have defined the notion of continuous for a subset $E \subseteq X$ the complement of E in X plays no role (unlike in the definition of limits of a function). Accordingly when talking about continuous functions we loose nothing by simply assuming that $E = X$.