Addition of Series

Theorem

If $\sum a_n = A$ *and* $\sum b_n = B$ *then* $\sum(a_n + b_n) = A + B$ *and for any* c, $\sum ca_n = cA$.

Cauchy Product

Definition

Given $\sum a_n$ and $\sum b_n$ we put

$$c_n = \sum_{k=0}^{n} a_k b_{n+k}$$

we call $\sum c_n$ the *Cauchy product* of $\sum a_n$ and $\sum b_n$

The definition is motivated by the fact that

$$\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n = (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots$$

$$= c_0 + c_1 z + c_2 z^2 + \cdots = \sum_{n=0}^{\infty} c_n z^n$$

Divergent Product

Example

Let

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

If $\sum_{n=0}^{\infty} c_n$ is the product of $\sum_{n=0}^{\infty} a_n$ with itself then

$$\sum_{n=0}^{\infty} c_n = 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{2}\sqrt{3}} + \right.$$

So

$$c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}}$$

Divergent Product Continued

Since

$$(n-k+1)(k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \leq \left(\frac{n}{2}+1\right)^2$$

we have

$$|c_n| \geq \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2}$$

So c_n doesn't converge to 0

Results on Products

Theorem

Suppose

(a) $\sum_{n=0}^{\infty} a_n$ *converges absolutely.*

(b) $\sum_{n=0}^{\infty} a_n = A$

(c) $\sum_{n=0}^{\infty} b_n = B$

(d) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

then

$$\sum_{n=0}^{\infty} c_n = AB$$

Cauchy Products

Theorem

If the series $\sum a_n, \sum b_n$ *and* $\sum c_n$ *converges to* A, B, C *and* $c_n = a_0 b_n + \cdots + a_n b_0$ *then* $C = AB$

Rearrangement

Definition

Let $\{k_n\}$ be a sequence in which every positive integer appears once and only once (i.e. it is a bijection from $\mathbb{N}$ to $\mathbb{N}$). Putting

$$a'_n = a_{k_n}$$

we say that $\sum a'_n$ is a *rearrangement* of $\sum a_n$.

If $\{s_n\}$ and $\{s'_n\}$ are the sequence of partial sums of $\sum a_n$ and $\sum a'_n$ then $\{s_n\}$ and $\{s'_n\}$ consist of completely different numbers.

Rearrangement

Theorem

Let $\sum a_n$ be a series of real numbers which converges, but not absolutely. Suppose

$$-\infty \leq \alpha \leq \beta \leq \infty$$

Then there exists a rearrangement $\sum a'_n$ with partial sums s'_n such that

$$\lim_{n\to\infty} inf\ s'_n = \alpha \quad \lim_{n\to\infty} sup\ s'_n = \beta$$

Absolute Convergence and Rearrangement

Theorem

Suppose $\sum a_n$ is a series of complex numbers which converges absolutely. Then every rearrangement of $\sum a_n$ converges and they all converge to the same value.