

# Series

All sequences/series will be complex valued

## Definition

Given a sequence  $\{a_n\}$  we associate with the sequence  $\{a_n\}$  a sequence  $\{s_n\}$  where

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

We call this the sequence of *partial sums*.

Notice that we can recover the sequence  $\{a_n\}$  from the sequence  $\{s_n\}$  by letting  $a_1 = s_1$  and  $a_{n+1} = s_{n+1} - s_n$

# Infinite Series

## Definition

Given a sequence  $\{a_n\}$ , if  $\{s_n\}$  converges to a limit  $s$  then we use the notation

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots = s$$

and say the series *converges*. If  $\{s_n\}$  diverges we say the series *diverges*.

We will also define  $\Sigma a_n = \sum_{i=1}^{\infty}$ .

# Condition for Convergence

## Theorem

$\sum a_n$  converges if and only if for every  $\epsilon > 0$  there is an integer  $N$  such that

$$\left| \sum_{k=n}^m a_k \right| \leq \epsilon$$

if  $m \geq n \geq N$

## Misc. Results

### Corollary

*If  $\sum a_n$  converges then for all  $\epsilon$  there is a  $N$  such that whenever  $n \geq N$  then  $|a_n| \leq \epsilon$*

### Theorem

*If  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$*

# Convergence Tests

## Theorem

*A series of nonnegative terms converges if and only if the partial sums form a bounded sequence*

# Comparison Tests

## Theorem

- (a) *If there is a  $N_0$  such that  $|a_n| \leq c_n$  for all  $n \geq N_0$  and  $\sum c_n$  converges, then  $\sum a_n$  also converges.*
- (b) *If there is a  $N_0$  such that  $a_n \geq d_n \geq 0$  for all  $n \geq N_0$  and  $\sum d_n$  diverges, then  $\sum a_n$  also diverges.*

# Geometric Series

## Theorem

If  $0 \leq x \leq 1$  then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

If  $x \geq 1$  then the series diverges.

# Geometric Series

## Theorem

Suppose  $a_1 \geq a_2 \geq a_3 \geq \dots \geq 0$ . Then the series  $\sum_{n=1}^{\infty} a_n$  converges if and only if

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + \dots$$

converges.

# Example Series

## Theorem

$\sum \frac{1}{n^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

## Theorem

If  $p > 1$  then

$$\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^p}$$

converges. If  $p \leq 1$  then the series diverges.

# Definition of $e$

## Definition

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

where  $n! = 1 \cdot 2 \cdot 3 \cdots n$  if  $n \geq 1$  and  $0! = 1$ .

## Theorem

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

# $e$ is irrational

Theorem  
 *$e$  is irrational.*