Limits of $\pm\infty$

Definition

Let $\{s_n\}$ be a sequence of real numbers. If

> For all real M there is an integer N where $s_n \geq M$ whenever $n \geq N$

then we write

$$s_n \longrightarrow +\infty$$

If

> For all real M there is an integer N where $s_n \geq M$ whenever $n \leq N$

then we write

$$s_n \longrightarrow -\infty$$

Upper and Lower Limits

Definition

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of x (in the extended real number system) such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$. (So E has all subsequential limits of $\{s_n\}$ plus, possibly, $-\infty$ or $+\infty$). Let

$$s^* = \sup E$$

$$s_* = \inf E$$

s^* is the *upper bound* of $\{s_n\}$ and s_* is the *lower bound* of $\{s_n\}$. We write

$$\lim_{n\to\infty} \sup s_n = s^*$$

$$\lim_{n\to\infty} \inf s_n = s_*$$

Properties of Lim Sup and Lim Inf

Theorem

Let $\{s_n\}$ be a sequence of real numbers, let $s^* = \lim_{n\to\infty} sup\ s_n$ and let E be the set of subsequential limits of $\{s_n\}$. Then

(a) $s^* \in E$

(b) *If $x > s^*$, then there is an integer N such that $n \geq N$ implies $s_n < x$.*

Further s^* is the only extended real number satisfying properties (a) and (b).

There is an analogous result for $s_* = \lim_{n\to\infty} inf\ s_n$

Examples

(a) Let $\{s_n\}$ be a sequence containing all rational numbers. Then every real number is a subsequential limit and

$$\lim_{n\to\infty} \inf s_n = -\infty, \quad \lim_{n\to\infty} \sup s_n = +\infty$$

(b) Let $s_n = \frac{(-1)^n}{1+(1/n)}$ Then

$$\lim_{n\to\infty} \inf s_n = -1, \quad \lim_{n\to\infty} \sup s_n = 1$$

(c) For a real valued sequence $\{s_n\}$ with $\lim_{n\to\infty} s_n = s$ if and only if

$$\lim_{n\to\infty} \inf s_n = \lim_{n\to\infty} \sup s_n = s$$

Ordering of Lim Sup/Lim Inf

Theorem

If $s_n \leq t_n$ for $n \geq N$ where N is fixed then

$$\lim_{n \to \infty} inf\ s_n \leq \lim_{n \to \infty} inf\ t_n$$

$$\lim_{n \to \infty} sup\ s_n \leq \lim_{n \to \infty} sup\ t_n$$

Squeeze Theorem

We now compute the limits of some special sequences. The proofs are based on the following lemma

Lemma

If $0 \leq x_n \leq s_n$ *for* $n \geq N$ *where* N *is some fixed number and* $s_n \longrightarrow 0$ *then* $t_n \longrightarrow 0$ *as well.*

Special Limits

Theorem

(a) *If* $p > 0$ *then* $\lim_{n\to\infty} \frac{1}{n^p} = 0$

(b) *If* $p > 0$ *then* $\lim_{n\to\infty} \sqrt[n]{p} = 1$

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

(d) *If* $p > 0$ *and* α *is real, then* $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

(e) *If* $|x| < 1$ *then* $\lim_{n\to\infty} x^n = 0$